Cosmic microwave background and large-scale structure

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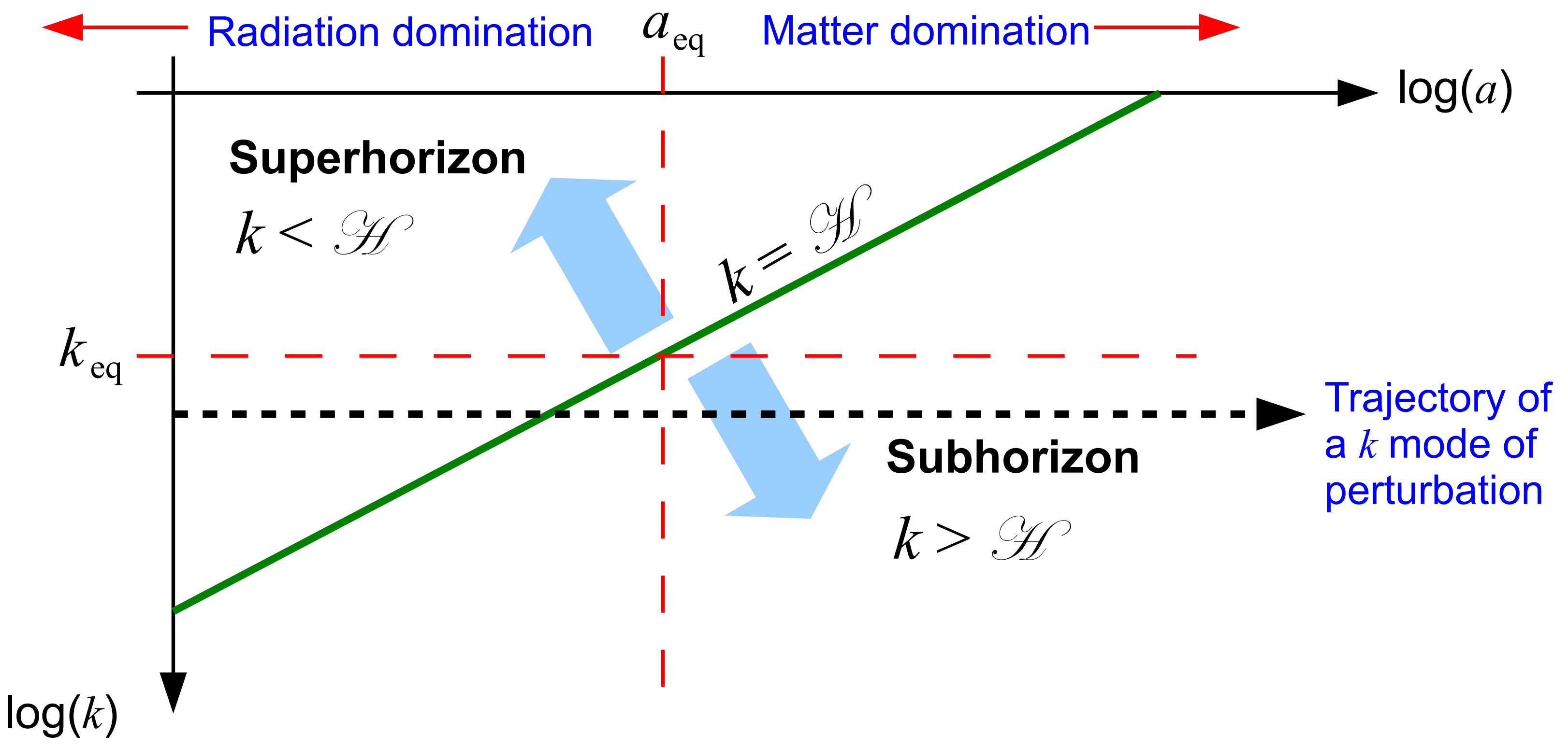
Plan...

1. Review: Homogeneous and isotropic universe
2. Inhomogeneities I: cosmological perturbation theory
3. Inhomogeneities II: Boltzmann equation
4. Initial conditions
5. Approximate solutions I: matter density perturbations
6. Approximate solutions II: CMB temperature fluctuations
7. Cosmological parameters from CMB temperature anisotropies

4. Initial conditions…

4.1 From superhorizon to subhorizon fluctuations...

- We observe **subhorizon** scales today. But all scales must have been **superhorizon** deep in the radiation era (since $\mathscr{H}$ decreases with time).



4.1 From superhorizon to subhorizon fluctuations...

- What are the initial superhorizon perturbations?
- Consider the scalar Boltzmann equations in the $k << \mathscr{H}$ limit:

Photons $\dot{\delta}_\gamma - 4\dot{\Phi} \simeq 0$

Massles neutrinos $\dot{\delta}_\nu - 4\dot{\Phi} \simeq 0$

$\dot{\delta}_c - 3\dot{\Phi} \simeq 0$ CDM

$\dot{\delta}_b - 3\dot{\Phi} \simeq 0$ Baryons

$$\longrightarrow \quad 4\dot{\delta}_b = 4\dot{\delta}_c = 3\dot{\delta}_\nu = 3\dot{\delta}_\gamma$$

- There are **two types of solutions**:
  - Adiabatic perturbations: $\delta_b = \delta_c = \frac{3}{4}\delta_\nu = \frac{3}{4}\delta_\gamma$
  - Isocurvature perturbations: $\delta_{b,c} = \frac{3}{4}\delta_\gamma + C_{b,c}, \quad \delta_\nu = \delta_\gamma + C_\nu$

Some constants

4.2 Adiabatic initial conditions...

- For ordinary particles:

$$\delta_b = \delta_c = \frac{3}{4}\delta_\nu = \frac{3}{4}\delta_\gamma \quad \longleftrightarrow \quad \frac{n_\alpha(x)}{n_\gamma(x)} \equiv \frac{\bar{n}_\alpha}{\bar{n}_\gamma}, \quad \alpha = b, c, \nu$$

  - i.e., local ratio of particle number densities = global ratio

- A **necessary consequence** of single-field inflation: If all perturbations come from the same source, they must be the same.

4.2 Adiabatic initial conditions...

- For ordinary particles:

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  - i.e., local ratio of particle number densities = global ratio

- A **necessary consequence** of single-field inflation: If all perturbations come from the same source, they must be the same.

- If different sources (e.g., multi-field inflation, curvaton), a mixture of adiabatic and isocurvature perturbations is possible, but...
  - … if equilibrium is established afterwards for **all interactions**, the particle number densities must obey either FD or BE statistics locally → **local ratio = global ratio**

4.2 Adiabatic initial conditions...

- Assuming adiabatic perturbations, we can relate the fluid perturbations to the metric perturbations in the superhorizon limit deep in RD:

Boltzmann equation

$$\dot{\delta}_\gamma - 4\dot{\Phi} = 4\dot{\Theta}_0^{(S)} - 4\dot{\Phi} \simeq 0$$

Assuming $\Phi \simeq \Psi$

Einstein equation

$$3\mathscr{H}(\dot{\Phi} + \mathscr{H}\Phi) \simeq -16\pi G a^2 (\bar{\rho}_\gamma + \bar{\rho}_\nu)\Theta_0^{(S)}$$

- Combine into a 2nd order DE:

We will be using this trick many more times...

$$\eta\ddot{\Phi} + 4\dot{\Phi} = 0$$

$$\Phi(k \ll \mathscr{H}, \eta \ll \eta_{eq}) = \text{time const.} \equiv \Phi_p(k)$$

$$\Theta_0^{(S)}(k \ll \mathscr{H}, \eta \ll \eta_{eq}) = -\Phi_p(k)/2$$

Photon temperature monopole fluctuation

4.2 Adiabatic initial conditions...

- "Primordial" superhorizon fluid perturbations:

$$\delta_b(k) = \delta_c(k) = \frac{3}{4}\delta_\nu(k) = \frac{3}{4}\delta_\gamma(k) = -\frac{3}{2}\Phi_p(k)$$

$$v_b^{(S)}(k) = v_c^{(S)}(k) = v_\nu^{(S)}(k) = v_\gamma^{(S)}(k) = -\frac{1}{2\mathscr{H}}\Phi_p(k)$$

- **But what is $\Phi_p(k)$???** → **INFLATION**

4.3 Some inflation basics...

- **Inflation** = a scalar-field driven phase of accelerating expansion, before the onset of radiation domination.

- Action for a scalar field $\varphi(x, \eta)$:

$$S = \int d^4x \sqrt{-g}\, \mathscr{L} = \int d^4x \sqrt{-g} \left[\frac{1}{2} \partial_\mu \phi \, \partial^\nu \phi + V(\phi) \right]$$

- **Stress-energy tensor**:

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - g_{\mu\nu} \mathscr{L}$$

By definition

$$T^{\mu\nu} = 2\sqrt{-g} \frac{\delta S}{\delta g_{\mu\nu}}$$

4.3 Some inflation basics...

- Split the field value into a homogeneous and a perturbed part:

$$\phi(x^i, \eta) = \bar{\phi}(\eta) + \delta\phi(x^i, \eta)$$

- Likewise for the stress-energy tensor: $T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$

- The **homogeneous** part is given by

$$-\bar{T}^0{}_0 = \frac{1}{2a^2}\left(\frac{d\bar{\phi}}{d\eta}\right)^2 + V(\bar{\phi}) \equiv \bar{\rho}_\phi, \quad \bar{T}^i{}_j = \left[\frac{1}{2a^2}\left(\frac{d\bar{\phi}}{d\eta}\right)^2 - V(\bar{\phi})\right]\delta^i_j \equiv \bar{P}_\phi$$

Energy density

Pressure

- Equation of motion (homogeneous part):

$$\ddot{\bar{\phi}} + 2\mathscr{H}\dot{\bar{\phi}} + a^2\frac{\partial V}{\partial\eta} = 0$$

4.3 Some inflation basics...

- To get a phase of accelerated expansion from **slow-roll inflation**:
  - Potential energy should dominate over kinetic energy.
  - Scalar field should dominate the energy density of the universe.

