Scattering Amplitudes at the Integrand Level

Tiziano Peraro

Max-Planck-Institut für Physik
Föhringer Ring 6, D-80805 München, Germany

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MAX-PLANCK-GESELLSCHAFT

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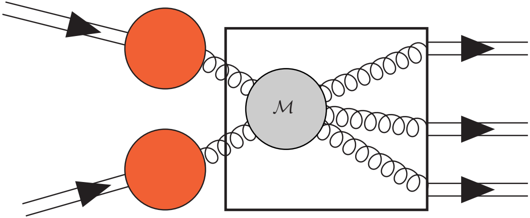
- P. Mastrolia, E. Mirabella, T. P., "Integrand reduction of one-loop scattering amplitudes through Laurent series expansion", *JHEP* 1206 (2012), arXiv:1203.0291
- P. Mastrolia, E. Mirabella, G. Ossola, T. P., "Scattering Amplitudes from Multivariate Polynomial Division", arXiv:1205.7087

Outline

Introduction

- Scattering amplitudes are the backbone of high–energy computations for colliders



- They can be computed in perturbation theory

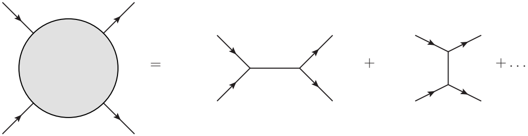
$$\mathcal{M} \sim \mathcal{M}_{\text{LO}} + \alpha\, \mathcal{M}_{\text{NLO}} + \alpha^2\, \mathcal{M}_{\text{NNLO}} + \dots$$

- Amplitudes with many external legs are of much interest
  - for testing QCD and the SM in different settings
  - as backgrounds to new physics processes

Trees and loops

The computation of scattering amplitudes at LO

- (usually) involves tree diagrams



- is relatively easy (pure algebra)
- the momentum flowing in all the internal lines is fixed by momentum conservation
- has a large uncertainty (sometimes of $O(100\%)$!!!)
- is hardly enough for a quantitative prediction

Trees and loops

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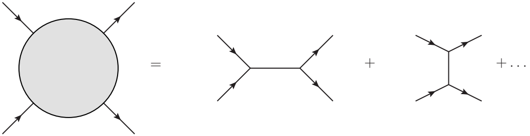
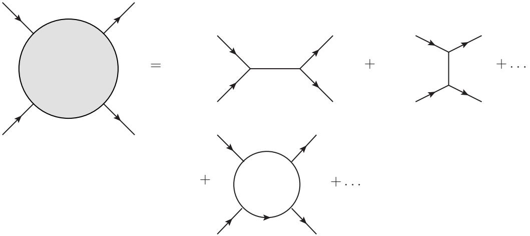


- is relatively easy (pure algebra)
- the momentum flowing in all the internal lines is fixed by momentum conservation
- has a large uncertainty (sometimes of $O(100\%)$!!!)
- is hardly enough for a quantitative prediction
  - $\Rightarrow$ we need at least NLO accuracy

Trees and loops

The computation of scattering amplitudes at NLO

- (usually) involves one-loop diagrams

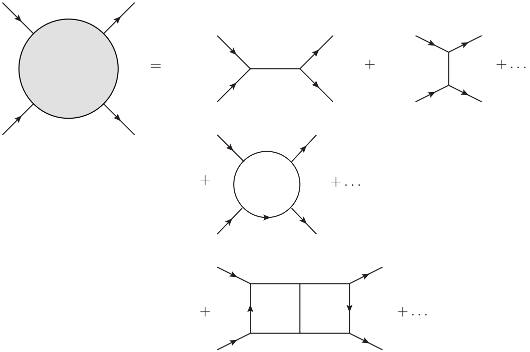


- is much more difficult
- involves an integration over the loop momentum (not fixed by momentum conservation)
- has a smaller uncertainty (maybe $\sim 10\%$)

