Stability of Generalized Kasner spacetimes

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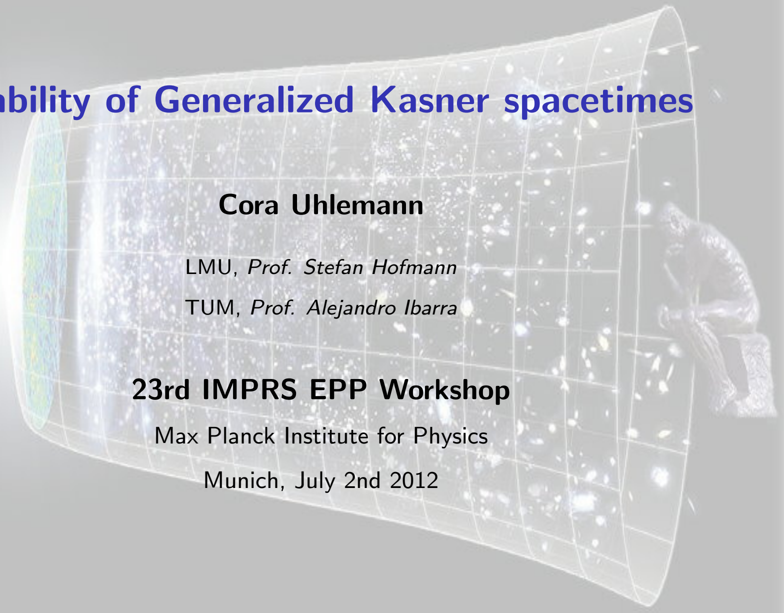

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Geometry near initial singularity

Relevance of singularities

Existence of singularities in GR ($\hookrightarrow$ Penrose-Hawking)

$\Rightarrow$ indicates required completion of gravity

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- role played by isometries
- possible destabilization due to perturbations

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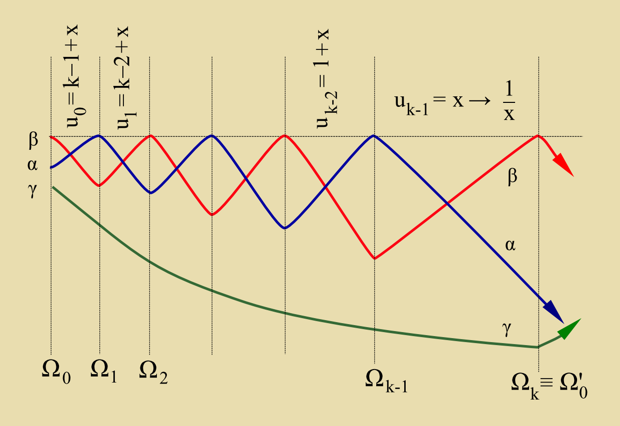

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Possibility of isotropization

- isotropic cosmologies may evolve
- via isotropization due to inflation

Guideline: Background

Bianchi I background in conformal time

$$ds^2 = a^2(\eta)\left[-d\eta^2 + \gamma_{ij}(\eta)dx^i dx^j\right] \tag{1}$$

$$\gamma_{ij}(\eta) = e^{2\beta_i(\eta)}\delta_{ij} \qquad \sum_i \beta_i = 0$$

$$\textbf{shear}: \; 2\sigma_{ij} := \gamma'_{ij} : (\sigma^i_j)' = (\gamma^{ik}\sigma_{kj})' \neq \gamma^{ij}\sigma'_{ij}$$

- consider minimally coupled scalar field φ with potential V

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- consider minimally coupled scalar field φ with potential V
- Friedmann equations and Klein Gordon equation

$$\kappa a^2 V = 2\mathcal{H}^2 + \mathcal{H}' \qquad \kappa(\varphi')^2 = 2\mathcal{H}^2 - 2\mathcal{H}' - \sigma^2$$

$$\varphi'' + 2\mathcal{H}\varphi' + a^2 V_\varphi = 0$$

- evolution of the shear $(\sigma^i_j)' = -2\mathcal{H}\sigma^i_j$

Guideline: Perturbations

Bianchi I perturbation in Newtonian gauge

$$ds^2 = a^2 \left[-(1 + 2\Phi) d\eta^2 + (\gamma_{ij} + h_{ij}) dx^i dx^j \right] \quad (2)$$

$$h_{ij} = -2\Psi \left(\gamma_{ij} + \frac{\sigma_{ij}}{\mathcal{H}} \right) + \overset{\partial_i E^i = 0}{2\partial_{(i} E_{j)}} + \overset{\partial_i E^{ij} = 0 = E^i_i}{2E_{ij}}$$