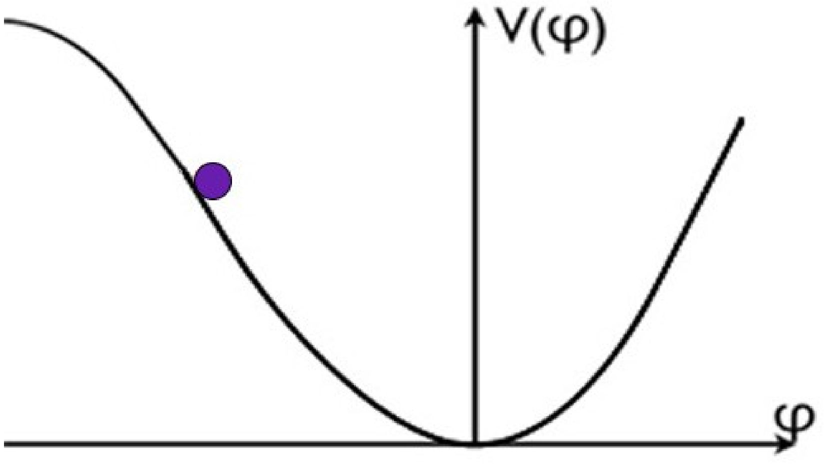
- Slow-roll parameters:

$$\epsilon_1 \equiv \frac{m_{\rm pl}^2}{16\pi}\left(\frac{V_{,\phi}}{V}\right)^2 = \frac{4\pi}{m_{\rm pl}^2}\left(\frac{\dot{\phi}}{\mathscr{H}}\right)^2$$

$$\epsilon_2 \equiv \frac{m_{\rm pl}^2}{8\pi}\frac{V_{,\phi\phi}}{V}$$

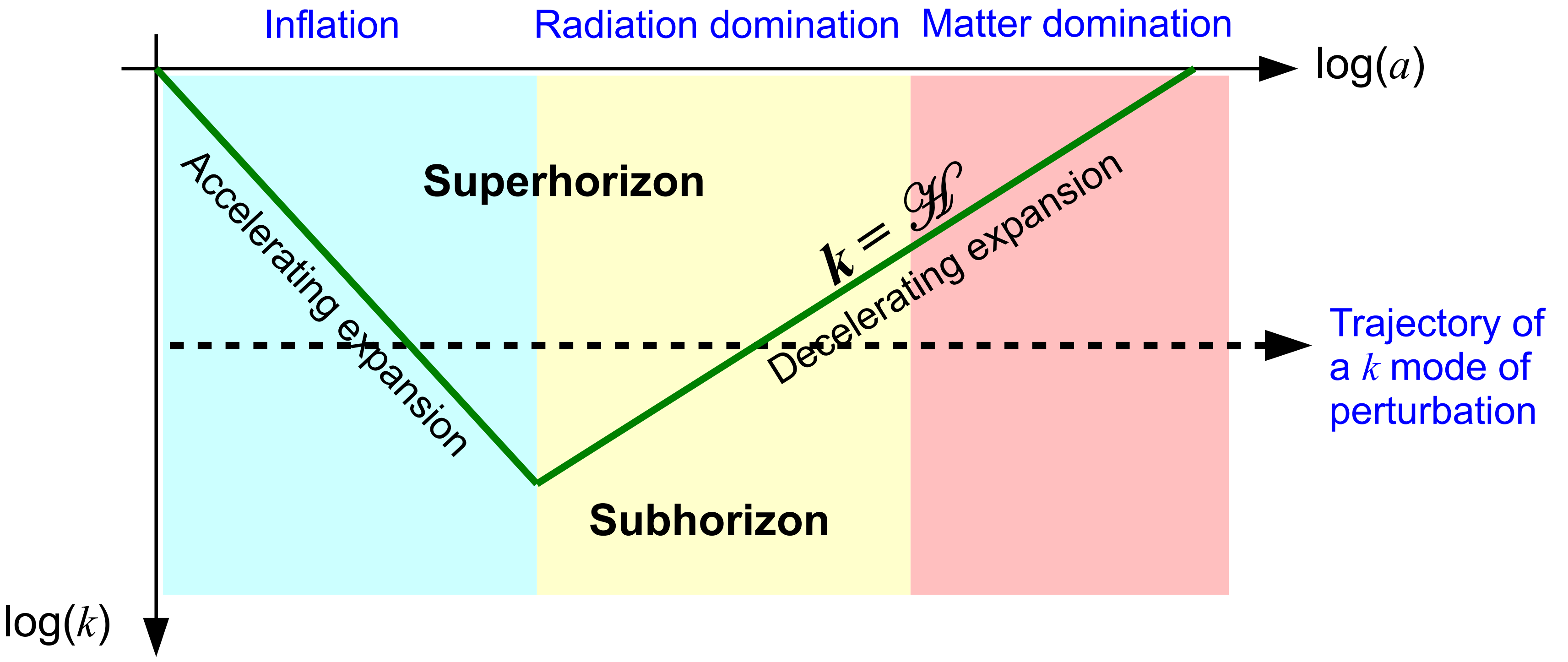
$\varepsilon \ll 1$ during inflation

- When $\varepsilon \to 1$, **inflation ends**. Energy in φ is somehow turned into a thermal bath of particles ("reheating") → RD begins.



4.4 Superhorizon fluctuations from inflation...

- **Quantum fluctuations** excited during inflation are stretched to cosmological scales, become frozen on superhorizon scales, and are imprinted on the spacetime metric.



4.4 Superhorizon fluctuations from inflation...

- **Slow-roll inflation predicts**:
  - Scalar perturbations (from quantum fluctuations of the inflaton)
  - **No** vector perturbations (no vector source at linear order in a scalar field theory)
  - Small amount of tensor perturbations (from quantum fluctuations of the spacetime metric itself, assuming it can be quantised)

4.4 Superhorizon fluctuations... : scalar perturbations...

- In the **spatially flat gauge** ($H_L = 0$, $H_T = 0$), the equation of motion for field perturbations $\delta\varphi$ (to linear order) is

$$\delta\ddot{\phi} + 2\mathscr{H}\delta\dot{\phi} + \left(a^2 V_{,\phi\phi} + k^2\right)\delta\phi = 0$$

- In the slow-roll limit: $a^2 V_{;\phi\phi} = 3\epsilon_2 \mathscr{H}^2 \ll 1$

$$\longrightarrow \quad \delta\ddot{\phi} + 2\mathscr{H}\delta\dot{\phi} + k^2\delta\phi = 0$$

η_{end} = a reference time; usually chosen to be the end of inflation

- Classical solution:

$$\delta\phi(k,\eta) = \frac{1}{a(\eta)\sqrt{2k}}\left(1 - \frac{i}{k(\eta - \eta_{\mathrm{end}})}\right) e^{ik(\eta - \eta_{\mathrm{end}})} a_k + \mathrm{c.c.}$$

4.4 Superhorizon fluctuations... : scalar perturbations...

- The solution again:

$$\delta\phi(k,\eta) = \frac{1}{a(\eta)\sqrt{2k}}\left(1 - \frac{i}{k(\eta-\eta_{\rm end})}\right) e^{ik(\eta-\eta_{\rm end})} a_k + \text{c.c.}$$

- Evolution history:

1. At early times, $|k(\eta - \eta_{\rm end})| >> 1$.

$\rightarrow$ **Subhorizon** evolution: oscillatory

2. When $|k(\eta - \eta_{\rm end})| = 1$, k mode **exits** the horizon.

3. When $|k(\eta - \eta_{\rm end})| << 1$, **superhorizon** evolution:

$$\delta\phi \rightarrow \frac{iH_{\rm inf}}{\sqrt{2k^3}}(a_k - a_k^*)$$

Time-independent: perturbations are "frozen" after horizon exit

From Friedmann equation

$$a = -H_{\rm inf}^{-1}(\eta-\eta_{\rm end})^{-1}$$

$$(\eta-\eta_{\rm end})^{-1} = -\mathscr{H}$$

4.4 Superhorizon fluctuations... : scalar perturbations...

- Since the classical solution is that of a harmonic oscillator, we know how to quantise it!
- Promote $\delta\varphi$ to an operator, and

$$a_k^* \rightarrow \hat{a}_k^\dagger , \qquad a_k \rightarrow \hat{a}_k$$

Creation and annihilation operators

- Vacuum expectation value:

$$\langle 0 | \delta\hat{\phi} | 0 \rangle = 0$$

- But has a variance:

$$\langle 0 | \delta\hat{\phi}^\dagger \delta\hat{\phi} | 0 \rangle = \frac{H_{\rm inf}^2}{2k^3}$$

In the superhorizon limit

4.4 Superhorizon fluctuations... : scalar perturbations...

- Define the $\delta\varphi$ power spectrum:

$$\langle 0 | \hat{\delta\phi}^{\dagger}(\boldsymbol{k}) \hat{\delta\phi}(\boldsymbol{k}') | 0 \rangle |_{k=\mathscr{H}} = (2\pi)^3 \delta_D^{(3)}(\boldsymbol{k}-\boldsymbol{k}') P_{\delta\phi}(k)$$

$$P_{\delta\phi}(k) = |\delta\phi(k)|^2 = \frac{H^2}{2k^3} |_{k=\mathscr{H}}$$

Evaluated at horizon exit

- For each k mode, the Hubble rate H is evaluated at **horizon exit** because it does in fact vary a little during inflation.

