Instantons in QFT and String Theory

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Motivation

Consider path-integral:

$$Z[J] = \int [\mathrm{dA}] \exp\left(\frac{-1}{e^2} S[A] + \int \mathrm{d}^4\mathrm{x}\, \mathrm{Tr}\left(J^\mu A_\mu\right) \right)$$

Semi-classical approximation:

$$S[A] = S[A_{cl}] + \frac{1}{2} \int \mathrm{d}^4\mathrm{x}_1 \mathrm{d}^4\mathrm{x}_2 \delta A_\mu(x_1) \Delta(x_1, x_2)^{\mu\nu} \delta A_\nu(x_2) + \dots$$

Note that different contributions are weighted with $\exp\left(-\frac{S_{cl}}{e^2}\right)$.

Finite action solutions

Let's look for *finite action* solutions to classical $SU(N)$ Yang-Mills theory.

$$\mathcal{L} = \frac{1}{2} \operatorname{Tr} F_{\mu\nu} F^{\mu\nu}$$

Finite action implies

$$F_{\mu\nu} \xrightarrow{r\to\infty} 0 \qquad A_\mu \xrightarrow{r\to\infty} ig^{-1}\partial_\mu g \, ,$$

where $g(x) = \mathrm{e}^{iT(x)} \in SU(N)$.

Homotopy

We are therefore dealing with maps from S^3 to $SU(N)$.
Mathematicians classify these according to their *homotopy class*.
In our case:

$$\pi_3(SU(N)) \cong \mathbb{Z}$$

$\Rightarrow$ Solutions are labelled by single integer k. Two solutions of the same homotopy class can be smoothly deformed into each other.

The easiest example is SU(2):

$$g^{(0)} = 1 \qquad g^{(1)} = \frac{x_4 + i x_j \sigma^j}{r} \qquad g^{(k)} = [g(1)]^k$$

The Bogomol'nyi bound

To obtain actual equations, we now exploit a trick. Let $\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. Then $\star F^2_{\mu\nu} = F^2_{\mu\nu}$ and

$$S = \frac{1}{2}\int \mathrm{d}^4\mathrm{x}\, F^2_{\mu\nu} = \frac{1}{4}\int \mathrm{d}^4\mathrm{x}\, (F_{\mu\nu} \mp \star F_{\mu\nu})^2 \pm 2F_{\mu\nu} \star F^{\mu\nu}$$
$$\geq \pm\frac{1}{2}\int \mathrm{d}^4\mathrm{x}\, \epsilon^{\mu\nu\rho\sigma}\partial_\nu \left(A_\nu F_{\rho\sigma} + \frac{2i}{3}A_\nu A_\rho A_\sigma\right) = 8\pi^2|k|$$

Hence S locally minimized $\Leftrightarrow$ $F_{\mu\nu} = \star F_{\mu\nu}$ for $k > 0$.

The $SU(2)$ solution

The $SU(2)$ solution for $k = 1$ was found by Polyakov et. al (1975):

$$A_\mu(x) = \frac{\rho^2 (x - X)^\nu}{(x - X)^2((x - X)^2 + \rho^2)} \bar{\eta}^i_{\mu\nu} (g \sigma^i g^{-1})$$

Note:

- Localized in 4 dimensions, hence the name
- 8 parameters called *collective coordinates*
- Singularity is only unphysical gauge artifact

ADHM

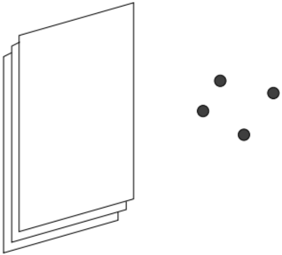
To find an instanton solution one has to solve $F_{\mu\nu} = \star F_{\mu\nu}$. But differential equations are hard!

Fortunately, there exists a very elegant solution by Atiyah, Drinfeld, Hitchin and Manin (1978). Originally constructed in terms of twistor spaces, they reduced the differential equation to an algebraic one.

Their construction admits a string theory interpretation in terms of a certain brane set-up.

Brane interpretation of ADHM I

- Stack N Dp-branes to obtain $U(N)$ SYM on their worldvolume. It couples to bulk fields e.g. via $\int_{Dp} \mathrm{d^{p+1}x}\, C_{p-3} \wedge F \wedge F$.
- Instanton gives $\int_{D(p-4)} \mathrm{d^{p-3}x}\, C_{p-3}$ $\Rightarrow$ identical to D$(p-4)$-brane!
- In $p+1$ dimensions, instanton is a $(p-3)$-dimensional object.



[stolen from D. Tong's lecture notes]

Brane interpretation of ADHM II

One now considers the gauge theory on the k D(p-4)-branes and looks for vacuum solutions.
It turns out that

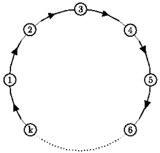
$$\mathcal{M}_{\text{Higgs}} \cong \mathcal{M}_{\text{instanton}} \ .$$

- ADHM works for all classical Lie groups.
- Instanton moduli space has Hyper-Kähler structure, the moment maps are obtained from gauge potential.
- For most QFT calculations one only needs the metric on the instanton moduli space.

Applications & my work

What is my project about?

- Calculated the Hilbert series using plethystics (Feng, Hanany, He (2007)) for certain class of quiver gauge theories:



$$g_N(\{t_i\}, k) = \frac{1}{\prod_{i=1}^{k}(1 - t_k^N)} \prod_{j=1}^{N-1} \frac{1}{1 - \prod_{i=1}^{k} t_i^j}$$

- Apply these techniques to study instanton moduli space, see Hanany, Mekareeya et al (2010 & 2012).

Sources

Sources

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- M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, "*Construction Of Instantons*," Phys. Lett. A 65 (1978) 185.
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