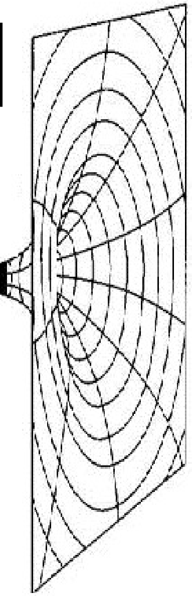
What Curves the Schwarzschild Geometry?



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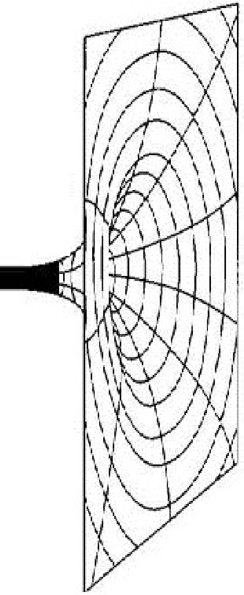
IMPRS workshop
July 2nd, 2012

Aim of my talk



- To present a different perspective on the Schwarzschild metric:

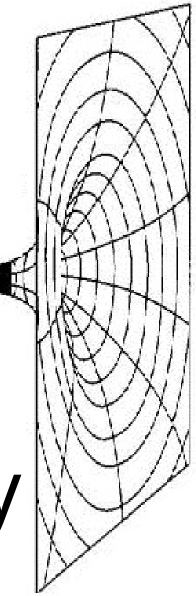
  The gravitational field of a **point mass**

- Problem of multiplication of distributions

  Gravity: nonlinear theory

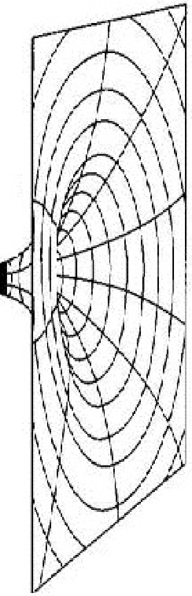
  Distributions: linear functionals

Literature I: Products of “Distributions”



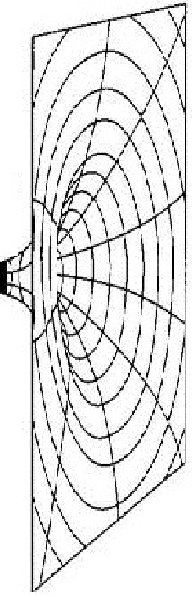
- **Schwarz** (1951): theorem of the impossibility of the multiplication of distributions
- **Colombeau** (1984): Colombeau algebra embedding generalized functions via convolution with smooth “mollifiers”
- **Kleinert** (2000): Definition of special products of distributions by claiming general coordinate invariance of path integrals

Literature II: “Distributions” in GR



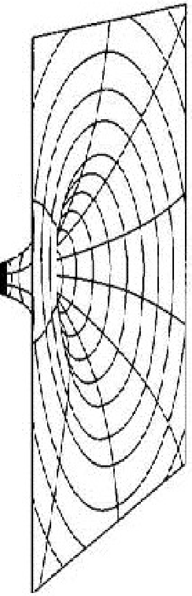
- **Geroch and Traschen** (1987): defined a class of metrics which can be treated with distributional methods
- Regularization techniques (1990s)
  (e.g. Balasin and Nachbagauer (1993))
- **Heinzle and Steinbauer** (2002) studied the Schwarzschild metric with Colombeau's theory of generalized functions
  → only possible in Eddington-Finkelstein coordinates

Content



- Theory of Gravitation
- Schwarzschild Metric
- Analogy to Electrostatics:
  Schwarzschild metric → point mass
- Perturbative approach:
  Point mass → Schwarzschild metric
- Conclusion

Theory of Gravitation



- General coordinate invariance:

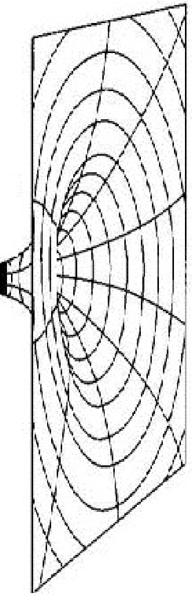
$$x^\mu \to x'^\mu\left(x^\nu\right)$$

$\rightarrow$ Transformation of the metric:

$$g_{\mu\nu} \to g'_{\mu\nu} = \frac{\partial x^\kappa}{\partial x'^\mu}\frac{\partial x^\lambda}{\partial x'^\nu} g_{\kappa\lambda}$$

$\rightarrow$ Christoffel symbols, covariant derivative...