- Because we are doing linear PT, the power spectrum **characterises completely** the fluctuation statistics → Gaussian fluctuations!
  - **Odd** correlators always vanish.
  - **Even** correlators can be constructed from the power spectrum.

4.4 Superhorizon fluctuations... : scalar perturbations...

- The $\delta\varphi$ power spectrum is useful, but what we really need is a prediction for the **Bardeen potential Φ_p** at the start of radiation domination...

- Find Φ using the gauge-invariant curvature perturbation:

$$\zeta \equiv -\Phi - i\mathscr{H}\left(i B^{(S)} + i\dot{H}_T + \frac{k^i k^{-2} T^0{}_i}{\bar{\rho}+\bar{P}}\right)$$

- In the spatially flat gauge:

$$\zeta = -\mathscr{H}\,\delta\phi/\dot{\bar{\phi}}$$ ← We just calculated this!

- Importantly, ζ is constant in time in the superhorizon limit.
  - It will remain the same even as inflation ends and the universe enters into the radiation domination era.

4.4 Superhorizon fluctuations... : scalar perturbations...

- Since ζ is constant outside the horizon, we can now evaluate it during radiation domination in the Newtonian gauge:

$$\zeta = -\frac{3}{2}\Phi_{\mathrm{p}}$$

For adiabatic initial conditions in the superhorizon limit

→ The Φ power spectrum can be now related to the δφ power spectrum:

$$P_{\Phi_{\mathrm{p}}}(k) = |\Phi_{\mathrm{p}}(k)| = \frac{4}{9}\left(\frac{\mathscr{H}}{\dot{\phi}}\right)^2 P_{\delta\phi}(k)$$

$$= \frac{1}{k^3}\frac{8\pi}{9 m_{\mathrm{pl}}^2}\frac{H^2}{\epsilon_1}\Big|_{k=\mathscr{H}}$$

Use the slow-roll parameter

4.4 Superhorizon fluctuations... : scalar perturbations...

- Introduce the dimensionless power spectrum:

$$\Delta^2(k) \equiv \frac{k^3 P_{\Phi_p}(k)}{2\pi^2} = \frac{4}{9\pi m_{\rm pl}^2} \frac{H^2}{\epsilon_1}\Big|_{k=\mathscr{H}}$$

- Because H and ε_1 are almost constant during inflation, $\Delta^2(k)$ is almost scale-invariant → Inflation produces **white noise fluctuations**.

- **Small deviation** from scale-invariance is expected because H^2/ε is evaluated at horizon crossing for each k mode.
  - A convenient parameterisation:

$$\Delta^2(k) = \Delta^2(k_0)(k/k_0)^{n_S - 1}$$

scalar spectral index

$$n_S = 1 + 2\epsilon_2 - 6\epsilon_1$$

k_0 = pivot scale of your choosing

4.4 Superhorizon fluctuations... : vector perturbations...

- Scalar-field inflation models do not produce vector perturbations because there is no vector source.

- But, even if you manage to cook up an inflation model that produces vector perturbations, the perturbations **will decay**, unless there is a source to maintain them.

$$\left(\frac{\partial}{\partial \eta}+2\mathscr{H}\right)\left(B^{(V)}+\dot{H}^{(V)}\right)=-8\pi G a^2 \Pi^{(V)}$$

$$\longrightarrow \quad \left(B^{(V)}+\dot{H}^{(V)}\right)\propto a^{-2} \quad \text{if} \quad \Pi^{(V)}=0$$

4.4 Superhorizon fluctuations... : tensor perturbations...

- Einstein-Hilbert action:

$$S = \bar{S} + \delta S = -\frac{1}{16\pi G}\int d^4x \sqrt{-(\bar{g}+\delta g)}(\bar{R}+\delta R)$$

→ Effective action for tensor perturbations (leading order):

$$\delta S = -\frac{1}{2\pi G}\int d^4x \frac{a^2}{2}[\partial_\mu H^{(T\times)}\partial^\mu H^{(T\times)} + \partial_\mu H^{(T\bullet)}\partial^\mu H^{(T\bullet)}]$$

Two polarisations

→ This is just the action for **two free scalar fields**:

$$\phi^\times \equiv \frac{1}{\sqrt{2\pi G}} H^{(T\times)}$$

$$\phi^\bullet \equiv \frac{1}{\sqrt{2\pi G}} H^{(T\bullet)}$$

→

Equation of motion

$$\ddot{\phi} + 2\mathscr{H}\dot{\phi} + k^2\phi = 0$$

We saw the same equation before for scalar perturbations → use the same tricks to compute the tensor power spectrum!

4.4 Superhorizon fluctuations... : tensor perturbations...

- The tensor power spectrum:

$$P_{H^{(T)}}(k) = \left|H^{(T\times)}(k)\right|^2 + \left|H^{(T\bullet)}(k)\right|^2 = 2\pi G\left(\left|\phi^{\times}(k)\right|^2 + \left|\phi^{\bullet}(k)\right|^2\right)$$
$$= \frac{1}{k^3}\frac{2\pi}{m_{\mathrm{pl}}^2} H^2\Big|_{k=\mathscr{H}}$$

- A convenient parameterisation:

$$k^3 P_{H^{(T)}}(k) = k_0^3 P_{H^{(T)}}(k_0)(k/k_0)^{n_T}$$

Tensor spectrum index

$$n_T = -2\epsilon_1$$

- Tensor-to-scalar ratio:

Not much tensor perturbations expected...

$$r \equiv \frac{P_{H^{(T)}}(k_0)}{P_\zeta(k_0)} = \frac{9}{4}\frac{P_{H^{(T)}}(k_0)}{P_{\Phi_{\mathrm{p}}}(k_0)} = \epsilon_1 \ll 1$$

4.5 Section summary...

- Slow-roll inflation provides a way to generate metric perturbations via quantum fluctuations.

    - Scalar perturbations from a scalar field.

    - Tensor perturbations quantum fluctuations of spacetime itself.

    - **No** vector perturbations.

- Tensor perturbations are highly suppressed relative to scalar perturbations.

- Adiabatic initial conditions are a necessary consequence of single-field inflation, but they are also quite generic if equilibrium for all possible interactions is established after inflation.

5. Approximate solutions I: matter density perturbations...

5.1 Downloadable codes...

- To study the evolution of matter density perturbations we must solve the full scalar Boltzmann-Einstein system of equations.

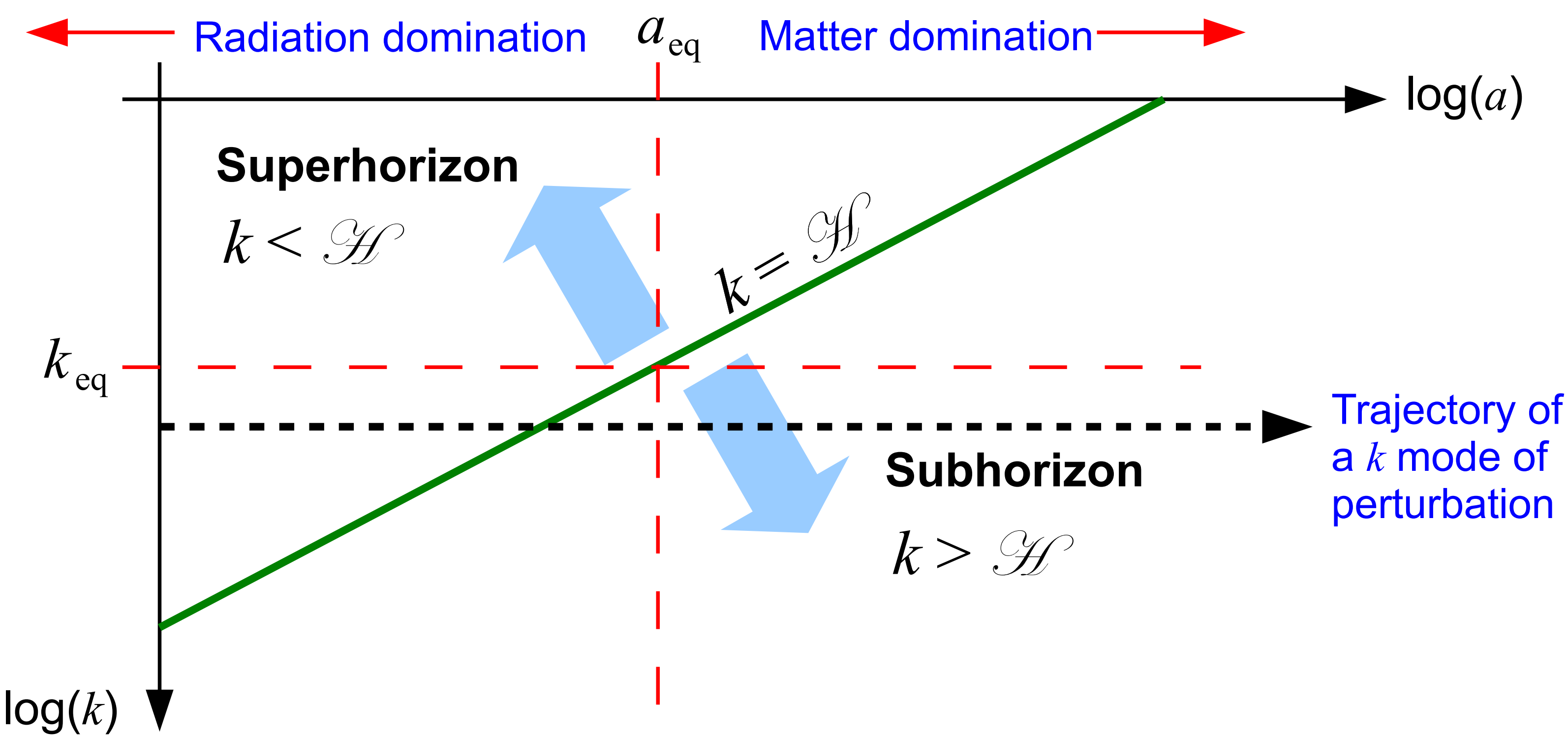
- Exact solutions are possible with **numerically**.

