Foundations of Quantum Mechanics - Some General Aspects

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Survey of my Thesis

- Detailed introduction to mathematical foundations of quantum mechanics, considering **abelian and non-abelian $C^\star$-algebras**, providing detailed proofs (some of my person).
- Discuss in this language **Hamiltonian mechanics, Quantum mechanics, Bohmian mechanics.**

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1. Introduction

2. Some Aspects
   - Correlation Matrices & Tsirel'son theorem
   - Consistent Histories

Introduction

In the 20th century four constitutional revolutions of physics took place, related $\hbar$, k, c^{-1} and l_p (Planck length).

New fundamental questions arose: Point out the general Difference between a **quantum theory** and a **classical theory**.

To answer these questions it was necessary to embed these theories into a **mathematical precise framework.**

Motivation

Why to recall this things? E.g. in **quantum information theory**, it became clear that before one can make a progress one has to understand better the fundamental processes.

Is there a fundamental mathematical structure that is inherent in all physical theories?
A mathematical language which applicates to both classical mechanics and Quantum mechanics?

Suitable language proposed by I. Segal, J. von Neumann (1947).

A dictionary:

physical observables $\Leftrightarrow$ $C^\star$-Algebra

physical states $\Leftrightarrow$ linear functionals on $C^\star$-Algebra

time evolution $\Leftrightarrow$ automorphism on $C^\star$-Algebra

C*-algebra (observables)

$\swarrow$ $\searrow$

Abelian

$\downarrow$

Functions on Phase Space Γ
e.g. $(p, q) \in \mathbb{R}^2 = \Gamma$

$\downarrow$

Hamiltonian Mechanics

Non-Abelian

$\downarrow$

e.g. $[q, p] = 1$ polynomials of p,q forming Heisenberg-algebra.

$\downarrow$

Quantum Mechanics

Correlation Matrices & Tsirel'son theorem

Theorem: The set of measurement results of a classical theory is a subset of those of quantum mechanics.

The physical system $\mathcal{S} = \mathcal{S}_1 \vee \mathcal{S}_2$, consists of two non-interacting but correlated subsystem $\mathcal{S}_1, \mathcal{S}_2$.

Classical System	Quantum System
$M_1 \times M_2$	$\mathscr{H}_1 \otimes \mathscr{H}_2$

	Classical System	Quantum System
States	Prob. measures on $M_1 \times M_2$	Density matrix $\mathcal{P}$ ($\mathscr{H}_1 \otimes \mathscr{H}_2$)
Obs.	a_k, b_l functions on M_1, M_2	$a_k \in B(\mathscr{H}_1)$ and $b_l \in B(\mathscr{H}_2)$
$[\cdot,\cdot]$	$[a_i, a_j] = 0$, $[a_k, b_l] = 0$	$[a_i, a_j] \neq 0$, $[a_k, b_l] = 0$
$\Gamma^{C/Q}$	$\Gamma^C_{kl} = \int_{M_1 \times M_2} \alpha_k \beta_l \, d\rho$	$\Gamma^Q_{kl} = tr(\mathcal{P} \, a_k b_l)$
Spec.	$\text{Image}(a_k(b_l)) = \{-1, 1\}$	$\text{Spectrum}(a_k(b_l)) = \{-1, 1\}$

Take into account **all possible states "the universe" can prepare** classically and quantum mechanically.

Classical System	Quantum System
$\max_{kl,\rho} \Gamma^C_{kl} = 1$	$\max_{kl,\mathcal{P}} \Gamma^Q_{kl} = K_G = 1,73 \pm 0.06$

Grothendieck Constant: $1,676 \leq K_G \leq 1,782$.

Compare to **Bell's inequalities:** We consider the concrete composite observable: $a_1(b_1 + b_2) + a_2(b_1 - b_2)$

Classical System	Quantum System
$\max_\rho \Gamma^C = 2$	$\Gamma^Q = \sqrt{2} \cdot 2$

The quantum expectation $\sqrt{2} \cdot 2$ computes for a spin singlett state.

Consistent Histories - Definitions

- Physical system $\mathcal{S}$ which is connected to e.g. the experimental devices, to a system E, Composed system: $S \vee E$.
- **Projection operators** $P^2 = P$: A possible event corresponds to the projection $P_a(\Delta)$ of measuring the observable a in the set $\Delta \subset \mathbb{R}$.