Theory of Gravitation



- Idea:
  - Masses deform space-time
  - curvature causes forces

- Einstein equation: $G_{\mu\nu} = \kappa T_{\mu\nu}$

  describes the deformation of space-time

  $G_{\mu\nu}$: Einstein tensor (nonlinear in the metric)

  $T_{\mu\nu}$: stress-energy-tensor (contains mass density)

  $\kappa = 8\pi G_N / c^4$: gravitational constant

Point Mass

- Stress-energy tensor of a point-mass at rest:

$$T_\mu{}^\nu = \begin{pmatrix} Mc^2\delta^{(3)}(\mathbf{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_\mu{}^\nu$$

- Einstein Equation: $G_\mu{}^\nu = \kappa T_\mu{}^\nu = \kappa M c^2 \delta_\mu{}^t \delta^\nu{}_t \delta^{(3)}(\mathbf{x})$

Schwarzschild Metric

- **Birkhoff's Theorem**:

  The Schwarzschild metric is the only nontrivial solution of the **vacuum** Einstein Equation:

$$G_{\mu}{}^{\nu} = \kappa T_{\mu}{}^{\nu} = 0$$

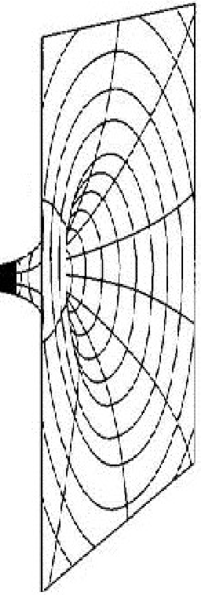
  of a spherically symmetric space-time.

- The line element is given by:

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\Omega^2$$

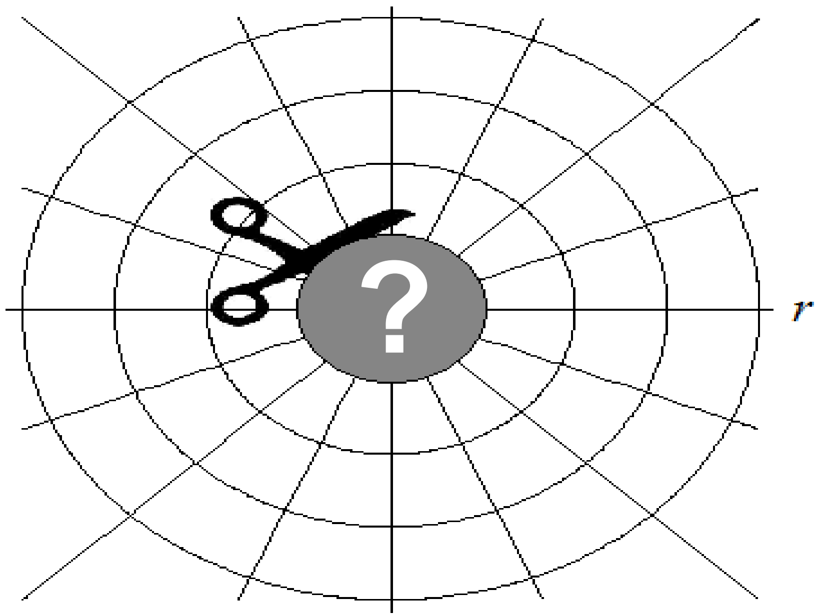
with: $r_s \equiv \frac{2G_N M}{c^2}$

Schwarzschild Metric



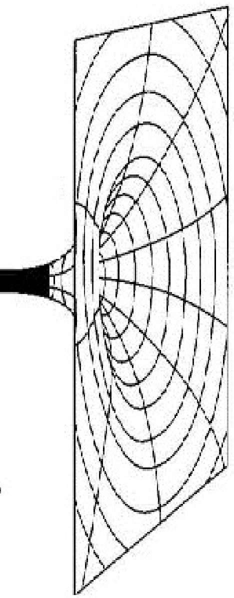
- Usual treatment:

  cut out the point r=0 of manifold



→ need not care about the divergency

Electrostatics



- Field of a positive point charge: $\mathbf{E}(\mathbf{x}) = \frac{e}{4\pi r^3}\mathbf{r}$