- Some publicly available codes:

  - **COSMICS**: web.mit.edu/edbert/ (F77)
  - **CMBFast**: lambda.gsfc.nasa.gov/toolbox/tb_cmbast_ov.cfm (F77)
  - **CAMB**: camb.info (F90) Maintained
  - **CMBEasy**: www.thphys.uni-heidelberg.de/~robbers/cmbeasy/ (C++)
  - **CLASS**: class-code.net (C) Maintained

5.2 Three stages of evolution...

- **Trajectory** of a k mode: Superhorizon → horizon crossing → subhorizon
- **Crucial point**: When? During matter or radiation domination?



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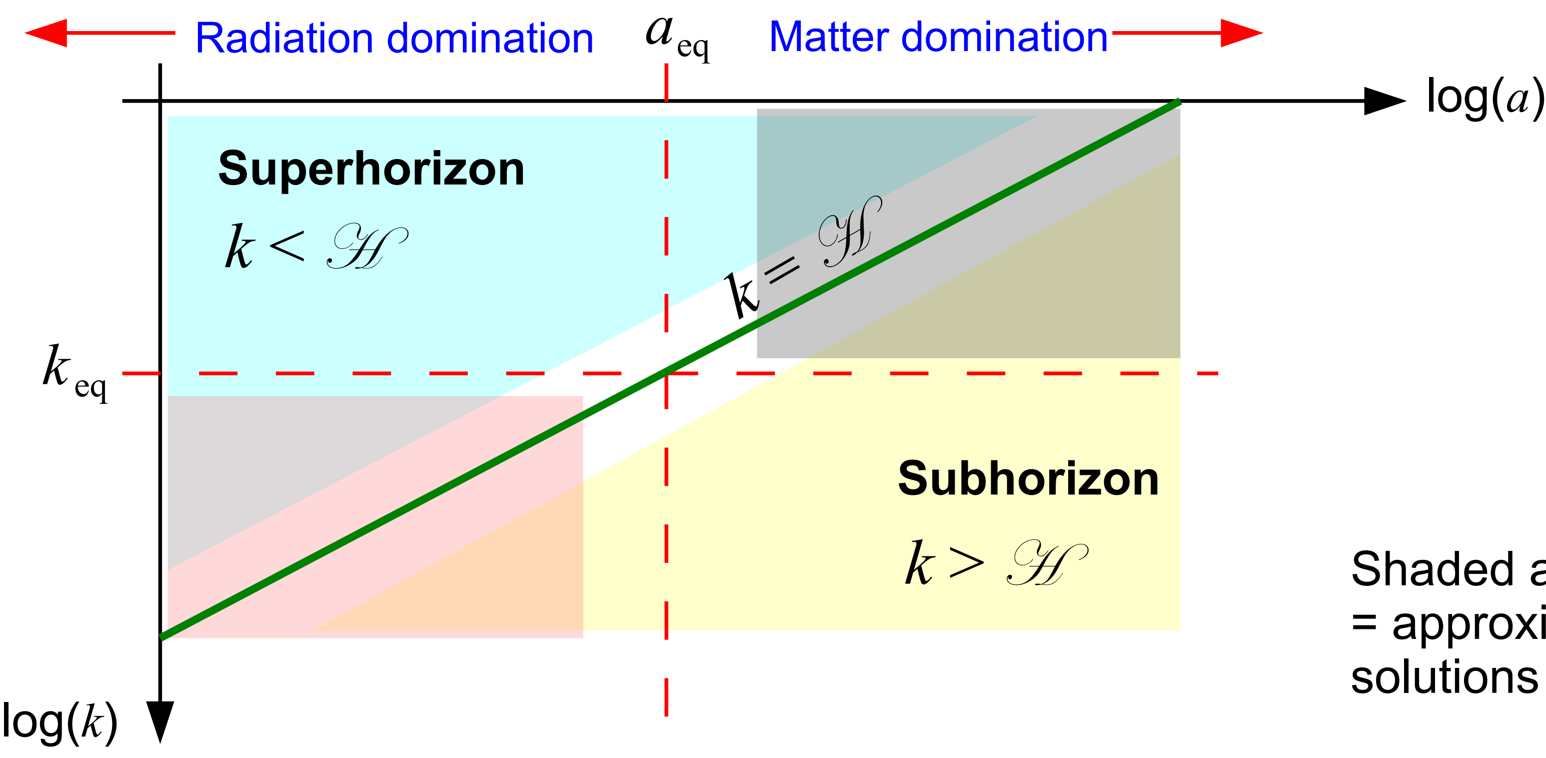


Shaded areas = approximate solutions exist

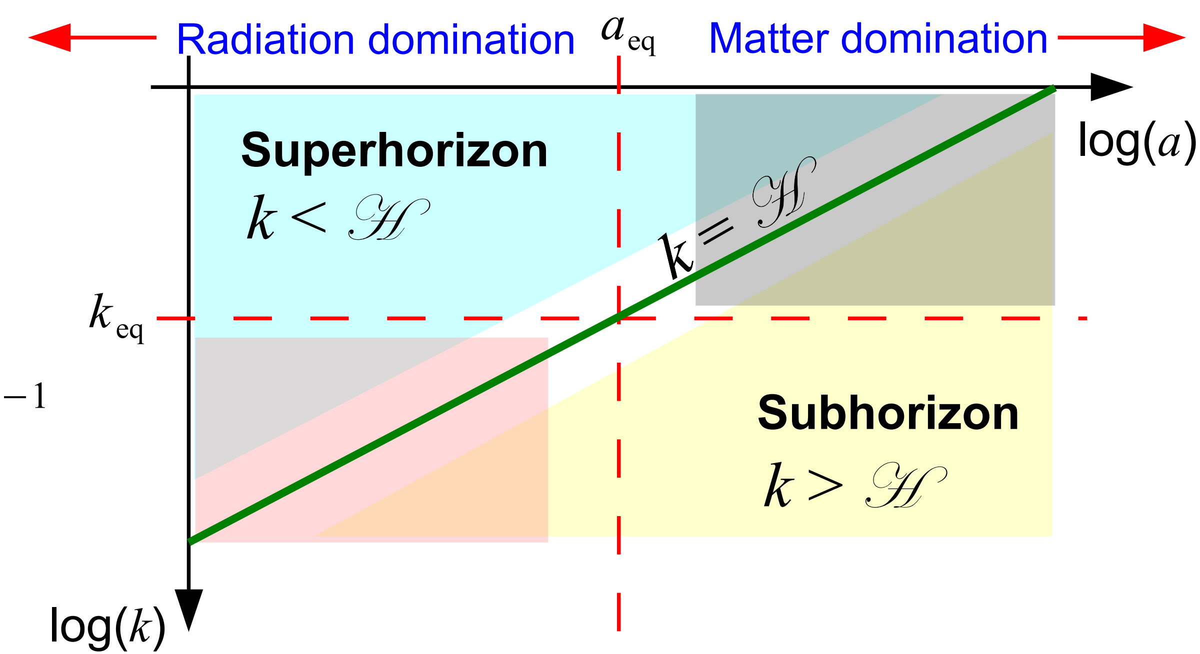
5.2 Three stages of evolution...

- The (comoving Hubble length)$^{-1}$ at matter-radiation equality:

$$k_{\mathrm{eq}} = \mathcal{H}(a_{\mathrm{eq}})$$

- Assuming radiation content = photons + 3 massless neutrinos:

$$k_{\mathrm{eq}} = \sqrt{2\Omega_m H_0^2 / a_{\mathrm{eq}}}$$
$$\simeq 0.073\,\Omega_m h \quad h\,\mathrm{Mpc}^{-1}$$



5.3 Simplified equations...

- Suppose the universe contains only photons and cold dark matter.
- **Boltzmann equations**:
  - CDM: $\dot{\delta}_c - k^2 v_c^{(S)} - 3\dot{\Phi} = 0 \qquad \dot{v}_c^{(S)} + \mathscr{H} v_c^{(S)} + \Psi = 0$
  - Photons $\dot{\delta}_\gamma - \frac{4}{3} k^2 v_\gamma^{(S)} - 4\dot{\Phi} = 0 \qquad \dot{v}_\gamma^{(S)} + \frac{1}{4}\delta_\gamma + \Psi = 0$

Truncated in the tightly-coupled limit

- **Einstein equation** assuming $\Psi = \Phi$:

$$k^2\Phi + 3\mathscr{H}(\dot{\Phi} + \mathscr{H}\Phi) = -4\pi G a^2(\bar{\rho}_c\delta_c + \bar{\rho}_\gamma\delta_\gamma)$$

OR

$$k^2\Phi = -4\pi G a^2[\bar{\rho}_c\delta_c + \bar{\rho}_\gamma\delta_\gamma - \mathscr{H}(3\bar{\rho}_c v_c^{(S)} + 4\bar{\rho}_\gamma v_\gamma^{(S)})]$$

