Diphoton plus jet production through graviton exchange at NLO

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Max-Planck-Institut für Physik, München

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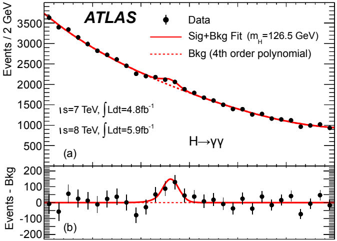
Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Overview

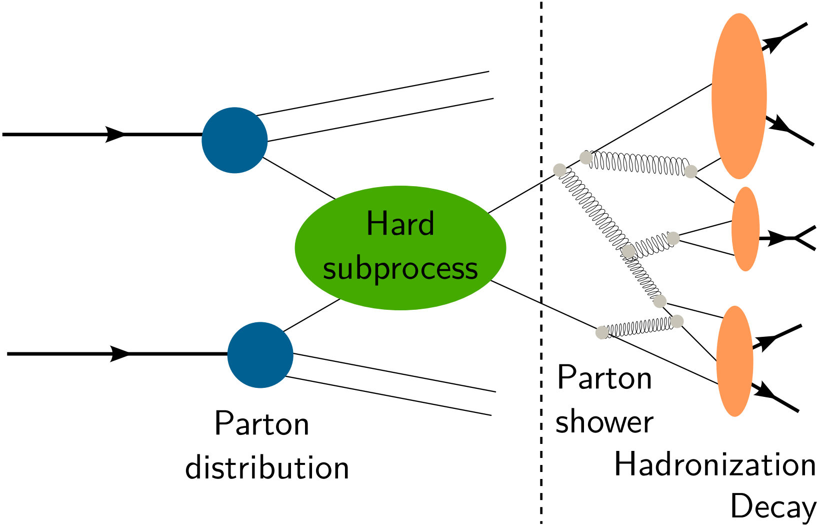
Introduction



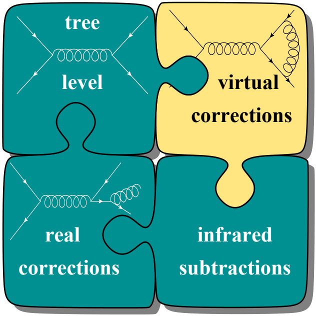
[ATLAS Collaboration (1207.7214)]

- New particle discovered at the LHC this year
- Is it the Higgs boson of the Standard Model?

$\Rightarrow$ high precision needed: *next-to-leading order calculations*


Hard
subprocess
Parton
distribution
Parton
shower
Hadronization
Decay

NLO calculation



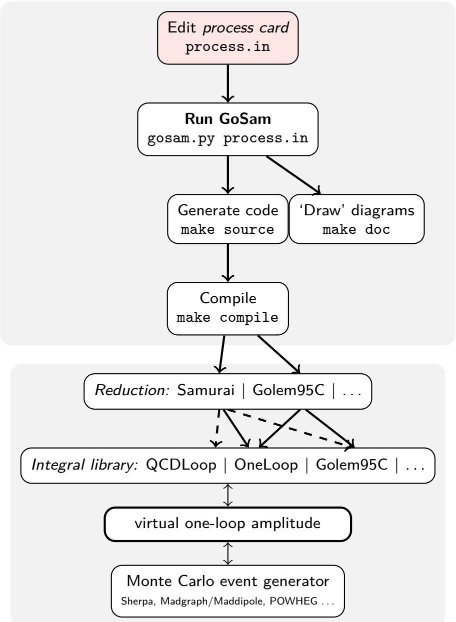
$$\sigma^{NLO} = \int_m d\sigma^B + \int_m \left[d\sigma^V + \int_1 d\sigma^A \right] + \int_{m+1} \left[d\sigma^R - d\sigma^A \right]$$

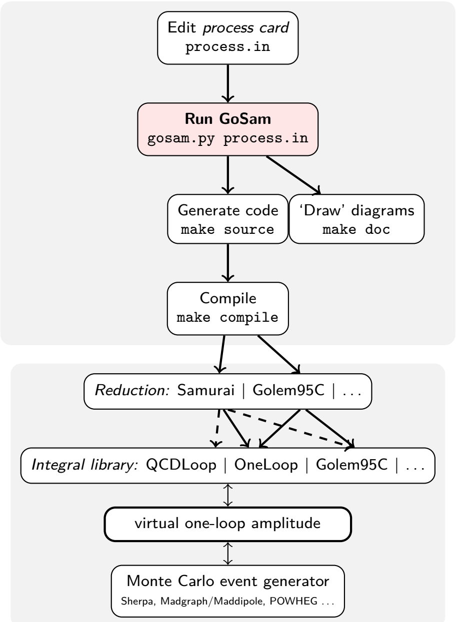
GoSam

- calculates virtual NLO corrections (one-loop multi-leg amplitudes)
- fully automated code generator
- core: integral libraries Samurai and Golem95C
- open source, uses only public tools