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- **Time ordered history:** Let $P_i := P_{a_{t_i}}(\Delta_i)$ be projections with $\Delta_i \subset \mathbb{R}$, $i = 1, ..., n$.

$$\{P_n, P_{n-1}, ..., P_1\} \quad \text{for} \quad t_0 < t_1 < ... < t_n$$

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- **Expectation value** $\langle\{P_n, P_{n-1}, ..., P_1\}\rangle$:

$$\mathcal{P}_{S\vee E}(P_1 P_2 \cdots P_{n-1} P_n P_{n-1} \cdots P_2 P_1), \quad \mathcal{P}_{S\vee E}\text{...Density-Matrix}$$

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We find for QM System in general:

$$\langle\{P_n, ..., P_j, ..., P_1\}\rangle + \langle\{P_n, ..., P_j^{\perp}, ..., P_1\}\rangle$$

$$\neq \langle\{P_n, ..., P_{j+1}, , P_{j-1}, ..., P_1\}\rangle$$

This is **not** seen in classical physics!!!

Be P_i operators asking if the quantum particle is in some region Ω_i at time t_i $i = 1, ..., n$. Thus symbolically:

$$\langle \text{particle transits } \Omega_j \text{ at } t_j \rangle + \langle \text{ particle transits } \Omega_j^c \text{ at } t_j \rangle \neq 1$$

In a consistent history the above is an equation
$\Rightarrow \{P_n, ..., P_j, ..., P_1\}$ behaves **"classically"**.

Lemma

A sufficient (but not necessary) condition for $\{P_n, ..., P_1\}$ *to be a consistent history is that* $[P_i, P_j] = 0 \ \forall i, j = 1, ..., n$

Lemma (Fröhlich-Weissenbacher)

Let $P_1, \cdots P_n$ be projections on a Hilbert space such that $||[P_i, P_j]|| < \epsilon \; \forall i, j$, $0 < \epsilon \in \mathbb{R}$. Then there exists a $C_n \in \mathbb{R}$, $C_n < \infty$ and projections $\tilde{P}_1, \cdots, \tilde{P}_n$ such that

(i) $||\tilde{P}_i - P_i|| < C_n \epsilon \quad \forall i = 1, ..., n$

(ii) $[\tilde{P}_i, \tilde{P}_j] = 0 \quad \forall i, j = 1, ..., n$

and $C_n = 2(2^{n+1} + 1)C_{n-1} + 2^{n+1}$ for $n \geq 3$ with $C_1 = 0$ and $C_2 = 10$.

Example for Observables that satisfy Lemma:

- Freely propagating 1-dimensional quantum wave function $\psi(q) \in \mathcal{L}^2(\mathbb{R})$.
- $P_1, ..., P_n$ asking the probability of $\psi(q)$ to be in the interval $[-a, a]$ at times $t_1, ..., t_n$.

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"If someone tells you they understand quantum mechanics, then all you've learned is that you've met a liar." *Richard Feynman*

Thanks for listening! Questions?!

Definition (C*-Algebra)

Let $\mathscr{B}$ be a Banach space over the field of complex numbers with norm $||\cdot||$. Then $\mathscr{B}$ is a *- **algebra**, if there additionally is

(A) an associative bilinear product, $\cdot : \mathscr{B} \times \mathscr{B} \to \mathscr{B}$

(B) a map, $^\star : \mathscr{B} \to \mathscr{B}$, called an involution

The image of an element $a \in \mathscr{B}$ under the involution is written $a^\star$, and obeys the following properties for all a, b in $\mathscr{B}$ and $\forall \lambda \in \mathbb{C}$:

(i) $(a + a)^\star = a^\star + b^\star$

(ii) $(ab)^\star = b^\star a^\star$

(iii) $(\lambda a)^\star = \bar{\lambda} a^\star$

(iv) $(a^\star)^\star = a$

A *-algebra is called a $C^\star$-**algebra** if the C*-identity holds

(C) $||aa^\star|| = ||a||^2 \,, \quad \forall a \in \mathscr{B}, \qquad \Longrightarrow \quad ||a^\star|| = ||a||$