5.4 Superhorizon evolution...

- **Trajectory** of a k mode: Superhorizon → horizon crossing → subhorizon
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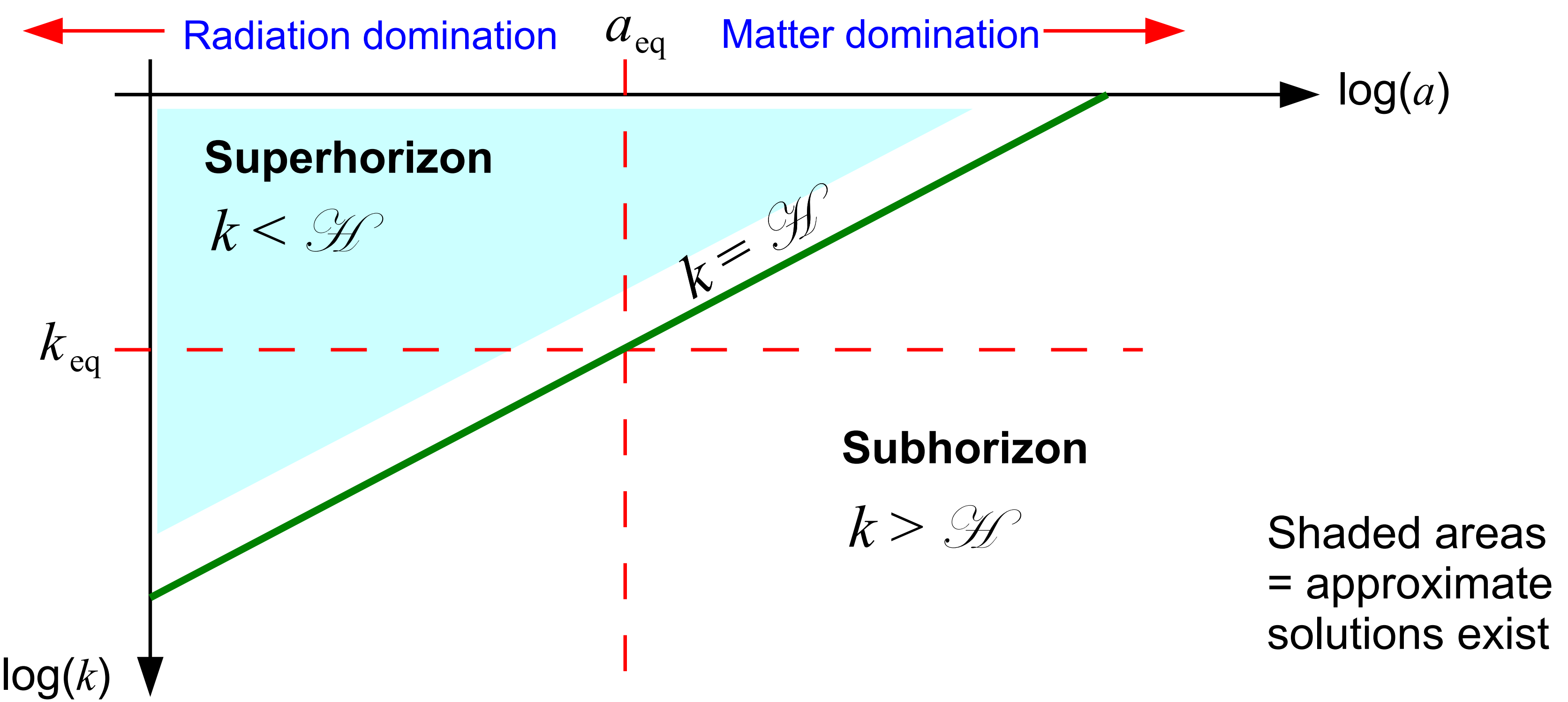


5.4 Superhorizon evolution...

- In the superhorizon limit ($k << \mathscr{H}$), the relevant equations are:

$$\dot{\delta}_\gamma - 4\dot{\Phi} \simeq 0 \qquad \dot{\delta}_c - 3\dot{\Phi} \simeq 0$$

$$3\mathscr{H}(\dot{\Phi} + \mathscr{H}\Phi) \simeq -4\pi G a^2 (\bar{\rho}_c \delta_c + \bar{\rho}_\gamma \delta_\gamma)$$

\+ adiabatic initial conditions $4\delta_c = 3\delta_\gamma$

- Combine into a 2nd order DE for Φ:

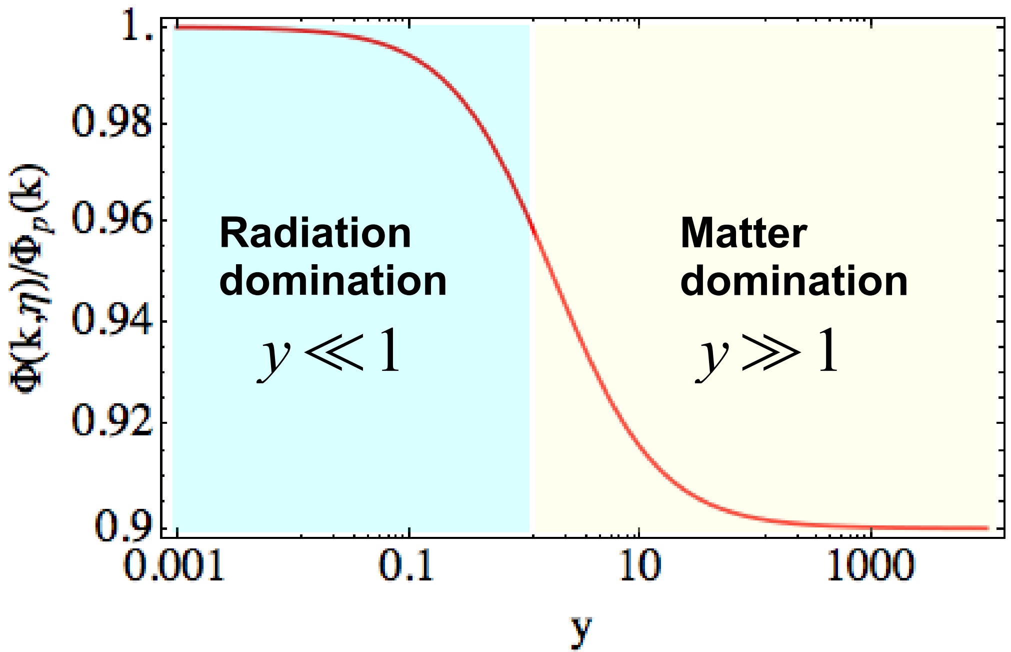
$$\frac{d^2\Phi}{dy^2} + \frac{21y^2 + 54y + 32}{2y(y+1)(3y+4)}\frac{d\Phi}{dy} + \frac{1}{y(y+1)(3y+4)}\Phi = 0$$

$$y \equiv a/a_{\text{eq}}$$

5.4 Superhorizon evolution...

- The growing solution:

$$\Phi(k,\eta)=\frac{\Phi_p(k)}{10\,y^3}\left[9\,y^3+2\,y^2-8\,y+16\sqrt{1+y}-16\right] \qquad y\equiv a/a_{eq}$$



Radiation domination

$$\Phi(k,\eta)=\Phi_p(k)$$

Matter domination

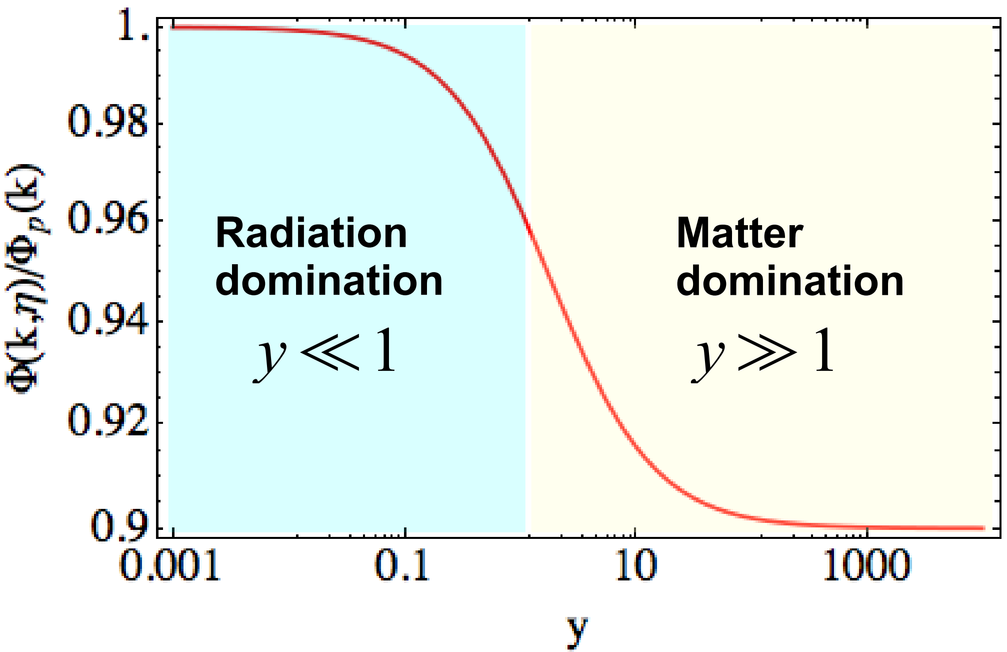
$$\Phi(k,\eta)=\frac{9}{10}\Phi_p(k)$$

Φ is constant deep in MD or RD but changes during transition.