Trees and loops

The computation of scattering amplitudes at NNLO

- (usually) involves two-loop diagrams

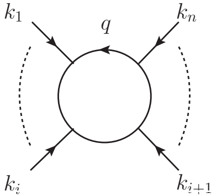


- is much^2 more difficult: integration over two loop momenta

Scattering amplitudes at one-loop

- A generic n-point one-loop amplitude

$$\mathcal{M}_n \equiv \int \mathcal{A}_n(q)\, d^4q \equiv \int \frac{N(q)}{D_1(q)\ldots D_n(q)} d^4q$$

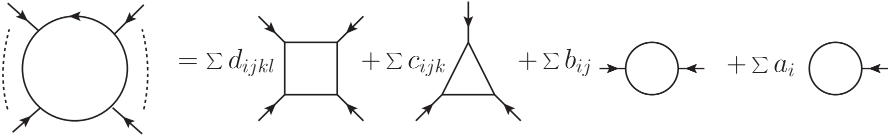


  - involves an integration over the loop momentum q
  - the Feynman denominators D_i have the form

$$D_i(q) = (q + p_i)^2 - m_i^2$$

- When the number n of external legs becomes large
  - the number of diagrams increases
  - the number of denominators increases
  - the numerator $N(q)$ becomes more complicated
  - performing the integration might seem a prohibitive task

Scattering amplitudes at one-loop



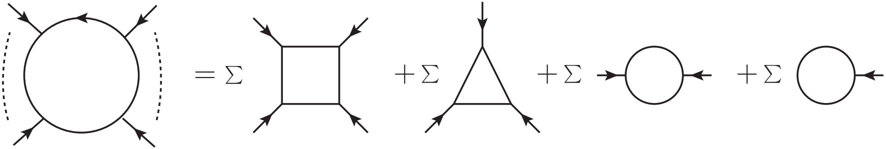
- Every one-loop amplitude in $d = 4$ can be decomposed as

$$\mathcal{M}_n = \sum_{ijkl} d_{ijkl}\, I_{ijkl} + \sum_{ijk} c_{ijk}\, I_{ijk} + \sum_{ij} b_{ij}\, I_{ij} + \sum_i a_i\, I_i$$

$$I_{ijk\ldots} = \int \frac{dq}{D_i D_j D_k \ldots}$$

  - the basis of Master Integrals (MIs) $I_{ijk\ldots}$ is known
  - the computation of the amplitude can be reduced to the problem of computing the coefficients of this decomposition

Integrand-level decomposition: OPP



$$\int \mathcal{A}_n(q) = \sum_{ijkl} \int \frac{d_{ijkl}}{D_i D_j D_k D_l} + \sum_{ijk} \int \frac{c_{ijk}}{D_i D_j D_k} + \sum_{ij} \int \frac{b_{ij}}{D_i D_j} + \sum_{i} \int \frac{a_i}{D_i}$$

- The previous decomposition holds at the integral-level

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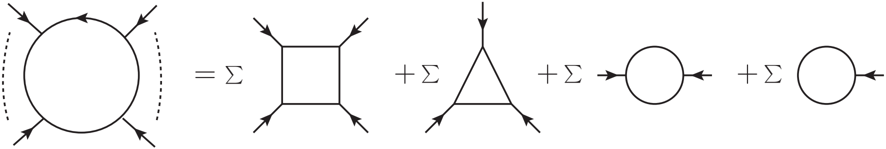

