The Goldstone Boson Equivalence Theorem (GBET)

for Massive Gravitons on Cosmological Backgrounds

Sophia Müller March 19th, 2012

The Standard Model GBET

Goldstone Theorem:
spontaneously broken global continuous symmetries
$$\Downarrow$$
massless Goldstone Boson (ϕ)

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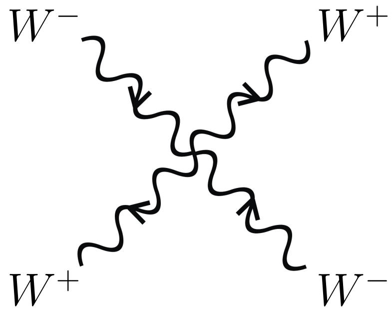
Goldstone Boson Equivalence Theorem:

$$\mathcal{M}_{\mathrm{L}}(A_\mu) = \mathcal{M}(\phi) + \mathcal{O}\left(\frac{m}{E}\right) \qquad \text{for } E \gg m$$

The Standard Model GBET

An Example: $W_L^+ W_L^- \to W_L^+ W_L^-$



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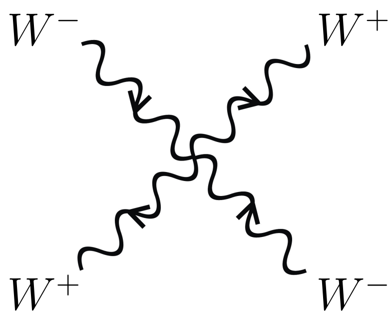