5.4 Superhorizon evolution...

- The growing solution:

$$\Phi(k,\eta)=\frac{\Phi_{\rm p}(k)}{10\,y^3}\left[9\,y^3+2\,y^2-8\,y+16\sqrt{1+y}-16\right] \qquad y\equiv a/a_{\rm eq}$$



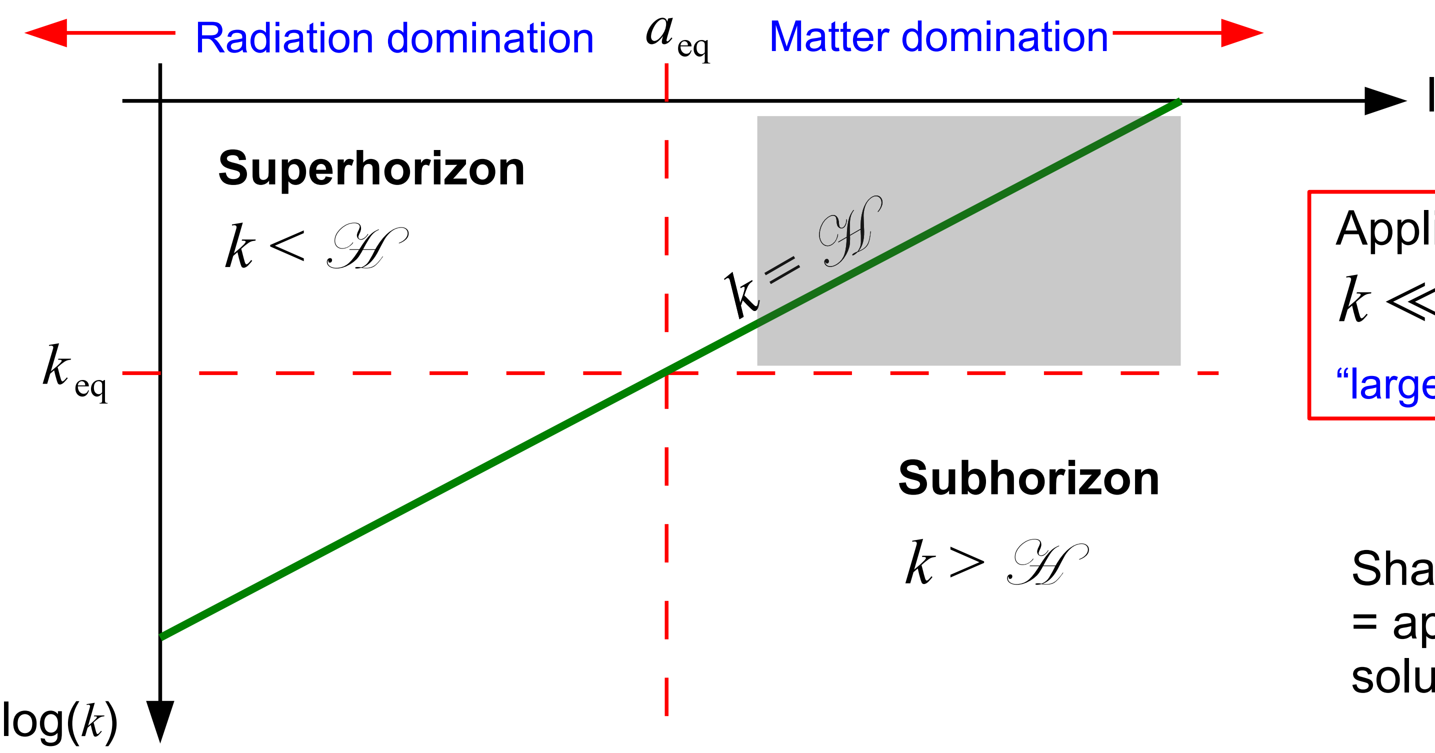
From the Einstein equation:

$$\delta_c=-\frac{3}{2}\Phi=-\frac{3}{2}\Phi_{\rm p} \qquad \text{RD}$$

$$\delta_c=-2\,\Phi=-\frac{9}{5}\Phi_{\rm p} \qquad \text{MD}$$

5.5 Horizon crossing during matter domination...

- **Trajectory** of a k mode: Superhorizon → horizon crossing → subhorizon
- **Crucial point**: When? During matter or radiation domination?



Applies to
$k \ll k_{\rm eq}$
"large scales"

Shaded areas = approximate solutions exist

5.5 Horizon crossing during matter domination...

- During matter domination, we neglect the radiation energy density:

$$\dot{\delta}_c - k^2 v_c^{(S)} - 3\dot{\Phi} = 0 \qquad \dot{v}_c^{(S)} + \mathscr{H} v_c^{(S)} + \Phi = 0$$

$$k^2 \Phi \simeq -4\pi G a^2 [\bar{\rho}_c \delta_c - 3\mathscr{H} \bar{\rho}_c v_c^{(S)}]$$

Combine → $\alpha \ddot{\Phi} + \beta \dot{\Phi} = 0$ → $\Phi(k \ll k_{\text{eq}}, \eta) = \text{constant in } \eta$

$$= \frac{9}{10} \Phi_{\text{p}}(k)$$

From section 5.4

- $\Phi(k, \eta)$ is **constant in time** during matter domination even as a k mode transits from super- to subhorizon.

5.5 Horizon crossing during matter domination...

- After a k mode has crossed into the subhorizon regime ($k >> \mathcal{H}$):

Einstein equation in the subhorizon limit

$$k^2 \Phi \simeq -4\pi G a^2 \bar{\rho}_c \delta_c$$

- Given:

$$\Phi(k, \eta) = \frac{9}{10} \Phi_{\mathrm{p}}(k)$$

$$\bar{\rho}_c \propto a^{-3}$$

$$\longrightarrow \quad \boxed{\delta_c(k \ll k_{\mathrm{eq}}, \eta) \sim a(\eta) \Phi_{\mathrm{p}}(k)}$$

$\rightarrow$ During matter domination, CDM density perturbations grow like the scale factor inside the horizon (optimal growth rate).

5.6 Horizon crossing during radiation domination...

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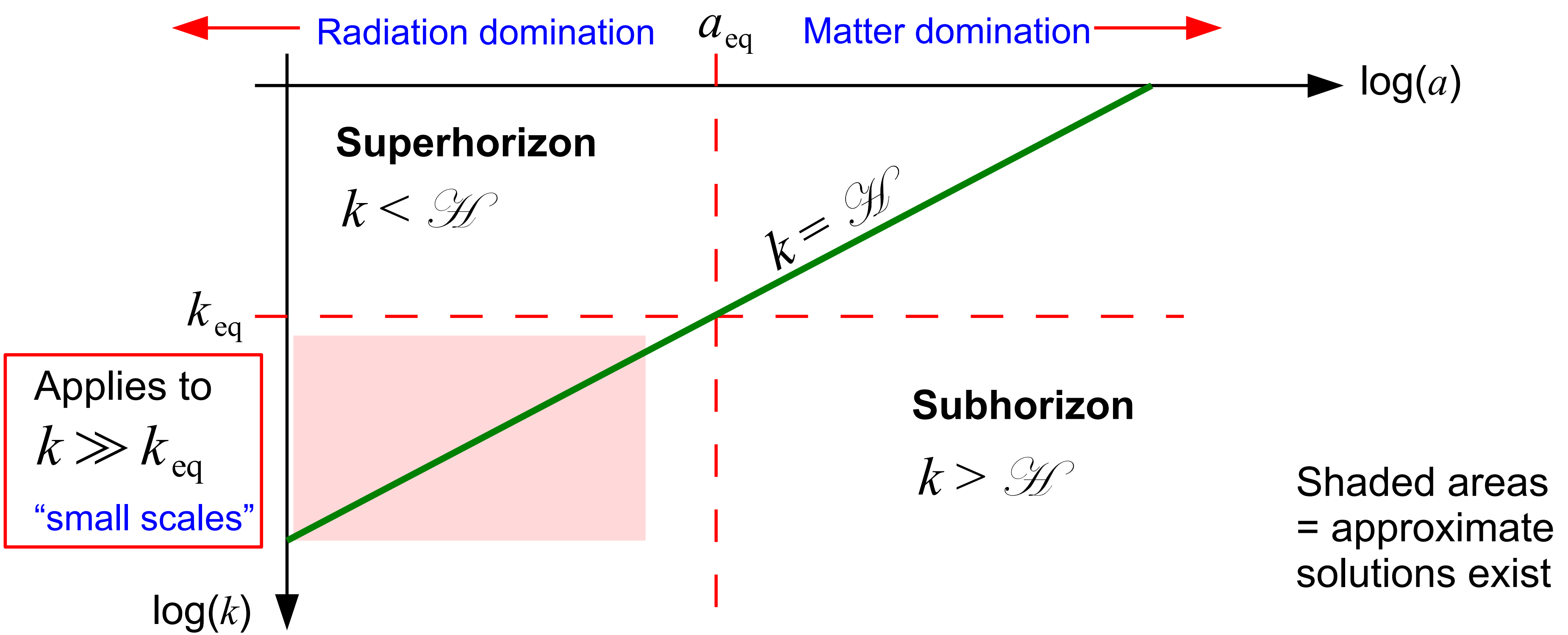


5.6 Horizon crossing during radiation domination...

- When radiation dominates, we can neglect the CDM component:

$$\dot{\delta}_\gamma - \frac{4}{3} k^2 v_\gamma^{(S)} - 4\dot{\Phi} = 0 \qquad \dot{v}_\gamma^{(S)} + \frac{1}{4}\delta_\gamma + \Psi = 0$$

$$k^2 \Phi \simeq -4\pi G a^2 [\bar{\rho}_\gamma \delta_\gamma - 4 \mathscr{H} \bar{\rho}_\gamma v_\gamma^{(S)}]$$