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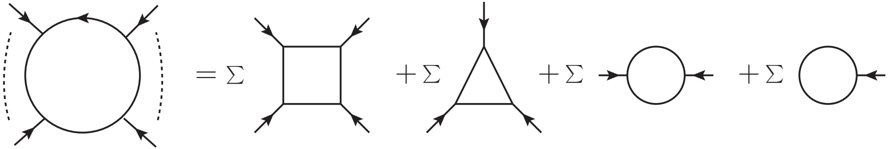
- The previous decomposition holds at the integral-level
- An analogous decomposition holds at the integrand-level
  [Ossola, Papadopoulos, Pittau (2007)]

$$\mathcal{A}(q) = \sum_{ijkl} \frac{\Delta_{ijkl}(q)}{D_i D_j D_k D_l} + \sum_{ijk} \frac{\Delta_{ijk}(q)}{D_i D_j D_k} + \sum_{ij} \frac{\Delta_{ij}(q)}{D_i D_j} + \sum_{i} \frac{\Delta_i(q)}{D_i}$$

Integrand-level decomposition: OPP



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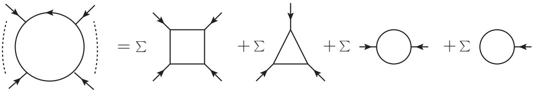
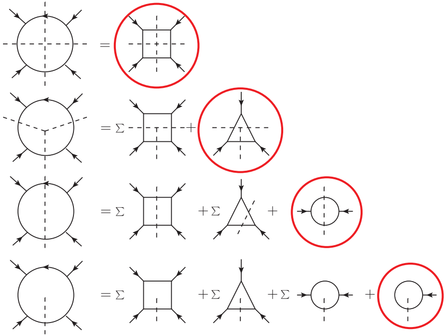
- The residues $\Delta_{ij\ldots}$
  - are polynomials in the components of q
  - have a known parametric form
  - contain the coefficients of the master integrals
  - $\Rightarrow$ they can be found by polynomial fitting

Finding the coefficients by cutting the loop



- How to fit the coefficients efficiently?
  - evaluate the integrand on multiple cuts



What is a cut?

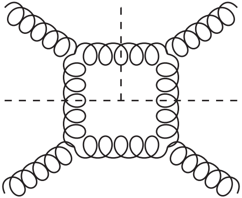
Cutting a loop propagator (roughly) means

$$\frac{1}{D_i} \to \delta(D_i)$$

i.e. putting it on-shell

Intermezzo: Loops from trees

- The coefficients on the one-loop decomposition can be found by evaluating the amplitudes on multiple cuts
  - the cut loop propagators are put on-shell



Intermezzo: Loops from trees

- The coefficients on the one-loop decomposition can be found by evaluating the amplitudes on multiple cuts
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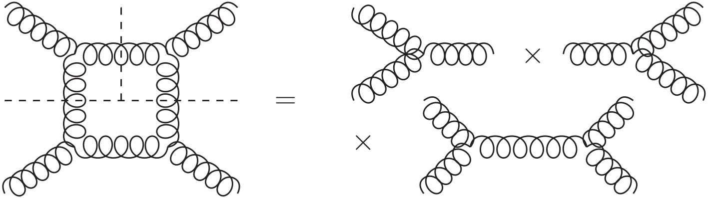
  - the integrand factorizes in a product of tree-level amplitudes



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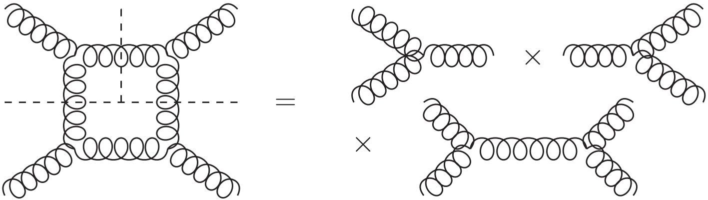


Loops from trees

If we want, we can compute a one-loop amplitude from products of tree-level amplitudes

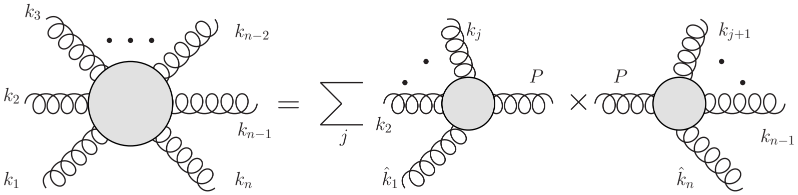
Intermezzo n. 2: Trees from smaller trees (BCFW)

