Dynamical Black Holes in Topologically Massive Gravity

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Topologically Massive Gravity

- Action:

$$S = \frac{1}{16\pi G_N} \int d^3x \sqrt{-g} \left(R + \frac{2}{l^2} \right) + \frac{1}{32\mu\pi G_N} \int d^3x \sqrt{-g} \epsilon^{\lambda\mu\nu} \Gamma^{\rho}_{\lambda\sigma} \left(\partial_\mu \Gamma^{\sigma}_{\nu\rho} + \frac{2}{3} \Gamma^{\sigma}_{\mu\tau} \Gamma^{\tau}_{\nu\rho} \right)$$

- Equations of motion (EOM):

$$G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \text{ with } G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{l^2} g_{\mu\nu}$$

$$\text{and } C_{\mu\nu} = \varepsilon_{\mu}{}^{\kappa\sigma} \nabla_\kappa \left(R_{\sigma\nu} - \frac{1}{4} g_{\sigma\nu} R \right)$$

[Deser, Jackiw, Tempelton 1982]

Background Metric and Linear Perturbations

- Stationary black holes similar to Kerr metric known [Bañados, Teitelboim, Zanelli 1992]
- $G_{\mu\nu} = 0$ and $C_{\mu\nu} = 0$ separately for these solutions
- For $M = 1$, $J = 0$ and $l = 1$:
  $ds^2 = -\sinh^2 \rho \, dt^2 + \cosh^2 \rho \, d\phi^2 + d\rho^2 \equiv \bar{g}_{\mu\nu} dx^\mu dx^\nu$ [Sachs 2011]
- Non trivial linear perturbations given by ${\epsilon_\mu}^{\alpha\beta} \nabla_\alpha h_{\beta\nu} + \mu h_{\mu\nu} = 0$ [Li, Song, Strominger 2008]

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Motivation

For one of these solutions the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ turns out to be a solution of the exact EOM of TMG:

$$G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = 0 \text{ with } G_{\mu\nu} = \frac{1-\mu^2}{2} h_{\mu\nu}$$

[Sachs 2011]

- What kind of metric is this?
- What properties does $g_{\mu\nu}$ inherit form the background $\bar{g}_{\mu\nu}$?
- What happens for the special values $\mu = \pm 1$?

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Global Coordinates

By introducing the coordinates

$$z = e^{-t}\frac{1}{\sinh(\rho)} \ , \ R = e^{-2t}\coth^2(\rho) \ , \ y = \phi + t + \log[\tanh(\rho)]$$

the metric takes the form:

$$ds^2 = \underbrace{\frac{1}{z^2}\left(dz^2 + dydR + Rdy^2\right)}_{\bar{g}_{\mu\nu}dx^\mu dx^\nu} + \underbrace{\frac{1}{z^{1+\mu}}dy^2}_{h_{\mu\nu}dx^\mu dx^\nu}$$

- y is the new angular coordinate, $\partial_\phi = \partial_y$, $y \sim y + 2\pi$, $z > 0$, $-\infty < R < \infty$
- R is a measure of time, smaller values of R correspond to the future
- Closed causal curves for $R \leq -z^{1-\mu}$