Combine $\longrightarrow$

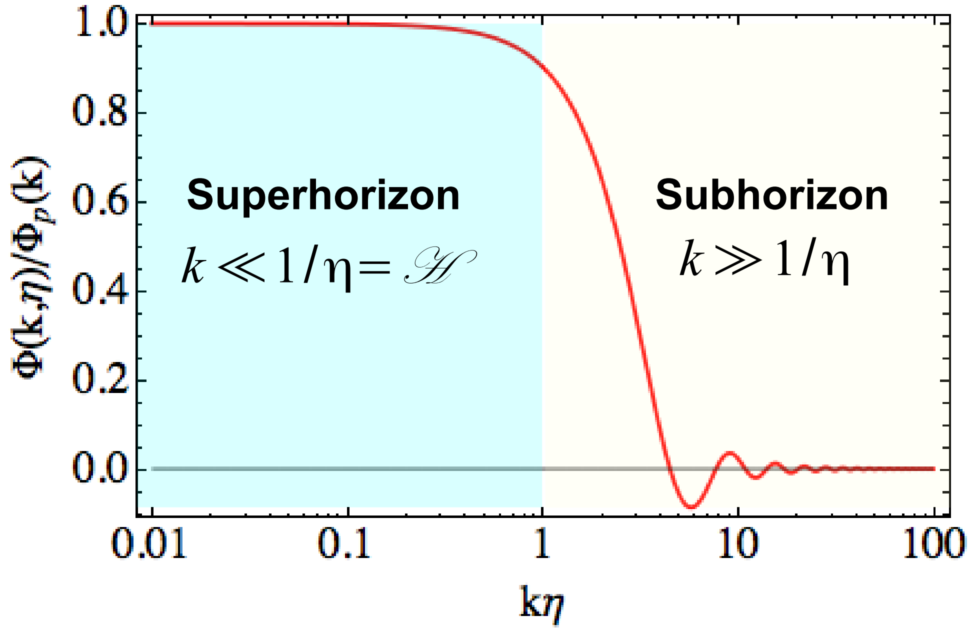
$$\ddot{\Phi} + \frac{4}{\eta}\dot{\Phi} + \frac{k^2}{3}\Phi = 0$$

- **Growing solution**:

$$\boxed{\Phi(k \gg k_{\rm eq}, \eta) = 3\,\Phi_{\rm p}(k)\left(\frac{\sin x - x\cos x}{x^3}\right)} \qquad x \equiv \frac{k\eta}{\sqrt{3}}$$

5.6 Horizon crossing during radiation domination...

- **Growing solution**:



$$\Phi(k,\eta) = 3\,\Phi_p(k)\left(\frac{\sin x - x\cos x}{x^3}\right)$$

$$x \equiv \frac{k\eta}{\sqrt{3}} = \frac{1}{\sqrt{3}}\frac{k}{\mathscr{H}}$$

During radiation domination

- During radiation domination, Φ decays away as soon as the k mode enters the horizon.

5.6 Horizon crossing during radiation domination...

- We can now feed $\Phi(k, \eta)$ into the dark matter EoM as an external source:

$$\ddot{\delta}_c + \frac{1}{\eta}\dot{\delta}_c = k^2 \Phi_p S(k\eta) \qquad S(x) = 3\frac{d^2\tilde{\Phi}}{dx^2} + \frac{3}{x}\frac{d\tilde{\Phi}}{dx} - \tilde{\Phi} \qquad \tilde{\Phi} \equiv \frac{\Phi}{\Phi_p}$$

- Formal solution:

~ constant at x>>1 ~ constant at x>>1

$$\frac{\delta_c(k,\eta)}{\Phi_p(k)} = -\frac{3}{2} - \left[\int_0^x dx' S(x') x' \ln(x')\right] + \left[\int_0^x dx' S(x') x'\right] \ln(k\eta)$$

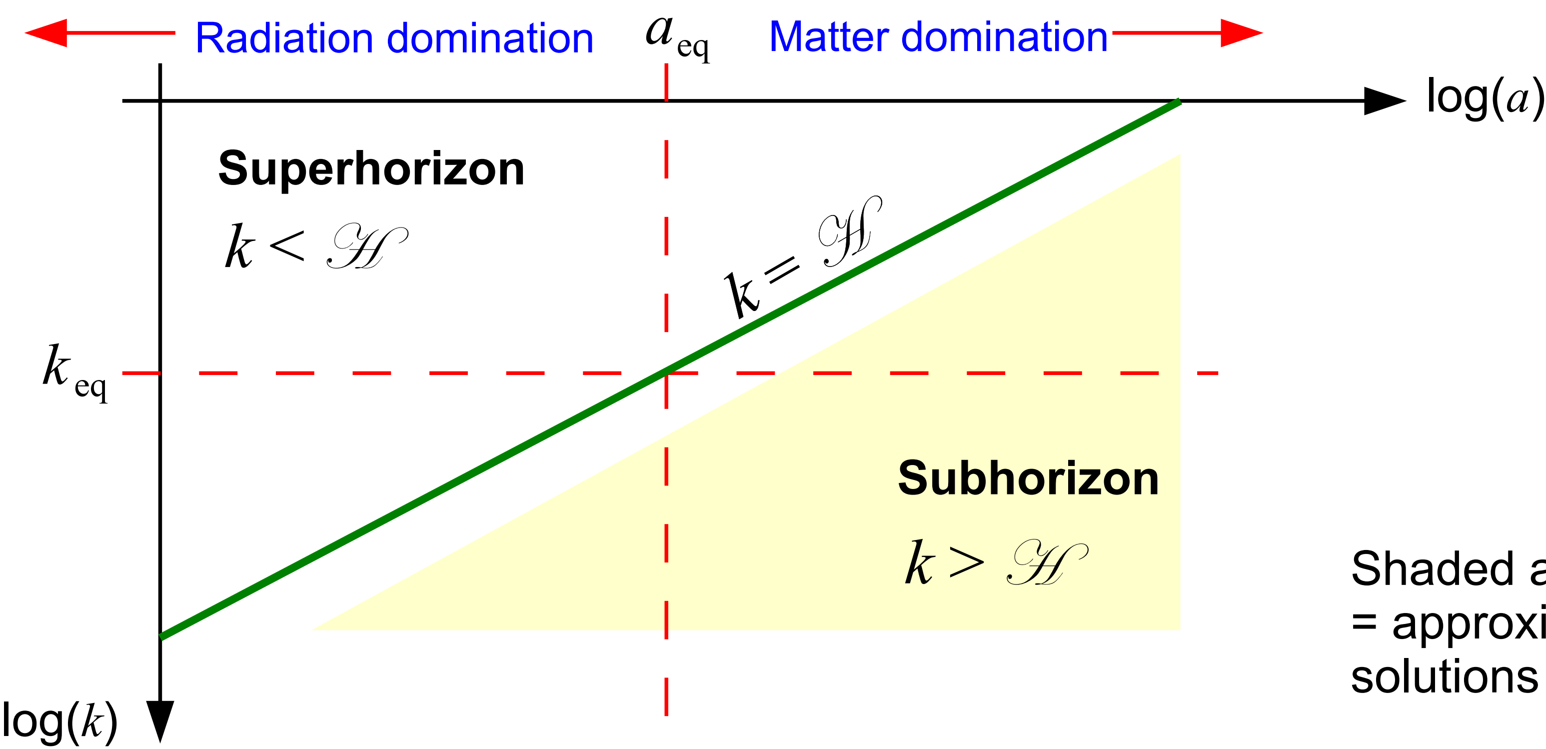
$$\rightarrow \quad \boxed{\delta_c(k\eta \gg 1) \simeq A\,\Phi_p(k) \ln(B k \eta)}$$

$a \propto \eta$
during RD

- δ_c grows logarithmically with a when inside the horizon during RD.
- **Physical reason**: Radiation pressure attenuates the growth of δ_c.

5.7 Subhorizon evolution...

- **Trajectory** of a k mode: Superhorizon → horizon crossing → subhorizon
- **Crucial point**: When? During matter or radiation domination?



Shaded areas = approximate solutions exist

5.7 Subhorizon evolution...

- During radiation domination and after horizon crossing:

$$\delta_c(k,\eta) \simeq A\,\Phi_{\mathrm{p}}(k)\ln(B\,k\,\eta)$$

$$\delta_\gamma(k,\eta) \sim \text{oscillatory}$$

From section 5.6

→ Matter perturbations will grow to be **larger** than photon perturbations even during radiation domination.

→ Ignoring photon perturbations, the relevant equations are:

$$\dot{\delta}_c - k^2 v_c^{(S)} - 3\,\dot{\Phi} = 0 \qquad \dot{v}_c^{(S)} + \mathscr{H}\,v_c^{(S)} + \Phi = 0$$

$$k^2\Phi \simeq -4\pi G\,a^2\bar{\rho}_c\,\delta_c$$

5.7 Subhorizon evolution...

- Combine into a 2nd order DE for δ_c:

$$\frac{d^2\delta_c}{dy^2}+\frac{2+3y}{2y(y+1)}\frac{d\delta_c}{dy}-\frac{3}{2y(y+1)}\delta_c=0 \qquad y\equiv a/a_{\mathrm{eq}}$$

- Formal solution:

$$\delta_c(k,\eta)=C_G(k)\,G(\eta)+C_D(k)\,D(\eta)$$

$$G(\eta)=y+2/3$$ Growing solution

$$D(\eta)=(y+2/3)\ln\left(\frac{\sqrt{1+y}+1}{\sqrt{1-y}-1}\right)-2\sqrt{1+y}$$ Decaying solution

- What are the constants $C_G(k)$ and $C_D(k)$? Initial conditions are set by the **logarithmic growth solution** from section 5.6.