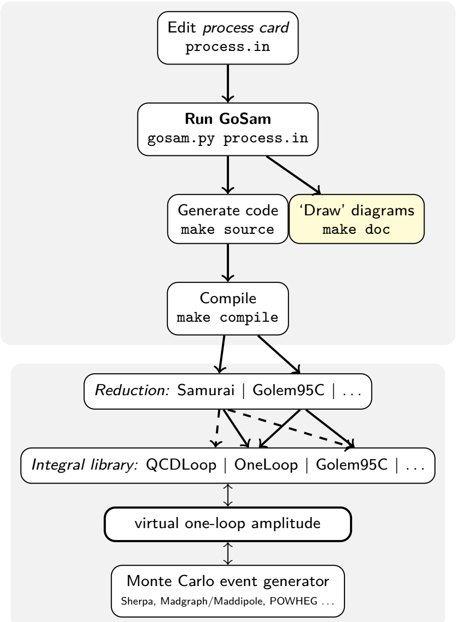
http://projects.hepforge.org/gosam/

[Cullen, Greiner, Heinrich, Luisoni, Mastrolia, Ossola, Reiter, Tramontano (1111.2034)]


Edit *process card*
process.in
Run GoSam
gosam.py process.in
Generate code
make source
'Draw' diagrams
make doc
Compile
make compile
Reduction: Samurai | Golem95C | ...
Integral library: QCDLoop | OneLoop | Golem95C | ...
virtual one-loop amplitude
Monte Carlo event generator
Sherpa, Madgraph/Maddipole, POWHEG ...



- Diagrams generated by QGraf [Nogueira]
- LaTeX documentation can be produced



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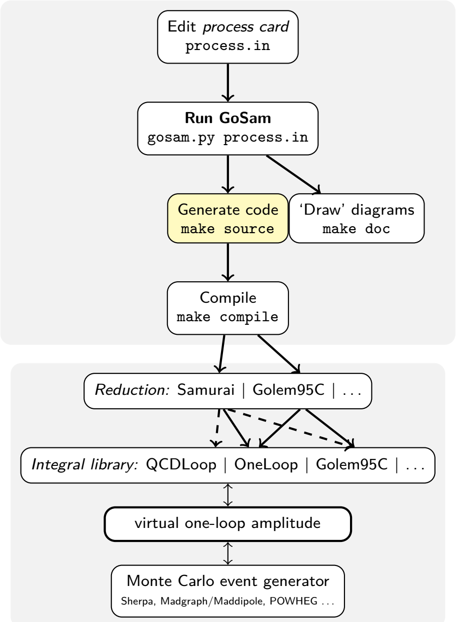
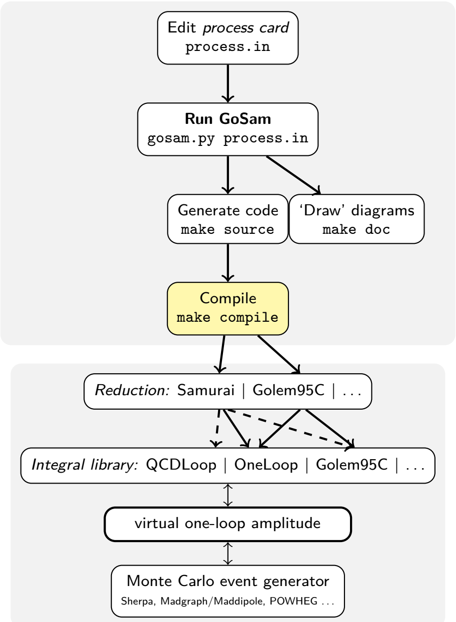


- Algebraic generation of optimized Fortran code on the fly using FORM

  [Vermaseren (math-ph/0010025)]
- Several reduction options
  - Samurai: unitarity based
  - Golem95C: tensorial reduction
- Standard interface to Monte Carlo event generators

  [Binoth et al. (1001.1307)]