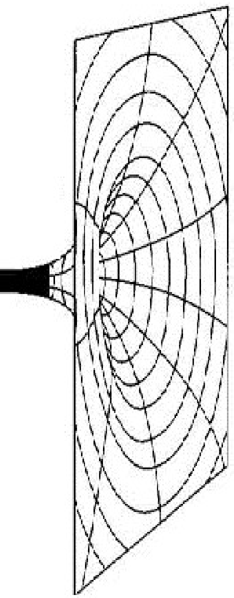
  diverges at the origin

- Charge density:
  - via distributional interpretation $\rho(\mathbf{x}) = \nabla \mathbf{E}(\mathbf{x}) = e\delta^{(3)}(\mathbf{x})$
  - or by applying Gauss' theorem:

$$Q = \int_{r<\rho} \mathrm{d}^3x \ \rho(\mathbf{x}) = \int_{r<\rho} \mathrm{d}^3x \ \nabla \mathbf{E}(\mathbf{x})$$

$$= \int_{r=\rho} \mathrm{d}^2\mathbf{S} \ \mathbf{E}(\mathbf{x}) = e$$

Electrostatic → Gravitystatic



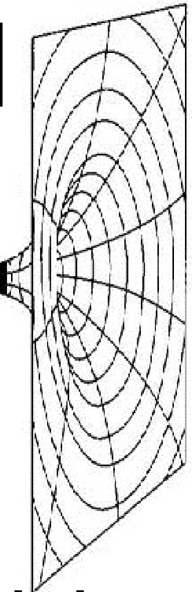
- Electric field becomes metric field

$$\mathbf{E}(\mathbf{x}) \to g_{\mu\nu}$$

- Maxwell equation becomes Einstein equation

$$\nabla \mathbf{E}(\mathbf{x}) = e\delta^{(3)}(\mathbf{x}) \quad \to \quad G_{\mu}{}^{\nu} = \kappa M c^2 \delta_{\mu}{}^{t} \delta^{\nu}{}_{t} \delta^{(3)}(\mathbf{x})$$

What Curves the Schwarzschild Geometry?



- **Corollary**:

  A spherically symmetric static space-time which obeys ${G_t}^t = {G_r}^r$ is described by the following line element:

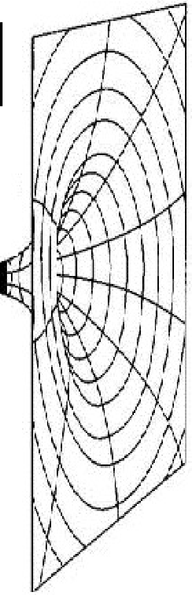
$$ds^2 = B(r)c^2\mathrm{d}t^2 - B(r)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2$$

  Its Einstein tensor is given by:

$${G_t}^t = {G_r}^r = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r - rB(r)\right]$$

$${G_\theta}^\theta = {G_\phi}^\phi = -\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{2}r^2B'(r)\right]$$

What Curves the Schwarzschild Geometry?



- See mass in ${G_t}^t$ with Gauss' theorem

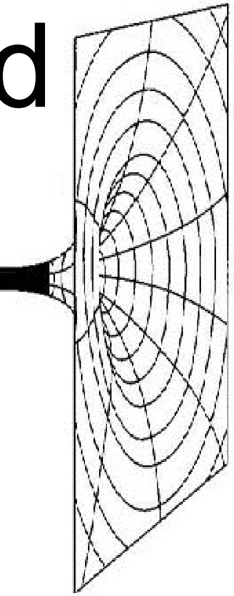
$$
\begin{aligned}
M_\rho &= \frac{1}{\kappa c^2}\int_{r<\rho} \mathrm{d}^3x \; {G_t}^t \\
&= \frac{1}{\kappa c^2}\int_{r<\rho} \mathrm{d}^3x \; \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[r - rB(r)\right] \\
&= \frac{1}{\kappa c^2}\int_{r<\rho} \mathrm{d}^3x \; \nabla\{\mathbf{e}_r \frac{1}{r}[1 - B(r)]\} \\
&= \frac{1}{\kappa c^2}\int_{r=\rho} \mathrm{d}S \; \frac{1}{r}[1 - B(r)] \\
&= \frac{1}{\kappa c^2}\rho[1 - B(\rho)] = M
\end{aligned}
$$

with:

$$B(\rho) = 1 - \frac{r_s}{\rho}$$

$$r_s = \frac{\kappa M c^2}{4\pi}$$

What Curves the Schwarzschild Geometry?