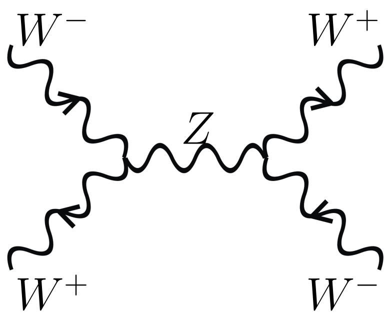

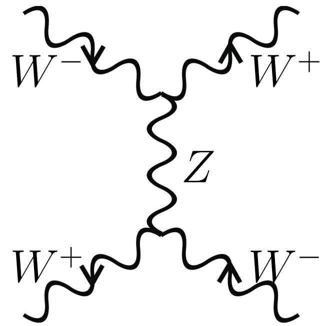

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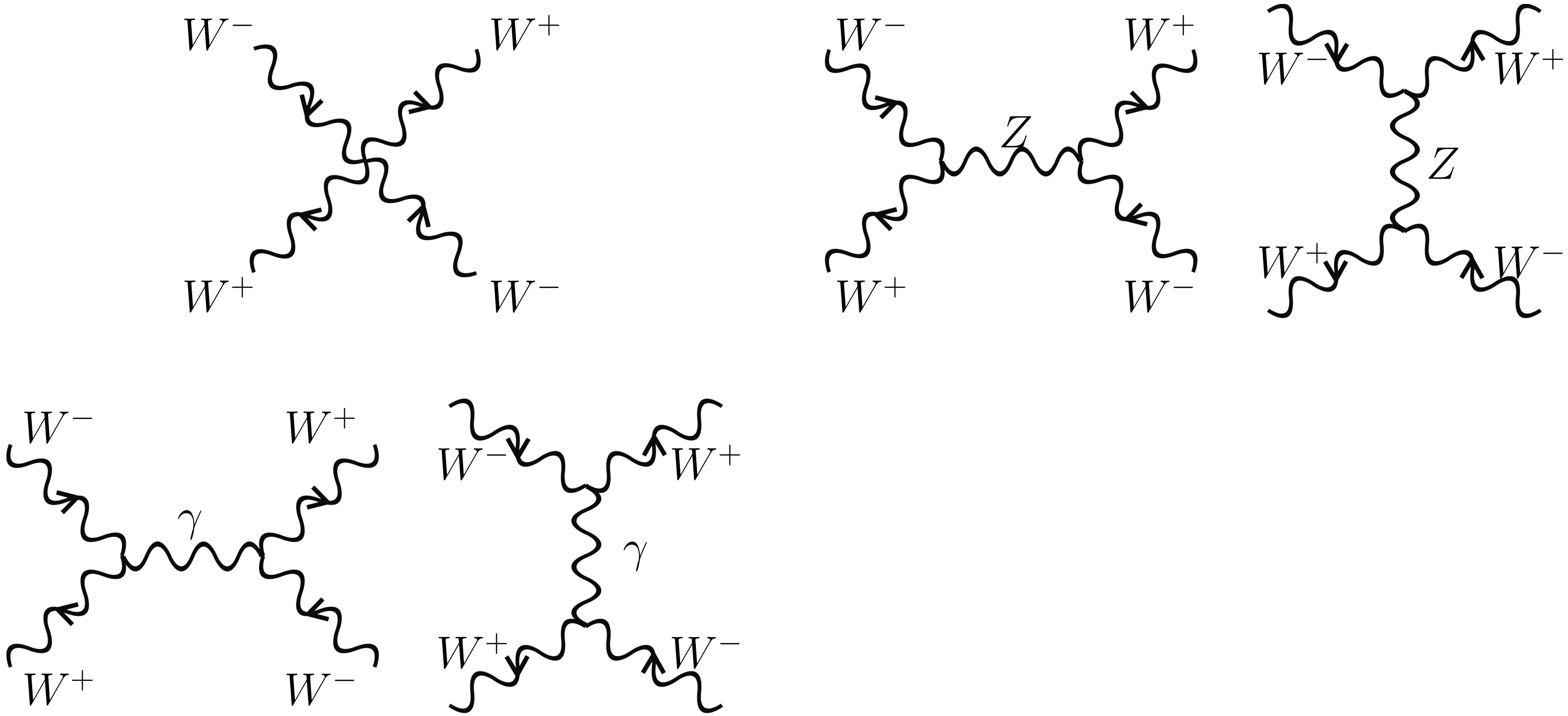

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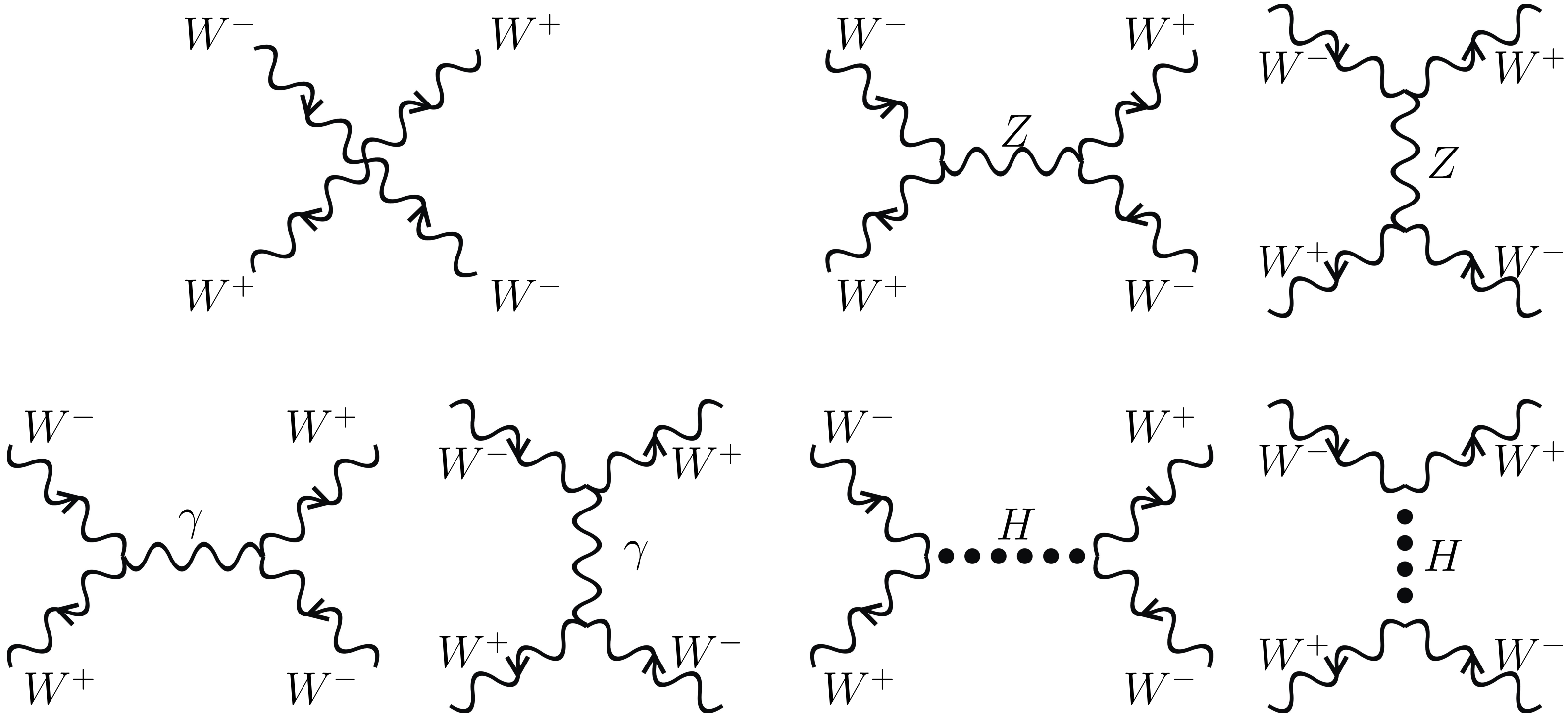