Britto, Cachazo, Feng(2004); Britto, Cachazo, Feng, Witten(2005)

- Consider a tree-level amplitude $\mathcal{M}(k_1, \ldots, k_n)$
- shift two external momenta of a complex amount, such that
  - the two external momenta remain on-shell
  - an internal propagator P goes on-shell

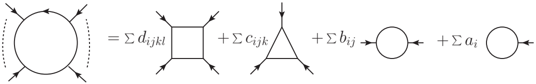
$$k_1 \to \hat{k}_1 = k_1 + z\,\eta, \qquad k_n \to \hat{k}_n = k_n - z\,\eta$$

- The original amplitude can be recursively factorized in products of smaller amplitudes with
  - shifted external momenta
  - a smaller number of external legs (down to 3)



Summary of OPP

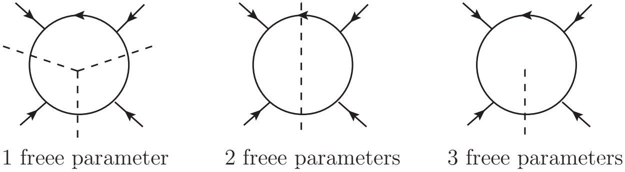
- Every one-loop amplitude is a linear combination of known master integrals



- The unknown coefficients of this linear combination can be found by polynomial fitting at the integrand level
  - requires to solve linear systems of equations
- An efficient way of doing the fit is by sampling the integrand on solutions of multiple cuts
  - some loop propagators are put on-shell
  - the systems of equations become much smaller
- The whole amplitude can be computed without actually performing the integration

Analytic and semi-analytic approach

- The computation of the coefficients of the integrand decomposition can be simplified by means of analytic methods
- In triple, double, and single cuts the loop momentum is not completely fixed by the on-shell constraints

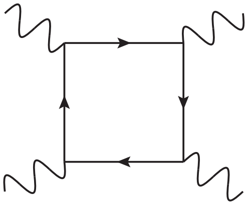


- Performing a Laurent expansion with respect to the free parameters not fixed by the cut
  - we obtain diagonal systems of equations
  - subtractions of higher-point contributions are simplified

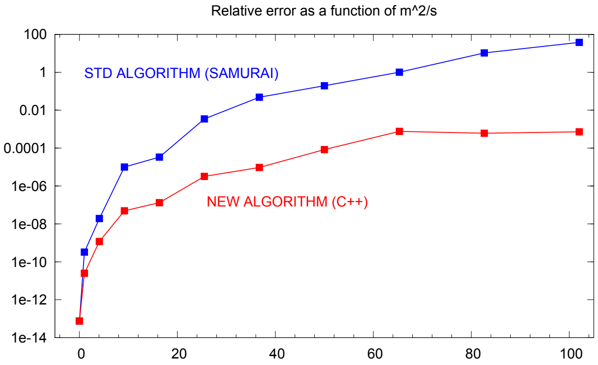
P. Mastrolia, E. Mirabella, T. P. (2012)

Semi-numeric implementation

- If the analytic expression of the integrand is known, we can perform the Laurent expansion (analytically or numerically) via polynomial division neglecting the remainder
  - first tests show an improved stability
- A very simple example



$m^2/s \to$ large



Extension to higher loops

How does this extend to higher loops?