- Solution in spherical coordinates:

$$G_\mu{}^\nu = \kappa \begin{pmatrix} Mc^2\delta^{(3)}(\mathbf{x}) & 0 & 0 & 0 \\ 0 & Mc^2\delta^{(3)}(\mathbf{x}) & 0 & 0 \\ 0 & 0 & -\frac{1}{2}Mc^2\delta^{(3)}(\mathbf{x}) & 0 \\ 0 & 0 & 0 & -\frac{1}{2}Mc^2\delta^{(3)}(\mathbf{x}) \end{pmatrix}_\mu{}^\nu$$

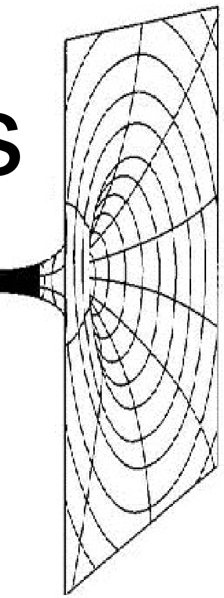
What Curves the Schwarzschild Geometry?

- Change to Cartesian coordinates:

$$G_\mu{}^\nu = \kappa \begin{pmatrix} Mc^2\delta^{(3)}(\mathbf{x}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_\mu{}^\nu = \kappa T_\mu{}^\nu$$

This gives the expected stress-energy tensor of a point mass

Perturbative Study of Point Mass



- Einstein equation for a point mass:

$$G_\mu{}^\nu = \kappa T_\mu{}^\nu = \kappa M c^2 \delta_\mu{}^t \delta^\nu{}_t \delta^{(3)}(\mathbf{x})$$

- Expand metric around the flat space-time:

$$g_{\mu\nu}(\mathbf{x}) \equiv \eta_{\mu\nu} + h_{\mu\nu}(\mathbf{x})$$

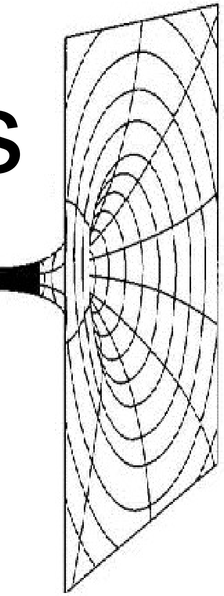
- Inverse metric:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^\mu{}_\sigma h^{\sigma\nu} - h^\mu{}_\sigma h^\sigma{}_\rho h^{\rho\nu} + \cdots$$

- Calculate Einstein tensor in order by order in $h_{\mu\nu}(\mathbf{x})$

$$G_\mu{}^\nu = G^{(1)}{}_\mu{}^\nu + G^{(2)}{}_\mu{}^\nu + \cdots$$

Perturbative Study of Point Mass



- Solve differential equations:

$$G^{(1)}{}_{\mu}{}^{\nu} = \kappa M c^2 \delta_{\mu}{}^{t} \delta^{\nu}{}_{t} \delta^{(3)}(\mathbf{x})$$

$$G^{(2)}{}_{\mu}{}^{\nu} = 0$$

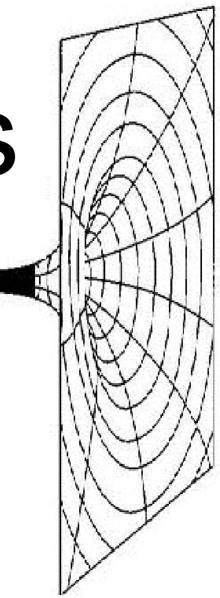
$$\ldots$$

- Obtain expansion of Schwarzschild metric in Schwarzschild coordinates order by order:

$$ds^2 = ds_{flat}{}^2 + \frac{r_s}{r}\left(c^2 \mathrm{d}t^2 - \mathrm{d}r^2\right) - \sum_{n=2}^{\infty}\left(\frac{r_s}{r}\right)^n \mathrm{d}r^2$$

- But: convergence radius of geometric series is r_s $\rightarrow$ no prediction for the origin

Perturbative Study of Point Mass



- Einstein tensor to first order in $h_{\mu\nu}$:

$$G^{(1)}_{\mu\nu} = \frac{1}{2}\partial_{\mu\kappa}h_{\nu}{}^{\kappa} + \frac{1}{2}\partial_{\nu\kappa}h_{\mu}{}^{\kappa} - \frac{1}{2}\partial^2 h_{\mu\nu} - \frac{1}{2}\partial_{\mu\nu}h - \frac{1}{2}\eta_{\mu\nu}\partial^{\kappa\lambda}h_{\kappa\lambda} + \frac{1}{2}\eta_{\mu\nu}\partial^2 h$$