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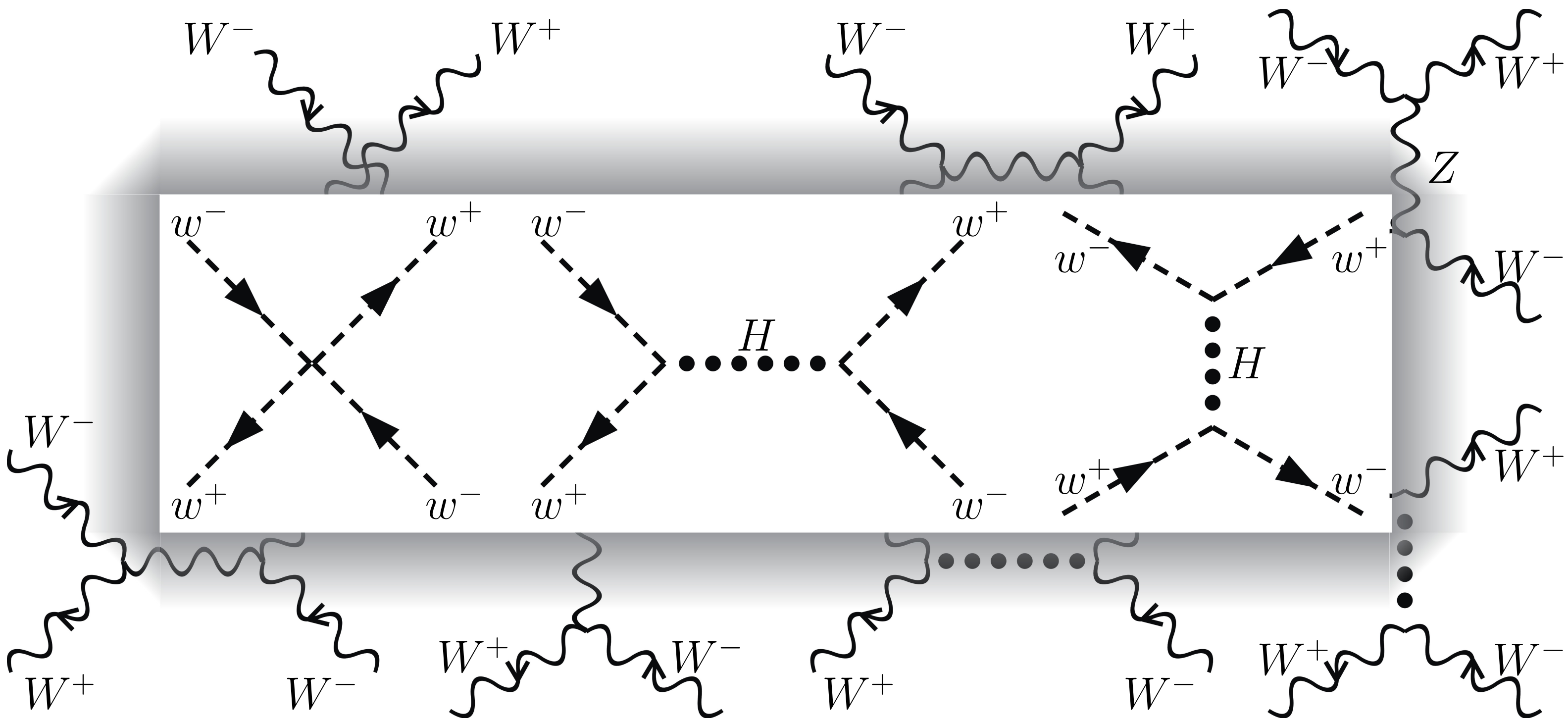