Let A be an $n \times n$ real square matrix with $n \geq 2$ such that

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, s_i\, t_j \right| \leq 1$$

for all real numbers $s_1, s_2, \ldots, s_n$ and $t_1, t_2, \ldots, t_n$ such that $|s_i|, |t_j| \leq 1$. Then Grothendieck showed that there exists a constant $k_R(n)$ satisfying

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, \mathbf{x}_i \cdot \mathbf{y}_j \right| \leq k_R(n)$$

for all vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ and $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n$ in a Hilbert space with norms $|\mathbf{x}_i| \leq 1$ and $|\mathbf{y}_j| \leq 1$. The Grothendieck constant is the smallest possible value of $k_R(n)$. For example, the best values known for small n are

$$k_R(2) = \sqrt{2}$$
$$k_R(3) < 1.517$$
$$k_R(4) \leq \frac{1}{2}\pi$$

(Krivine 1977, 1979; König 1992; Finch 2003, p. 236).

Now consider the limit

$$k_R \equiv \lim_{n \to \infty} k_R(n),$$

Figure:

The classical system:

$$\Gamma^C(\rho) = \int_{M_{S_1}\times M_{S_2}} (a_1(b_1+b_2)+a_2(b_1-b_2))(p)d\rho$$

$$\leq \int_{M_{S_1}\times M_{S_2}} |a_1(b_1+b_2)+a_2(b_1-b_2)|(p)d\rho$$

$$\leq \int_{M_{S_1}\times M_{S_2}} (\underbrace{|a_1|}_{=1}|b_1+b_2|(p)+\underbrace{|a_2|}_{=1}|b_1-b_2|(p))d\rho$$

$$= \int_{M_{S_1}\times M_{S_2}} (|b_1+b_2|(p)+|b_1-b_2|(p))d\rho = \int_{M_{S_1}\times M_{S_2}} 2d\rho = 2 \quad (1)$$

Since $|b_1+b_2|(p)+|b_1-b_2|(p) = 2\ \forall p \in M_{S_1}\times M_{S_2}$ since by definition b_1, b_2 have values $\{\pm 1\}$. Thus,

$$\Gamma^C(\rho) \leq 2 \quad (2)$$

The quantum mechanical system:

Theorem

Let $a_i \in \mathscr{B}(\mathscr{H}_1)$ and $b_i \in \mathscr{B}(\mathscr{H}_2)$ be self-adjoint, and $||a_i|| \leq 1, ||b_i|| \leq 1$, $i = 1, 2$. Furthermore, $a_i b_j = b_j a_i$ $i \neq j = 1, 2$, and the quantum mechanical correlation matrix be

$$\Gamma^Q(\mathcal{P}) := tr[\mathcal{P}(a_1(b_1 + b_2) + a_2(b_1 - b_2))] \tag{3}$$

Then,

$$\max_{\mathcal{P}, a_i, b_i, i=1,2} \Gamma^Q(\mathcal{P}) = 2\sqrt{2} \tag{4}$$

This can be shown in general. We consider one concrete realization of the above case.

Let us consider two spin $\frac{1}{2}$ particle in a spin singlett state.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle) \tag{5}$$

We define the observables to be

$$a_1 := \sigma_z \otimes \mathbb{1}, \qquad a_2 := \sigma_x \otimes \mathbb{1}$$

and

$$b_1 := -\frac{1}{\sqrt{2}}(\mathbb{1} \otimes \sigma_z + \mathbb{1} \otimes \sigma_x) \qquad b_2 := \frac{1}{\sqrt{2}}(\mathbb{1} \otimes \sigma_z - \mathbb{1} \otimes \sigma_x)$$

Therefore we get

$$\langle\Psi|a_1(b_1+b_2)+a_2(b_1-b_2)|\Psi\rangle = \underbrace{\langle\Psi|a_1 b_1|\Psi\rangle}_{=1/\sqrt{2}} + \underbrace{\langle\Psi|a_1 b_2|\Psi\rangle}_{=1/\sqrt{2}}$$

$$+\underbrace{\langle\Psi|a_2 b_1|\Psi\rangle}_{=1/\sqrt{2}} - \underbrace{\langle\Psi|a_2 b_2|\Psi\rangle}_{=-1/\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \tag{6}$$

The expectation values follow after a short calculation.