- Gauge freedom of linear gravity:

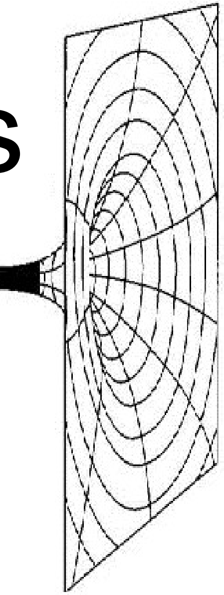
$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\Lambda_\nu + \partial_\nu\Lambda_\mu$$

$$G^{(1)}_{\mu\nu} \to G'^{(1)}_{\mu\nu} = G^{(1)}_{\mu\nu}$$

Λ_μ: arbitrary vector field

→ **Gauge invariance broken in 2nd order**

Perturbative Study of Point Mass

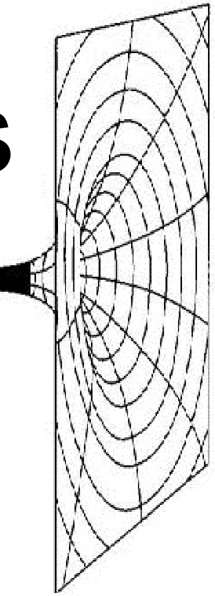


- Solve linear Einstein equation in (d >3) dimensions:

$$
{h_\mu}^\nu = \left(\frac{r_s^{(D)}}{r}\right)^{d-2} \times \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & \dfrac{1}{d-2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \dfrac{1}{d-2} \end{pmatrix}_\mu^{\ \nu}
$$

$$
+ \partial_\mu \Lambda^\nu + \partial^\nu \Lambda_\mu
$$

Perturbative Study of Point Mass



- Find an appropriate gauge field:

$$\Lambda_0 = \pm \frac{r}{d-3} \left(\frac{r_s^{(D)}}{r} \right)^{d-2} \quad \Lambda_i = \frac{q^i}{2} \frac{1}{d-2} \left(\frac{r_s^{(D)}}{r} \right)^{d-2}$$

- Derivative of :

$$\partial_i \Lambda_j = \frac{1}{2} \left(\frac{r_s^{(D)}}{r} \right)^{d-2} \left(\frac{\delta^{ij}}{d-2} - \frac{q^i q^j}{r^2} \right)$$

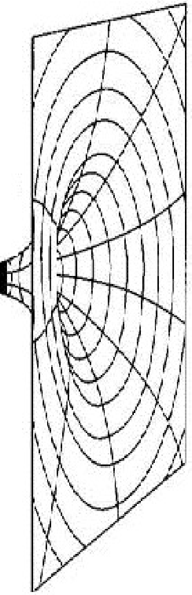
$$\partial_i \Lambda_0 = \mp \left(\frac{r_s^{(D)}}{r} \right)^{d-2} \frac{x_i}{r}$$

$$\partial_0 \Lambda_\mu = 0$$

with:

$$i = 1, 2, \cdots, d$$

Perturbative Study of Point Mass



- Get the **full Schwarzschild solution** in Eddington-Finkelstein coordinates:

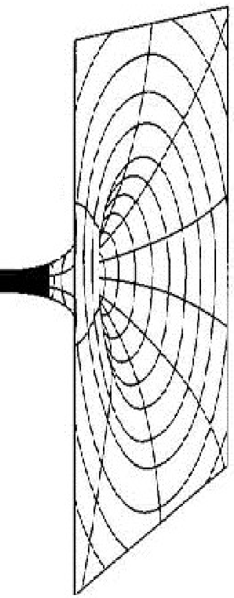
$$h_{\mu\nu} = \left(\frac{r_s^{(D)}}{r}\right)^{d-2} \times \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & -\frac{1}{d-2} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{d-2} \end{pmatrix}_{\mu\nu} + \partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu$$

$$= \left(\frac{r_s^{(D)}}{r}\right)^{d-2} \times \begin{pmatrix} -1 & \mp\frac{q^1}{r} & \mp\frac{q^2}{r} & \cdots & \mp\frac{q^d}{r} \\ \mp\frac{q^1}{r} & -\frac{(q^1)^2}{r^2} & -\frac{q^1 q^2}{r^2} & \cdots & -\frac{q^1 q^d}{r^2} \\ \mp\frac{q^2}{r} & -\frac{q^1 q^2}{r^2} & -\frac{(q^2)^2}{r^2} & \cdots & -\frac{q^2 q^d}{r^2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mp\frac{q^d}{r} & -\frac{q^1 q^d}{r^2} & -\frac{q^2 q^d}{r^2} & \cdots & -\frac{(q^d)^2}{r^2} \end{pmatrix}_{\mu\nu}$$