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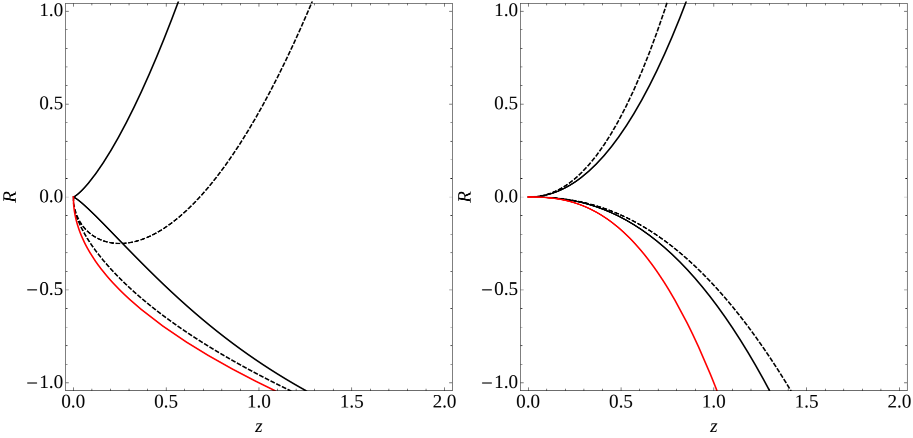
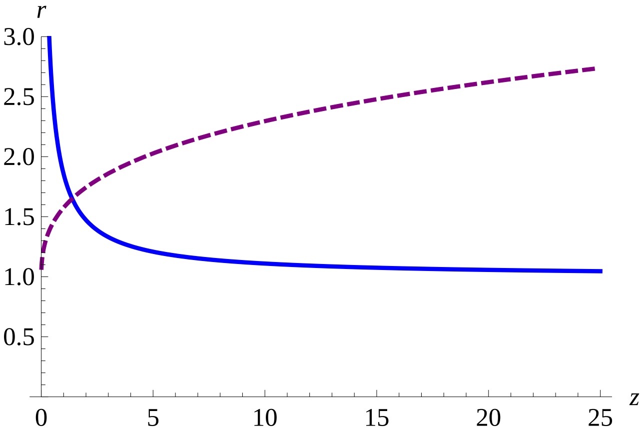
Diagrams for $\mu = 1/2$ (left) and $\mu = -3/2$ (right)

Singularity: red, Event and Cauchy horizons: solid black, Marginally trapped surfaces: dashed black



Circumference of Outer Horizon

Horizon radius ($r \equiv \frac{1}{2\pi} \cdot \textit{circumference}$) as function of z for $\mu = 1/2$ (solid), and $\mu = -3/2$ (dashed):



Entropy

- In TMG, the entropy of stationary black holes is $S = S_{EH} + S_{CS}$ with $S_{EH} = \frac{1}{4G_N}\mathcal{A}$ and $S_{CS} = \frac{1}{8\mu G_N}\int_{\Sigma} \epsilon^{\nu\mu}\Gamma_{\mu\nu\rho}dx^{\rho}$. [Tachikawa 2007]
- Ansatz for the calculation of the entropy of dynamic black holes: [Wald, Iyer 1994]
  - Introduce a transformation that creates a new metric in which the horizon cross section Σ is embedded as a bifurcation surface of a stationary black hole.
  - Calculate the entropy with respect to Σ using the appropriate formula for the stationary case.

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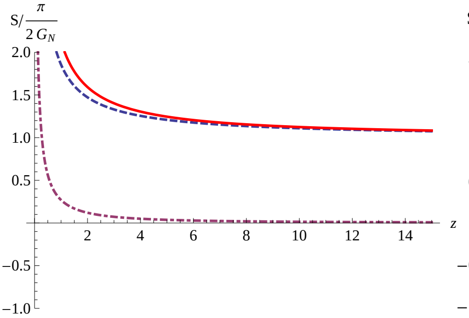
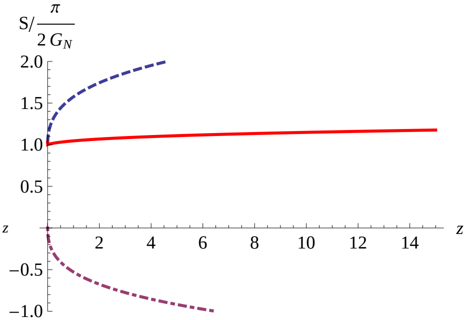
Dynamic Entropy

Numerical calculations of dynamical entropy using this ansatz for $\mu = 1/2$ (left) and $\mu = -3/2$ (right).