- Only few papers on the subject

  [the first one in 2011 (Mastrolia, Ossola), at least other five in 2012 by several authors]

- We have a similar integrand decomposition

$$
\left[\frac{N(q)}{D_1 \ldots D_n}\right]_{1\text{ loop}} = \sum_{i_1,\ldots,i_4} \frac{\Delta_{i_1\ldots i_4}}{D_{i_1}\ldots D_{i_4}} + \sum_{i_1,i_2,i_3} \frac{\Delta_{i_1 i_2 i_3}}{D_{i_1} D_{i_2} D_{i_3}} + \ldots
$$

$$
\left[\frac{N(q_1,q_2)}{D_1 \ldots D_n}\right]_{2\text{ loops}} = \sum_{i_1,\ldots,i_8} \frac{\Delta_{i_1\ldots i_8}}{D_{i_1}\ldots D_{i_8}} + \sum_{i_1,\ldots,i_7} \frac{\Delta_{i_1\ldots i_7}}{D_{i_1}\ldots D_{i_7}} + \ldots
$$

  - at one-loop the residues $\Delta_{i_1 i_2\ldots}$ sit over 4 or less denominators
  - at two-loop the residues $\Delta_{i_1 i_2\ldots}$ sit over 8 or less denominators
  - …

Integrand reduction at 2 loops

- The decomposition at 2-loops (in $d = 4$ dimensions) is

$$\left[\frac{N(q_1, q_2)}{D_1 \dots D_n}\right]_{2\text{ loops}} = \sum_{i_1,\dots,i_8} \frac{\Delta_{i_1 \dots i_8}}{D_{i_1} \dots D_{i_8}} + \sum_{i_1,\dots,i_7} \frac{\Delta_{i_1 \dots i_7}}{D_{i_1} \dots D_{i_7}} + \dots$$

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- The parametric form of the residues $\Delta_{i_1 i_2 \ldots}$ is not known
  - it depends on the topology of the diagram
  - it can be found by techniques of algebraic geometry (Gröbner bases, multivariate polynomial division, . . .)

[Y. Zhang (2012); P. Mastrolia, E. Mirabella, G. Ossola, T. P. (2012)]

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- We still evaluate the integrand on multiple cuts
  - we start from 8-cuts to determine $\Delta_{i_1\dots i_8}$
  - we proceed with 7-cuts to determine $\Delta_{i_1\dots i_7}$
  - ...

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  - the reduction tells you which MIs you need
  - the number of independent MIs can be further reduced with techniques such as IBPs...

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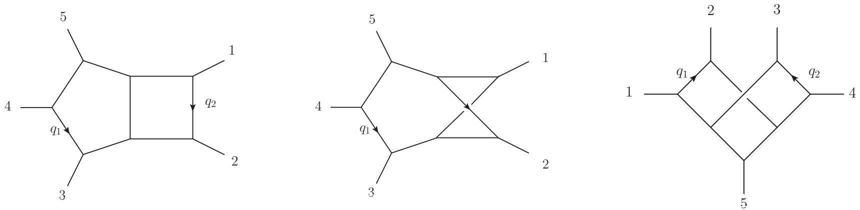
  - the number of independent MIs can be further reduced with techniques such as IBPs...
  - ...but eventually you have to compute some of them

5-point amplitude in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ SG

G. Ossola, P. Mastrolia, E. Mirabella, T. P. (to be published)



- 5-point amplitude in $\mathcal{N} = 4$ SYM
  - we decomposed it in terms of 8-cut and 7-cut residues
- 5-point amplitude in $\mathcal{N} = 8$ SG
  - we decomposed it in terms of 8-cut, 7-cut and 6-cut residues
- We found analytic and numeric results for the coefficients of the integrand decomposition

Summary and conclusions

- The reduction at the integrand level is a general method we can apply to any amplitude in any QFT
- At one-loop
  - allows to compute the amplitude without performing any (new) integration
  - has been implemented in several codes [e.g. SAMURAI]
  - is already producing results for LHC [GoSam, FormCalc, . . .]
  - a simplified reduction via Laurent expansion can provide improved stability
- At higher loops
  - the first results look promising
  - applied to both planar and non-planar diagrams
  - analytic techniques such as the Laurent expansion and polynomial division of the integrand can also simplify the computation at two (and more?) loops
  - . . . work is still in progress!