The Standard Model GBET: Advantages

Advantages of $\mathcal{M}_{\rm L}(A_\mu) = \mathcal{M}(\phi) + \mathcal{O}\left(\frac{m}{E}\right)$

- **computational simplifications:**
  - **less diagrams**
  - **simpler Feynman rules**
- **focus on those d.o.f. that limit the regime of validity**
  - **easier cut-off estimation**
  - **conclusions for possible UV-completions**

$E_{\rm cut-off}$

- **easier check for ghost-free theories**

The GBET: Foundations

LSZ reduction formula for $E \gg m$:

$$\mathcal{M}(A_\mu) \;=\; \mathrm{FT}\left\{\partial_\mu \mathcal{K}^{\mu\nu}\langle out|A_\nu|in\rangle\right\} + \mathcal{O}\left(\frac{m}{E}\right)$$

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Standard Model:		
S.S.B./ Higgs Mechanism	$\Leftrightarrow$	introduction of ϕ $\Downarrow$ gauge invariance $\Downarrow$
BRST quantization	$\Leftrightarrow$	gauge fixing

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<table>
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The GBET for Massive Gravitons

LSZ reduction formula for Massive Gravitons ($E \gg m$):

$$\mathcal{M}(h_{\mu\nu}) \; = \; \mathrm{FT}\left\{\partial^\alpha\partial^\beta\mathcal{K}^{\mu\nu}_{\alpha\beta}\langle out|h_{\mu\nu}|in\rangle\right\} + \mathcal{O}\left(\frac{m}{E}\right)$$

$$\Longrightarrow \quad \mathcal{M}_{\mathrm{L}}(h_{\mu\nu}) = \mathcal{M}(\phi) + \mathcal{O}\left(\frac{m}{E}\right)$$

$\Longrightarrow$**Transparent understanding of**

- ✧ **Fierz-Pauli form of the Lagrangian** $\quad m^2(h_{\mu\nu}h^{\mu\nu} - a h^2)$

$$m^2(1-a)(\Box\phi)^2$$

- ✧ **vDVZ discontinuity** $\quad \Longrightarrow a \stackrel{!}{=} 1$

Massive Gravitons on Cosmological Backgrounds

Problem:

S-Matrix formulation not possible a priori

$\Longrightarrow$ Restriction to spacetimes

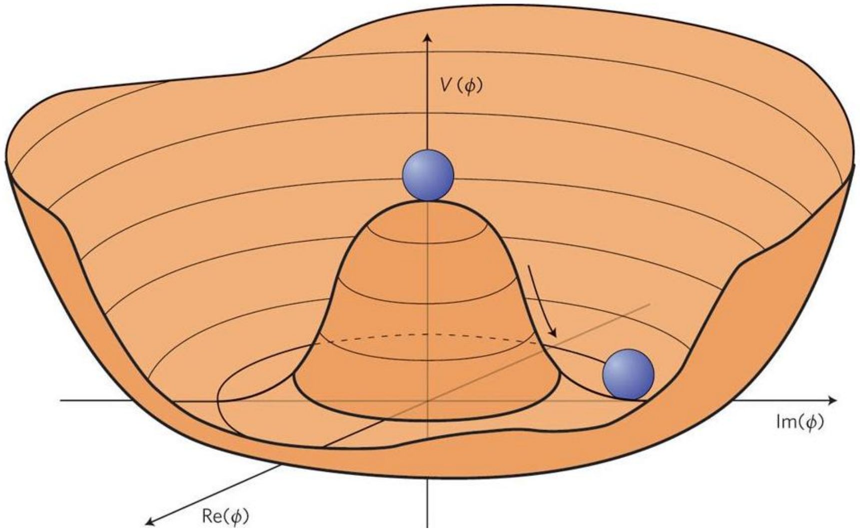
- ✧ **which are globally hyperbolic**
- ✧ **where the fields are asymtotically free**

e.g.: FRW-background

$$\mathcal{M}_{\mathrm{L}}(h_{\mu\nu}) = \mathcal{M}(\phi) + \mathcal{O}\left(\frac{m}{E}\right)$$

Additional Material / Discussion

Spontaneous Symmetry Breaking and the Goldstone Theorem



degenerate vaccum

with non-zero expectation value

$\Downarrow$

excitation of zero energy can take one to another vacuum

$\Downarrow$

expectation of particles with zero energy

$\Downarrow$

massless particles

Cut-off estimation

Strong coupling scale: higher order terms cannot be neglected anymore

$\Longrightarrow$ it can be derived by comparing the different orders

e.g. Massive Gravity:

$\frac{1}{m_g^4 M_{\mathrm{Pl}}}(\partial^2\phi^c)^3$ **is the strongest coupling term**

$\Longrightarrow$ **it becomes large at** $\Lambda_5 \sim (m_g^4 M_{\mathrm{Pl}})^{\frac{1}{5}}$

Polarization vectors in the high energy limit

- vector boson at rest:
  - four momentum $k^\mu = (m, \mathbf{0})$
  - polarization vectors $\varepsilon_\mu^{\mathrm{T1}} = (0,1,0,0)$, $\varepsilon_\mu^{\mathrm{T2}} = (0,0,1,0)$

$$\varepsilon_\mu^{\mathrm{L}} = (0,0,0,1)$$

$$(\,\varepsilon_\mu k^\mu = 0 \;\; ; \;\; \varepsilon^2 = -1\,)$$

- boosted vector boson (along the 3-axis)
  - four momentum $k^\mu = (E, \boldsymbol{k}) = (E, 0, 0, |\boldsymbol{k}|)$
  - polarization vectors $\varepsilon_\mu^{\mathrm{T1}} = (0,1,0,0)$, $\varepsilon_\mu^{\mathrm{T2}} = (0,0,1,0)$

$$\varepsilon_\mu^{\mathrm{L}} = \tfrac{1}{m}(|\boldsymbol{k}|, 0, 0, E) = \frac{k_\mu}{m}$$

$|\boldsymbol{k}| \to \infty \implies \varepsilon_\mu^{\mathrm{T}}$ become increasingly negligible

The LSZ reduction formula for spin 1

- **S-matrix element** $\mathcal{M} = \langle \mathrm{out}|\mathrm{in}\rangle$
- $|\mathrm{in}\rangle = a^\dagger_{\mathrm{in}}(k_1)|k_2, ..., k_r, \mathrm{in}\rangle$ **for an r-particle initial state**
- **expressions for** a **and** $a^\dagger$ **from field expansions**

e.g. $A^\mu(x) = \sum_n \left(a_n f_n^{\mu *}(x) + a_n^\dagger f_n^\mu(x)\right)$

$$\Longrightarrow a_n = \mathrm{i}\int \mathrm{d}^\nu x\, A^\mu(x) \overleftrightarrow{\partial}_\nu f_{\mu n}(x)$$

$$a_n^\dagger = -\mathrm{i}\int \mathrm{d}^\nu x\, A^\mu(x) \overleftrightarrow{\partial}_\nu f^*_{\mu n}(x)$$

$$\begin{aligned}\mathcal{M} = \langle \mathrm{out}|\mathrm{in}\rangle \;&=\; \mathrm{i}^{(r+s)} \int \mathrm{d}^4 x_1 ... \mathrm{d}^4 y_s\, f^*_{\mu k_1}(x_1) ... f_{\varrho p_s}(y_s) \\ &\times \mathcal{K}^{\mu\nu}(x_1) ... \mathcal{K}^{\varrho\gamma}(y_s) \\ &\times \langle out|\mathrm{T}\left[A_\nu(x_1) ... A_\gamma(y_s)\right]|in\rangle\end{aligned}$$