$S_{EH} \sim \mathcal{A}$: dashed blue, S_{CS}: dot-dashed purple, $S = S_{EH} + S_{CS}$: solid red.





Results

- For TMG, some solutions of the linerized EOM also seem to describe exact solutions of the full EOM.
- $g_{\mu\nu}$ describes a dynamical black hole, with a BTZ-like global stucture.
- The event horizon circumference is non-constant for generic values of μ.
- For certain values of μ, the time-dependent entropy is decreasing.

Outlook

- Why does S decrease?
- What is the correct definition for dynamical entropy?
- Is there a connection between solutions of the linearized and exact EOMs?

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Thank you for your attention.

Details of the Solution

For the metric $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ with $\bar{g}_{\mu\nu} = \begin{pmatrix} -\sinh^2\rho & 0 & 0 \\ 0 & \cosh^2\rho & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$h_{\mu\nu} = e^{(1+\mu)t}(\sinh\rho)^{1+\mu} \begin{pmatrix} 1 & 1 & \frac{2}{\sinh(2\rho)} \\ 1 & 1 & \frac{2}{\sinh(2\rho)} \\ \frac{2}{\sinh(2\rho)} & \frac{2}{\sinh(2\rho)} & \frac{4}{\sinh^2(2\rho)} \end{pmatrix}$, it follows:

$$R = \bar{R} = -6 \ , \ G_{\mu\nu} = -\frac{1}{\mu} C_{\mu\nu} \ , \ G_{\mu\nu} = \frac{1-\mu^2}{2} h_{\mu\nu}$$

Note that we can write $h_{\mu\nu} = l_\mu l_\nu = f(t,\phi,\rho)\ \xi_\mu \xi_\nu$ where l_μ is a null vector, and ξ_μ is a null Killing vector of $\bar{g}_{\mu\nu}$ and $g_{\mu\nu}$ as well. [Sachs 2011]

A Note on Energy Conditions

Because of the equations of motion $G_{\mu\nu} = -\frac{1}{\mu} C_{\mu\nu}$, we can use theorems that were derived for Einstein gravity with $\Lambda = 0$ (e.g. the area theorem) when we use $-\frac{1}{\mu} C_{\mu\nu} + \frac{1}{l^2} g_{\mu\nu}$ as a substitute for the energy-momentum tensor $T_{\mu\nu}$.
For an arbitrary any null vector k^μ we therefore find

$$T_{\mu\nu} k^\mu k^\nu = \frac{1 - \mu^2}{2} h_{\mu\nu} k^\mu k^\nu$$

Since we can write $h_{\mu\nu} = l_\mu l_\nu$ it follows that $h_{\mu\nu} k^\mu k^\nu = (l_\mu k^\mu)^2 \geq 0$. For this reason, the weak energy condition for example is only satisfied for $|\mu| \leq 1$ while it is violated for $|\mu| > 1$.

Special Values: $\mu = +1$

Because of $G_{\mu\nu} = \frac{1-\mu^2}{2} h_{\mu\nu}$, the metric is a solution not only of TMG, but also of Einsteinian gravity for $\mu = \pm 1$. It is natural to ask whether it is a new solution for these cases or whether it reduces to already known ones.

For $\mu = +1$ the metric takes the form:

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dydR + Rdy^2\right) + \frac{1}{z^2}dy^2$$

This can be related to the background metric (BTZ with $M = 1$, $J = 0$) by a simple shift in the coordinate R.

Special Values: $\mu = -1$

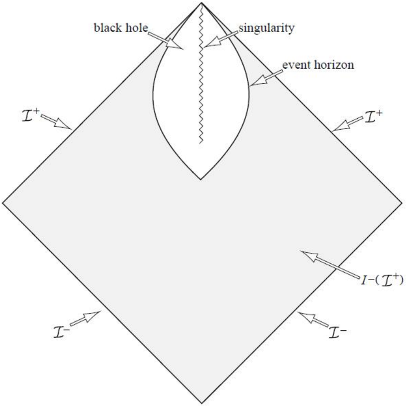
For $\mu = -1$, the relation to already known solutions is not that obvious. When calculating the horizons, we find:

- Horizons and marginally trapped surfaces are identical
- The outer horizon has radius $r_+ = const. \approx 1.61803$ while the inner horizon has $r_- = const. \approx 0.61803$
- The metric now has additional Killing vectors and is stationary.