Conclusion

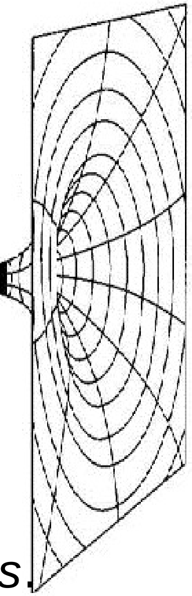
- Problem solved with Colombeau algebra
  But only in Eddington-Finkelstein coordinates
- Regularization independent technique
  to see mass in Schwarzschild metric
- Perturbative approach:
  Gauge 1st order solution → Schwarzschild metric in Eddington-Finkelstein coordinates
- Motovation for gauge via calculation in d>3 dimensions

Conclusion



- Eddington-Finkelstein coordinates are the natural coordinates of a point mass
  - Results from the perturbative study of the Einstein equation
  - Only this choice of coordinates could be treated by Colombeau's theory of generalized functions
- Different coordinates of the Schwarzschild metric describe different space-times since coordinate transformations diverge at r_s

Literature



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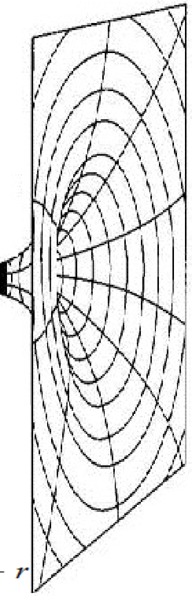
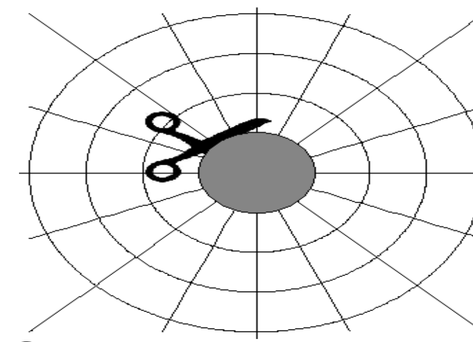
Appendix
Closing the manifold



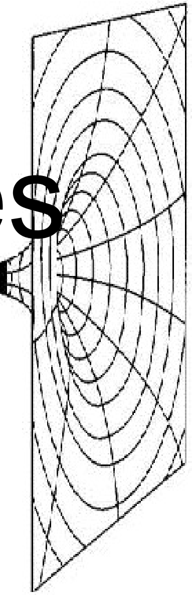


- Origin of coordinates is cut out
- Coordinate invariance

  → Infinity of different possibilities
- But: Which differentiable structure?