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- identify suitable set of gauge-invariant variables

$$\left\{Q = \chi + \frac{\Psi}{\mathcal{H}}\varphi', \Phi, \Psi, \Phi^i = -(E^i)', E_{ij}\right\}$$

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- identify suitable set of gauge-invariant variables
$$\left\{ Q = \chi + \frac{\Psi}{\mathcal{H}} \varphi', \Phi, \Psi, \Phi^i = -(E^i)', E_{ij} \right\}$$
- mode decomposition via Fourier transform
  - comoving coordinate system $\{x^i\} \leftrightarrow \{k_i\} : k_i' = 0$
  - pick local basis: $\{e^1, e^2\} \perp k_i$
  - e.g shear: $\sigma_{\|} = \sigma_{ij} \hat{k}^i \hat{k}^j \quad \sigma_{V^a} = \sigma_{ij} \hat{k}^i e^j_a \quad \sigma_{T^\lambda} = \sigma_{ij} \epsilon^{ij}_\lambda$

Imprints of anisotropy

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- **coupling: scalar perturbations $\leftrightarrow$ gravitational waves** ($+ \leftrightarrow \times$)

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- vector and remaining scalar modes cannot be ignored
  - $\Phi_a = \Phi_a(Q, E_+, E_\times)$
  - $\Phi = \Phi(Q, E_+, E_\times), \Psi = \Psi(Q, E_+, E_\times)$

Kasner as Bianchi I

Generic Kasner

$$\gamma_{ij} = (\eta/\eta_0)^{2Q_i}\,\delta_{ij} \qquad \sigma_{ij} = (Q_i/\eta)\gamma_{ij} = (2\mathcal{H}Q_i)\gamma_{ij} \tag{4}$$

$$Q_1 \leq Q_2 \leq Q_3 \qquad \sum_i Q_i = 0 \quad \sum_i Q_i^2 = 3/2$$

- vacuum solution ($\varphi = 0$, $V = 0$) characterized by Weyl tensor

$$R = 0 \quad R_{\mu\nu} = 0 \quad R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} \quad C^2 = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$$

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- asymptotically modes align $k \stackrel{\eta\to 0}{\longrightarrow} k_3(\eta/\eta_0)^{-Q_3}$

principal axis $\Downarrow$ **simplification**

$$\Sigma_{\parallel} = 2Q_3 \quad \Sigma_{V^a} = 0 \quad \Sigma'_{T^\lambda} = 0 \tag{5}$$

$$\Sigma_T^2 = \Sigma_{T^+}^2 + \Sigma_{T^\times}^2 = 6(1 - Q_3^2)$$

Application of CPT

- system for E_λ **can be decoupled and solved analytically** with combined Bessel functions $Z_\nu = AJ_\nu + BN_\nu$

$$\begin{pmatrix} E_+ \\ E_\times \end{pmatrix} [\eta] = \begin{pmatrix} \Sigma_+ & \Sigma_\times \\ \Sigma_\times & -\Sigma_+ \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_{\sqrt{3\frac{1+Q_3}{1-Q_3}}} \end{pmatrix} \left[\frac{k_3\eta_0}{1-Q_3} \left(\frac{\eta}{\eta_0} \right)^{1-Q_3} \right]$$

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- effect on Weyl square (simplest case $\Sigma_\times = 0$)