All of this seems to imply that for $\mu = -1$, the metric is just a BTZ black hole with $M = 3$ and $|J| = 2$. Indeed, a coordinate transformation relating the two metrics can be found.

Event Horizon - Definition

In an asymptotically flat spacetime, there is a future null infinity $\mathcal{I}^+$. The *event horizon* of a black hole is then defined to be the boundary of the past of $\mathcal{I}^+$.



[Hawking 1994]

Horizons

- As the given metric is not asymptoticallly flat, there is no $\mathcal{I}^+$
- For the (asymptotically AdS) background-metric, the line $z = 0$, $R \geq 0$ corresponds to infinity
- Our new metric is asymtotically AdS for $\mu < -1$
- We use the line $z = 0$, $R \geq 0$ as our infinity for all $\mu < 1$.

Marginally Trapped Surfaces - Definition

- A *trapped surface* is a closed, orientable, smooth, spacelike, co-dimension two submanifold S, s.t. both families of future directed null geodesics orthogonal to S (i.e. "ingoers" n^μ and "outgoers" k^μ) have *expansions* $\theta_n = n^\alpha_{\ ;\alpha} < 0$, $\theta_k = k^\alpha_{\ ;\alpha} < 0$.
- Here, we define a *marginally trapped surface* to be a closed, orientable, smooth, spacelike, co-dimension two submanifold S, s.t. one of the families of future directed null geodesics orthogonal to S has expansion $\theta = 0$ while the other family has $\theta < 0$.

[Frolov, Novikov 1998]

Generating the new space time à la [Wald, Iyer 1994]

- Introduce new coordinates U and V s.t. ∂_U and ∂_V are the null vectors orthogonal to Σ, and $U = V = 0$ on Σ. In old coordinates, we denote these by l^μ and n^μ.
- The Taylor expansion in U, V of the metric around Σ reads in old coordinates:

$$g_{ab} = \sum_{n,m=0}^{\infty} \frac{U^m V^n}{m!n!} \underbrace{\sum_{\alpha\beta} \left(\frac{\partial^{m+n} g_{\alpha\beta}}{\partial U^m \partial V^n} (dx^\alpha)_a (dx^\beta)_b \right) \Bigg|_{U=V=0}}_{l^{c_1} \dots l^{c_m} n^{c_{m+1}} \dots n^{c_{m+n}} \partial_{c_1} \cdots \partial_{c_{m+n}} g_{ab}|_\Sigma}$$

- Then, in this series, the terms $\partial_{c_1} \cdots \partial_{c_{m+n}} g_{ab}|_\Sigma$ are replaced by their *boost invariant parts*. This gives the new metric g^I_{ab}.

Generating the new space time à la [Wald, Iyer 1994]

- For the thereby constructed metric g^I_{ab}, the vector field $\eta = U\partial_U - V\partial_V$ is a Killing vector field ($\mathcal{L}_\eta g^I_{ab} = 0$).
- η obviously vanishes on Σ, where $U = V = 0$.
- In the new metric g^I_{ab}, the horizon cross section Σ therefore is the bifurcation surface of a Killing horizon.
- Because of linearity, we can calculate $(\partial_c g_{ab})^I$ instead of $\partial_c \left(g^I_{ab}\right)$ in our expression for $\Gamma_{\alpha\beta\gamma}$.
- Also, because of $g^I_{ab}|_\Sigma = g_{ab}|_\Sigma$ we can use the original metric to raise and lower indices in our expression for the entropy.

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