- Choose the simplest/most physical one

Appendix
Eddington-Finkelstein coordinates



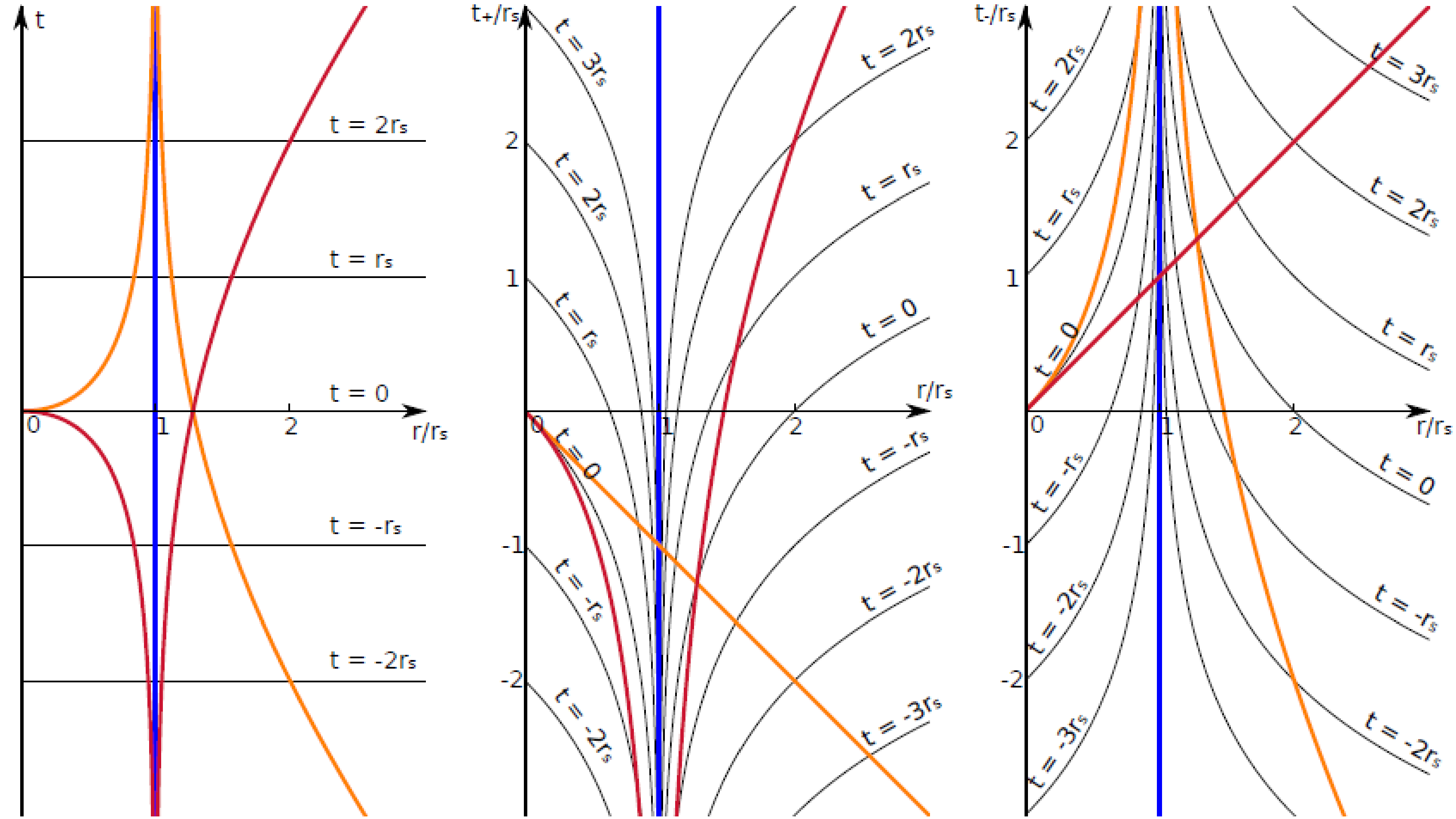
- Line element given by:

$$ds^2 = \mathrm{d}t_{\pm}^2 - \mathrm{d}r^2 - \frac{r_s}{r}\left(\mathrm{d}r \pm \mathrm{d}t_{\pm}\right)^2 - r^2\mathrm{d}\Omega^2$$

- Transformation from Schwarzschild to Eddington-Finkelstein coordinates:

$$t \to t_{\pm} = ct \pm r_s \log\left|1 - \frac{r}{r_s}\right|$$

Appendix
Eddington-Finkelstein coordinates

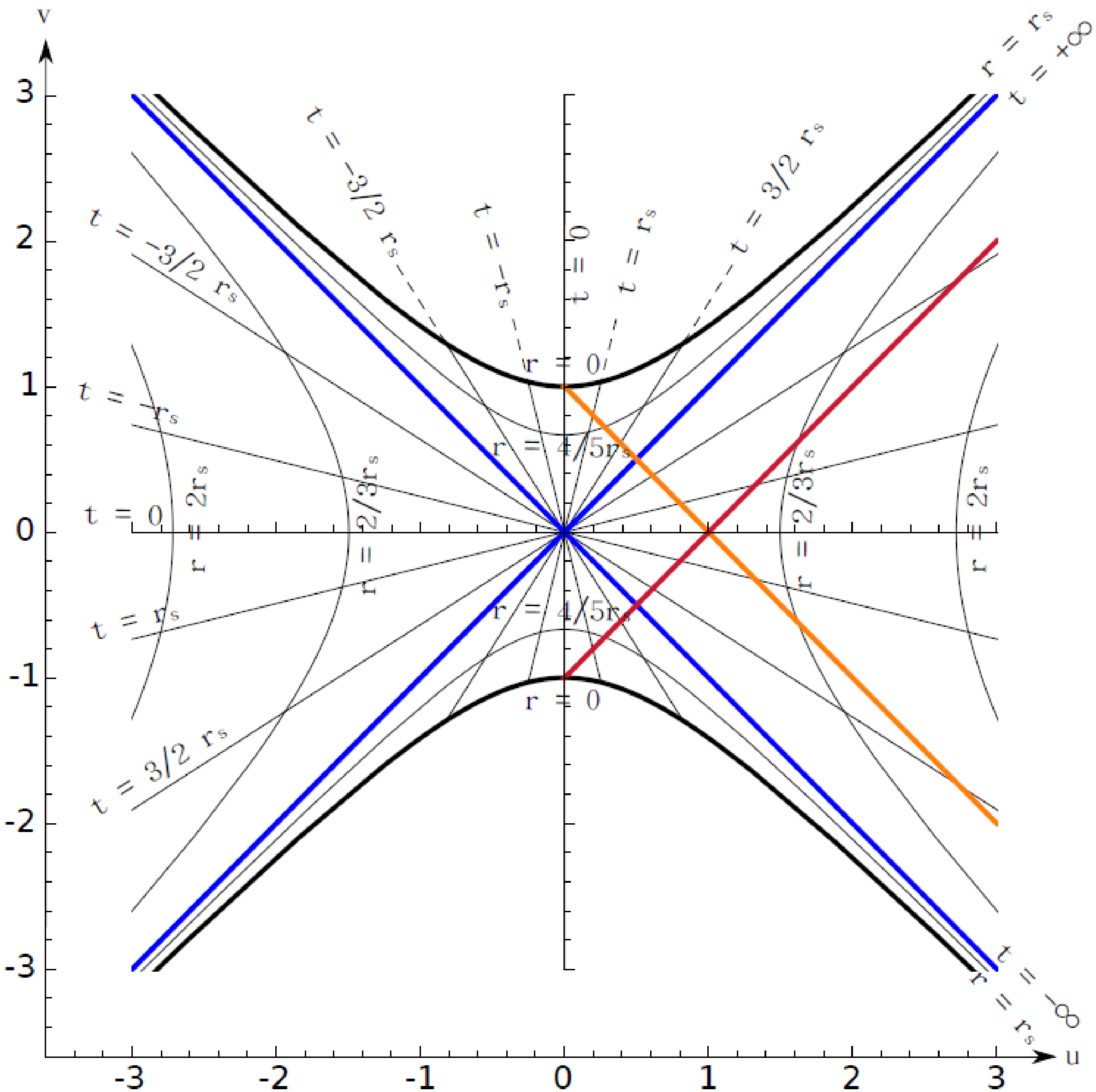


Schwarzschild

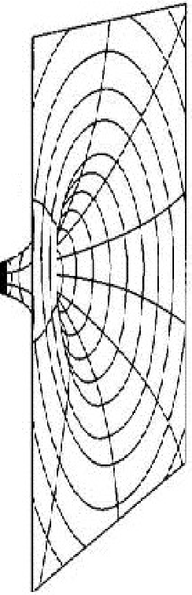
Ingoing Eddington-Finkelstein

Outgoing Eddington-Finkelstein

Appendix
Kruskal coordinates



Appendix
Distributional calculation



$$\int \mathrm{d}^3x \; G_t{}^t \phi(\mathbf{x}) = \int \mathrm{d}^3x \; \nabla[\mathbf{e}_r f(\mathbf{x})]\phi(\mathbf{x})$$

$$= -\int \mathrm{d}^3x \; f(\mathbf{x}) \, (\mathbf{e}_r \nabla) \, \phi(\mathbf{x})$$

$$= -4\pi \int_0^\infty \mathrm{d}r \; r^2 f(\mathbf{x}) \; \partial_r S_\Phi(r)$$

$$= -4\pi r_s \int_0^\infty \mathrm{d}r \; \partial_r S_\Phi(r)$$

$$= 4\pi r_s \Phi(0)$$

$$f(\mathbf{x}) := \frac{1}{r}[1 - B(r)] = \frac{r_s}{r^2}$$

$$S_\Phi(r) := \frac{1}{4\pi} \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \sin\theta \; \Phi(\mathbf{x})$$