Introducing Stückelberg fields into Massive Gravity

$$S_{\text{massive gravity}} = \int \mathrm{d}^4x \, (-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h^{\mu\lambda}\partial^\nu h_{\nu\lambda} - \partial_\mu h^{\mu\lambda}\partial_\lambda h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \\ -\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h^{\mu\nu}T_{\mu\nu})$$

Restoring gauge invariance / introducing a Stückelberg field A_μ:

$h_{\mu\nu} \to h_{\mu\nu} + \frac{\partial_{(\mu}A_{\nu)}}{2m}$

$$\implies \mathcal{L} = \mathcal{L}_{m=0} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} - h^2) + \kappa h^{\mu\nu}T_{\mu\nu} \\ -\frac{1}{8}F_{\mu\nu}F^{\mu\nu} + m(h\partial A - h_{\mu\nu}\partial^\mu A^\nu)$$

The resulting Lagrangian is invariant under

$$\delta h_{\mu\nu} = \frac{\partial_{(\mu}\Upsilon_{\nu)}}{2m}, \quad \delta A_\mu = -\Upsilon_\mu$$

Quantization in R_ξ-gauge

- ✦ **abelian gauge theories:** **Gupta Bleuler quantization**
- ✦ **non-abelian gauge theories:** **demand for Faddeev-Popov ghosts (unitarity, renormalizability)**

$$\Downarrow$$

BRST symmetry

$$\Downarrow$$

BRST quantization

$\Longrightarrow$ **Quantization of non-abelian gauge theories demands a more restrictive state selection**

Additional Material / Discussion

The vDVZ discontinuity

Massive Gravity

$$\not\Downarrow \quad m \to 0$$

General Relativity

But: Within the Vainshtein radius $r_V = \left(\frac{G_N M}{m_g^4}\right)^{\frac{1}{5}}$

the linear approximation breaks down

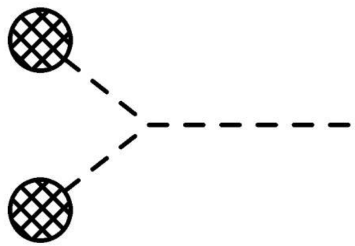
For $m \to 0 \quad r_V \to \infty \implies$ no trustworthy information

$\implies$ non-linear solution shows a smooth massless limit

Additional Material / Discussion

Derivation of the Vainshtein radius

The potential set up for ϕ^c by a source of mass M can be diagrammatically represented as



(coupling to source $\sim \frac{M}{M_{\mathrm{Pl}}}$)

$\phi^{c(1)} \sim \frac{M}{M_{\mathrm{Pl}}}\frac{1}{r}$

$\phi^{c(2)} \sim \left(\frac{M}{M_{\mathrm{Pl}}}\right)^2 \frac{1}{\Lambda_5}\frac{1}{r^6}$

$$\Longrightarrow \quad \phi^{c(1)} \stackrel{!}{=} \phi^{c(2)}$$

$$\Longrightarrow \quad r_V = \left(\frac{G_N M}{m_g^4}\right)^{\frac{1}{5}}$$ **using** $\Lambda_5 = (m_g^4 M_{\mathrm{Pl}})^{\frac{1}{5}}$ **and** $G_N = \frac{1}{M_{\mathrm{Pl}}^2}$

A simple plausibility check

Introducing Stückelberg fields in Massive Gravity

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{\partial_\mu \partial_\nu \phi}{m^2}$$

going to momentum space

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{k_\mu k_\nu \phi}{m^2}$$

argument: in the high energy limit the last term dominates

But: ϕ is also k-dependent