$\Xi^{(2)}(k_3,\eta) \propto$	$\uparrow$ growing	$\downarrow$ decaying
$+$ polarization	η^0	$\eta^{-4(1-Q_3)}$
$\times$ polarization	$\eta^{\sqrt{12(1-Q_3^2)}}$	$\eta^{-\sqrt{12(1-Q_3^2)}}$

decaying modes destabilize generic Kasner

Conclusions

- **Relevance of BKL and Cosmological Billiard**
  raised the fundamental question of stability for Kasner
- **Cosmological Perturbation Theory for Bianchi I**
  has been developed and allows for dynamical investigation
  - anisotropy causes coupling of perturbations
  - interplay with 'derived' modes, dependence on wavenumber
- **Application of CPT to generic Kasner**
  - allows for asymptotic investigation
  - shows that decaying modes destabilize generic Kasner
  - $\Rightarrow$ **potentially threats Stability of Generalized Kasner**
    further work is required to provide more details
    Generalized Kasner $\overset{?}{\leftrightarrow}$ Generic Kasner

Literature

- PENROSE, HAWKING: *The singularities of gravitational collapse and cosmology*
- BELINSKII, LIFSHITZ, KHALATNIKOV: *Oscillatory Approach to a Singular Point in the Relativistic Cosmology*
- DAMOUR, HENNEAUX, NICOLAI: *Cosmological Billiards*
- MUKHANOV, FELDMAN, BRANDENBERGER: *Theory of cosmological perturbations*
- PEREIRA, PITROU, UZAN:
  - *Theory of cosmological perturbations in an anisotropic universe*
  - *Predictions from an anisotropic inflationary era*
- KOFMAN,PITROU, UZAN: *Perturbations of generic Kasner spacetimes and their stability*

Penrose-Hawking Singularity Theorem

No spacetime $\mathcal{M}$ can satisfy all of the following 3 requirements:

1. $\nexists$ closed timelike curve $\gamma \in \mathcal{M}$
2. $\forall$ inextendible causal geodesic $\gamma \in \mathcal{M}$ $\exists p_{1/2}$ conjugate
3. $\exists$ future/past-trapped set $S \in \mathcal{M}$

Corollary in more physical terms

A spacetime $\mathcal{M}$ cannot satisfy causal geodesic completeness, if together with Einstein's equations the following conditions hold:

1. $\mathcal{M}$ contains no closed timelike curves
2. $\mathcal{M}$ satisfies energy and generality conditions
3. $\mathcal{M}$ contains either a trapped surface, a compact spacelike hypersurface or a point p for which the convergence of all the null geodesics through p changes sign in the past of p

BKL singularity

Generalized Kasner in synchronous gauge

$$ds^2 = -dt^2 + \left(a^2 l_i l_j + b^2 m_i m_j + c^2 n_i n_j\right) dx^i dx^j$$

Kasner exponents $a = t^{p_l}, b = t^{p_m}, c = t^{p_n}$ Kasner axes l, m, n

- near singularity: anisotropic, homogeneous, chaotic solution
- **epoch**: time interval in which order of p_l, p_m, p_n is fixed
  - 3dim Ricci negligible compared to terms with time-derivatives
  - 'dangerous' terms can be included in a new system
    $\Rightarrow$ asymptotic solution can be described in full details and description is valid and stable up to the singularity!
- **era**: time interval in which largest p remains the same

$$V'' + k^2 V - \begin{pmatrix} \frac{z_s''}{z_s} & 0 & 0 \\ 0 & \frac{z_+''}{z_+} & 0 \\ 0 & 0 & \frac{z_\times''}{z_\times} \end{pmatrix} V = \begin{pmatrix} 0 & \aleph_+ & \aleph_\times \\ \aleph_+ & 0 & \beth \\ \aleph_\times & \beth & 0 \end{pmatrix} V$$