5.7 Subhorizon evolution...

- Match logarithmic growth solution to the new solution at some time η_m deep in radiation domination (but k should be subhorizon):

$$A\Phi_{\rm p}(k)\ln(Bk\eta_m)=C_G(k)G(\eta_m)+C_D(k)D(\eta_m)$$

+ first derivative

- Outcome:

Constants depending logarithmically on k

$$\frac{\delta_c(k,\eta)}{\Phi_{\rm p}(k)}=\tilde{C}_G(\ln k)G(\eta)+\tilde{C}_D(\ln k)D(\eta)$$

$$\to\tilde{C}_G(\ln k)a/a_{\rm eq},\quad \eta\gg\eta_{\rm eq}$$

Matter domination

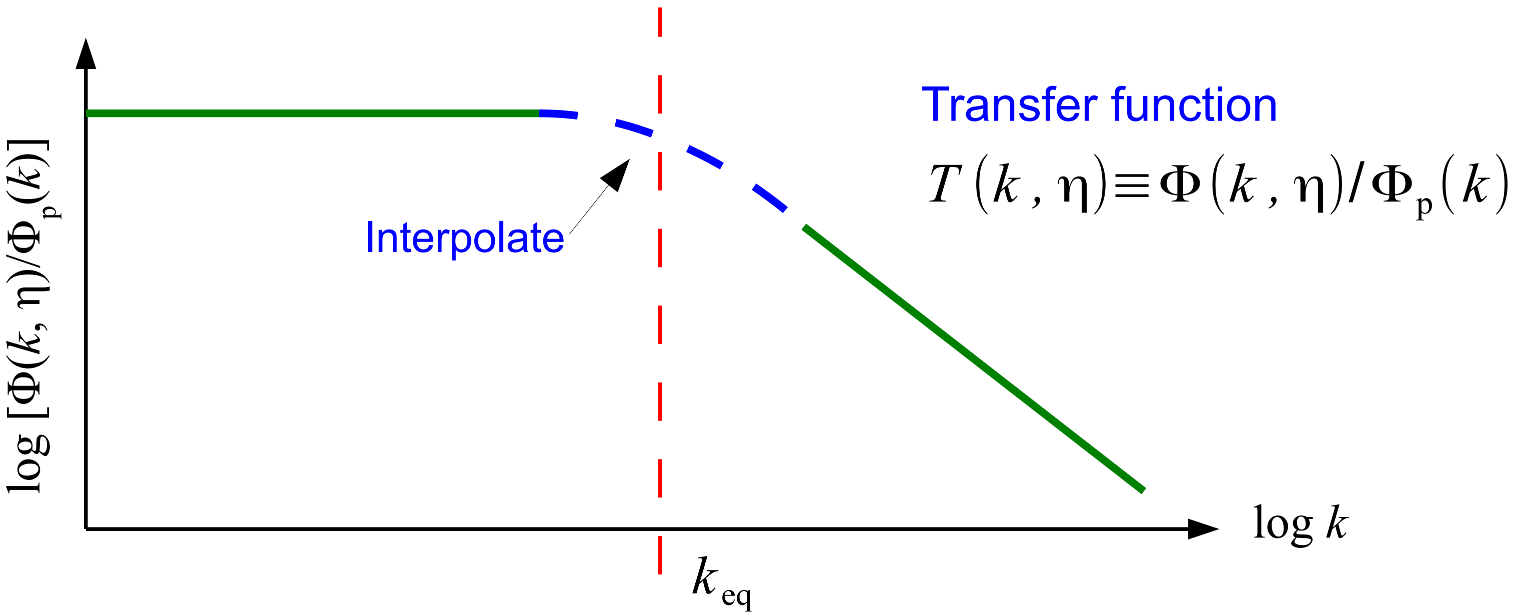
Einstein equation →

$$\Phi(k\gg k_{\rm eq},\eta\gg\eta_{\rm eq})\sim\frac{\tilde{C}_G(\ln k)}{k^2}\Phi_{\rm p}(k)$$

5.8 Putting it all together: the transfer function...

- Pick a time η deep in **matter domination**.
  - The relevant solutions are:

$$\Phi(k \ll k_{\text{eq}}, \eta) = \frac{9}{10}\Phi_{\text{p}}(k) \qquad \Phi(k \gg k_{\text{eq}}, \eta) \sim \frac{\tilde{C}_G(\ln k)}{k^2}\Phi_{\text{p}}(k)$$



5.9 What if we include baryons...

- At early times, baryons and photons form a **tightly-coupled** fluid.
  - Like photons, baryon density perturbations oscillate around 0.

- Replacing some of CDM with baryons means we have **less total matter perturbations** for those k modes inside the horizon before decoupling.

Total matter density perturbations

- Define: $\delta_m \equiv (1 - f_b)\delta_c + f_b \delta_b$ $\qquad f_b \equiv \frac{\Omega_b}{\Omega_b + \Omega_c}$ Baryon fraction

- Effective EoM for subhorizon evolution:

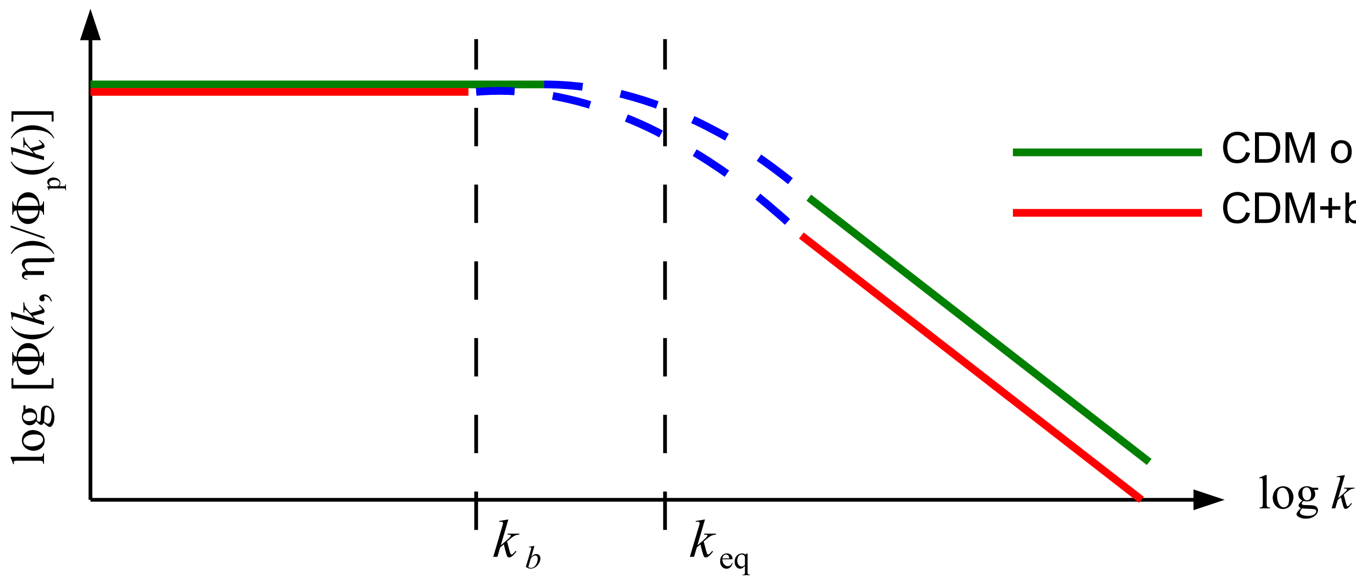
$$\dot{\delta}_m - k^2 v_m^{(S)} - 3\dot{\Phi} = 0 \qquad \dot{v}_m^{(S)} + \mathscr{H} v_m^{(S)} + \Phi = 0$$

$$k^2 \Phi \simeq -4\pi G a^2 \bar{\rho}_m (1 - f_b) \delta_m$$

While baryons & photons are coupled

5.9 What if we include baryons: the transfer function...

- Baryons decouple from photons at during early MD.
  → Reduced amplitude for k modes crossing horizon while baryons are **coupled** to photons.
- Scale k_b at which suppression begins fixed by recombination physics.
- **Amount** of suppression at $k >> k_{eq}$ depends on the baryon fraction f_b.



5.10 What if we include massive neutrinos...

- Like baryons, neutrinos also do not cluster on small scales, but for a different reason.
  - Neutrinos have too much thermal motion to cluster efficiently, even today.

- Neutrino **thermal speed** at low redshift:

$$c_\nu \simeq 81(1+z)\left(\frac{\mathrm{eV}}{m_\nu}\right) \ \mathrm{km\ s^{-1}}$$

  - Compare with velocity dispersion of a galaxy (~ O(100) km s^{-1}) and dwarf galaxy (~ O(10) km s^{-1}).

- Massive neutrinos must make up some of the dark matter in the universe, but **cannot** be the dominant dark matter component .

5.10 What if we include massive neutrinos...

- If some CDM is replaced with neutrinos:
    - Reduced fluctuation amplitude for those k modes crossing the horizon while the neutrinos are still relativistic.
- **Scale** at which suppression begins is fixed by the neutrino mass.
- **Amount** of suppression depends on the neutrino fraction: $f_\nu \equiv \frac{\Omega_\nu}{\Omega_m}$

$$\Omega_\nu h^2 = \sum \frac{m_\nu}{94\,\text{eV}}$$