We define a new operator P_2' by

$$P_2' := \underbrace{P_1 P_2 P_1}_{=:Q_1} + \underbrace{P_1^{\perp} P_2 P_1^{\perp}}_{=:Q_2}$$

To construct $\tilde{P}_2$ we consider the operator $Q_i^2 - Q_i$, $i = 1, 2$.

$$Q_1^2 - Q_1 = P_1 P_2 P_1^2 P_2 P_1 - P_1 P_2 P_1 = P_1 [P_2, P_1] P_2 P_1 + P_1^2 P_2^2 P_1 - P_1 P_2 P_1$$

$$= P_1 [P_2, P_1] P_2 P_1 \qquad (7)$$

Analogous $Q_2^2 - Q_2 = P_1^{\perp} [P_2, P_1^{\perp}] P_2 P_1^{\perp}$.

$$||Q_1^2 - Q_1|| \leq \underbrace{||P_1||}_{\leq 1} \underbrace{||[P_2, P_1]||}_{<\epsilon} \quad \underbrace{||P_2 P_1||}_{\leq ||P_2||||P_1|| \leq 1} \quad \overset{def.}{<} \epsilon$$

With $P_1^{\perp}[P_2, P_1^{\perp}]P_2 P_1^{\perp} = P_1^{\perp}[P_2, \mathbb{1} - P_1]P_2 P_1^{\perp} = P_1^{\perp}[P_2, P_1]P_2 P_1^{\perp}$ we also find $||Q_2^2 - Q_2|| < \epsilon$.

$$\tilde{P}_2 = \left. E_1(\Delta_>)\right|_{P_1 \mathscr{H}} + \left. E_2(\Delta_>)\right|_{P_1^{\perp} \mathscr{H}} \tag{8}$$

Quantum Eraser:

Definition (Which-path detector)

An apparatus W, which marks the path of the particle without disturbing it's wave functions $|\psi_1\rangle, |\psi_2\rangle$ can be implemented by

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\psi_1\rangle \otimes |W_1\rangle + |\psi_2\rangle \otimes |W_2\rangle] \tag{9}$$

hence by composing the free system with the system of the which-path marker by the tensor product. $|W_i\rangle$ is the state of the apparatus corresponding to the possibility of the particles to pass the slit $i = 1, 2$. Such an apparatus is called a which-path marker.

A which-path marker which completely determines the path of the particle destroys the interference pattern.

Quantum eraser In the case that the wave functions $|\psi_1\rangle$ and $|\psi_2\rangle$ are only less pertubated by the which-path markers $|W_1\rangle$ and $|W_2\rangle$, one can restore interference terms by correlating the particle measurement of $|\psi_1\rangle$ and $|\psi_2\rangle$ with an appropriate measurement of the which-path markers. This procedure of recovering the interference terms, and thus the interference pattern, is called quantum eraser. The word eraser refers to the influence of the which-path marker on the experiment outcome, which is "erased".

Bohmian mechanics is a deterministic theory of point particles, which tries to describe the effects in nature related to quantum mechanics.

The N-particle system, with masses $m_1, \cdots, m_N$, is described by the configuration space $q := (\vec{q}_1, \cdots, \vec{q}_N) \in \mathbb{R}^{3N}$. **The observable algebra of Bohmian mechanics is an abelian $C^\star$-algebra.**

The dynamics of the theory is generated by a time-dependent vector field $X^\psi : \mathbb{R} \times M_{\mathscr{B}} \longrightarrow TM_{\mathscr{B}}$,
$(t, q) \longmapsto X^\psi_q(t) = X^\psi(t, q)$, which is defined by the equation

$$X^\psi := \frac{\vec{j}^\psi}{|\psi(q,t)|^2}, \quad \text{with} \quad \vec{j}^\psi = \sum_k \frac{\hbar}{2im_k} \left(\psi^\star \vec{\nabla}_k \psi - \psi \vec{\nabla}_k \psi^\star \right),$$

$\psi(q,t)$ denoting the solution of the usual Schrödinger equation. Therefore, $\vec{j}^\psi$ is the well known quantum mechanical probability current.