Details on relevant functions

$$\frac{z_s''}{z_s} = \frac{a''}{a} - a^2 V_{,\varphi\varphi} + \frac{1}{a^2}\left(\frac{2a^2\kappa\varphi'^2}{2\mathcal{H} - \sigma_\parallel}\right)'$$

$$\frac{z_\lambda''}{z_\lambda} = \frac{a''}{a} + 2\sigma^2_{T(1-\lambda)} + \frac{(a^2\sigma_\parallel)'}{a^2} + \frac{1}{a^2}\left(\frac{2a^2\sigma^2_{T\lambda}}{2\mathcal{H} - \sigma_\parallel}\right)'$$

$$\aleph_\lambda = \frac{\sqrt{\kappa}}{a^2}\left(\frac{2a^2\varphi'\sigma_{T\lambda}}{2\mathcal{H} - \sigma_\parallel}\right)' \qquad \beth = \frac{1}{a^2}\left(\frac{2a^2\sigma_{T\times}\sigma_{T+}}{2\mathcal{H} - \sigma_\parallel}\right)' - 2\sigma_{T\times}\sigma_{T+}$$

$$e^1_i = \begin{pmatrix} e^{\beta_1}\left[\cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma\right] \\ e^{\beta_2}\left[\cos\alpha\cos\beta\sin\gamma + \sin\alpha\cos\gamma\right] \\ -e^{\beta_3}\cos\alpha\sin\beta \end{pmatrix} \quad \hat{k}_i = \begin{pmatrix} e^{\beta_1}\sin\beta\cos\gamma \\ e^{\beta_2}\sin\beta\sin\gamma \\ e^{\beta_3}\cos\beta \end{pmatrix}$$

$$e^2_i = \begin{pmatrix} -e^{\beta_1}\left[\sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma\right] \\ e^{\beta_2}\left[-\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma\right] \\ e^{\beta_3}\sin\alpha\sin\beta \end{pmatrix}$$

Connection between Euler angles

$$(e^1_i e^i_2)' = 0 : \alpha' = -\gamma'\cos\beta$$

$$k'_i = 0 : \tan\gamma = e^{\beta_1-\beta_2}\tan\gamma_0 \quad \tan\beta = e^{\beta_3-\beta_2}\frac{\sin\gamma_0}{\sin\gamma}\tan\beta_0$$

Coupled system for gravitational waves

$$E_+'' + \frac{1}{\eta}E_+' + \left[k_I^2\left(\frac{\eta}{\eta_0}\right)^{-2Q_I} - \frac{1}{2\eta^2}\Sigma_{T^\times}^2\right]E_+ = -\frac{1}{2\eta^2}\Sigma_{T^+}\Sigma_{T^\times}E_\times$$

$$E_\times'' + \frac{1}{\eta}E_\times' + \left[k_I^2\left(\frac{\eta}{\eta_0}\right)^{-2Q_I} - \frac{1}{2\eta^2}\Sigma_{T^+}^2\right]E_\times = -\frac{1}{2\eta^2}\Sigma_{T^+}\Sigma_{T^\times}E_+$$

General solution

$$E_+ = \Sigma_{T^+}Z_0 + \Sigma_{T_\times}Z_{\sqrt{3\frac{1+Q_I}{1-Q_I}}} \qquad E_\times = \Sigma_{T^\times}Z_0 - \Sigma_{T_+}Z_{\sqrt{3\frac{1+Q_I}{1-Q_I}}}$$

Bessel functions $Z_\nu(x) = AJ_\nu(x) + BN_\nu(x)$

Bianchi I asymptotically becomes Kasner

- early times (prior to inflation): shear σ dominates
- contribution φ to energy density mainly given by V
  $\hookrightarrow$ pure cosmological constant $V = V_0 \quad \dot{\varphi} = 0$

Bianchi I as Generalized Kasner

$$ds^2 = -dt^2 + X_i(t)^2 (dx^i)^2$$

$$X_i(t) = a_* \left[\sinh\left(\frac{t}{t_*}\right)\right]^{\frac{1}{3}} \left[\tanh\left(\frac{t}{2t*}\right)\right]^{\frac{2}{3}\sin\left(\beta + \frac{2\pi}{3} i\right)}$$

$$\cong \left(\frac{t}{t*}\right)^{\frac{2}{3}Q_i + \frac{1}{3}} \left[1 + \frac{1}{18}(1 - Q_i)\left(\frac{t}{t_*}\right)^2\right]$$

$\Rightarrow$