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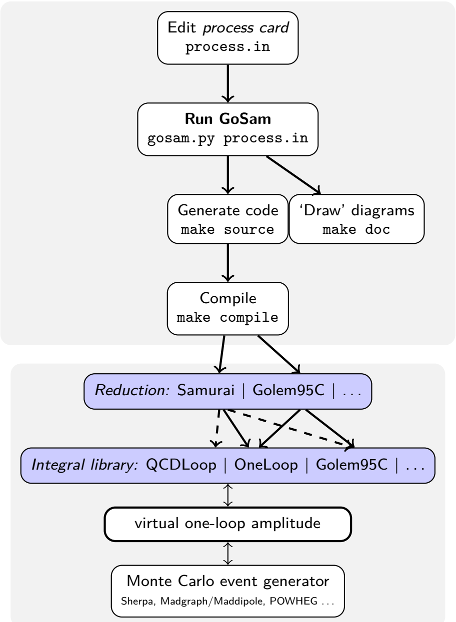

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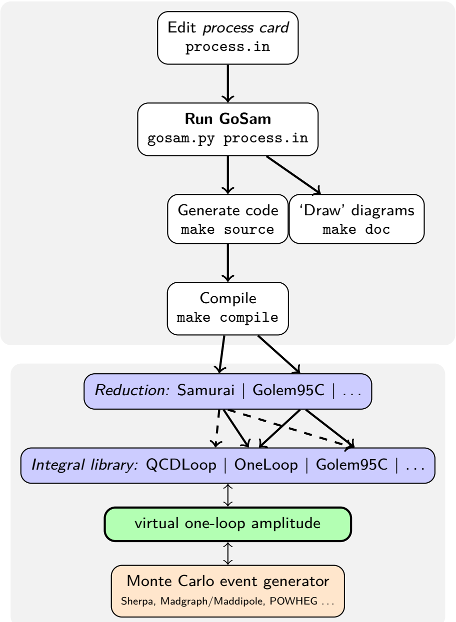

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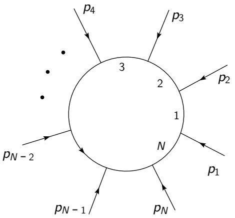
Golem95C integral library

- integral library
  - tensor coefficients up to rank 6, 6-point massive and massless integrals
  - scalar master integrals
  - support complex masses

`http://projects.hepforge.org/~golem/95/`

[Cullen, Guillet, Heinrich, Kleinschmidt, Pilon, Reiter, Rodgers (1101.5595); Binoth, Guillet, Heinrich, Pilon, Reiter (0810.0992)]

Form factor representation



$$I_N^{n,\mu_1\dots\mu_r}(\mathcal{S}) = \int \mathrm{d}^n k \frac{k^{\mu_1}\dots k^{\mu_r}}{\prod_{i=1}^{N}\left((k+r_i)^2 - m_i^2 + i\delta\right)}$$

Form factor representation of tensor integrals:

$$I_N^{n,\mu_1,\dots,\mu_r}(\mathcal{S}) = \sum_m A^{\mu_1\cdots\mu_r}(m, r_1, \dots, r_r) \underbrace{I_N^{n+2m}(j_1, \dots, j_{r-2m}; \mathcal{S})}_{\text{Form factors}}$$

with $r_i = p_1 + \cdots + p_i$
and the kinematic matrix: $\mathcal{S}_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2$

Method

- Reduction to certain set of 'basis' integrals
- special treatment of difficult regions $\Rightarrow$ numerical stability
- only $r \leq N$ implemented (renormalizable theories)
  r: tensor rank, N: # of propagators

[Binoth, Guillet, Heinrich, Pilon, Schubert (hep-ph/0504267)]

Higher tensor rank extension

$$I_N^{n,\mu_1\ldots\mu_r}(S) = \int \mathrm{d}^n k \frac{k^{\mu_1}\cdots k^{\mu_r}}{\prod_{i\in S}\left((k+r_i)^2 - m_i^2 + i\delta\right)}$$

with $\boxed{r \geq N+1}$ (especially $r = N+1$)

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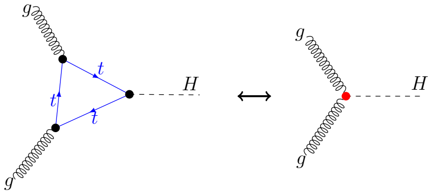
- effective couplings like ggH-vertices (top-loops integrated out)



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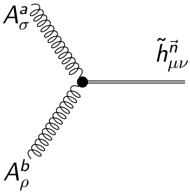
with $\boxed{r \geq N+1}$ (especially $r = N+1$)

- effective couplings like ggH-vertices (top-loops integrated out)
- spin-2 particles (gravitons)



$$\propto \delta^{ab}\left((m_A^2 + k_1\cdot k_2)C_{\mu\nu,\rho\sigma} + \dots\right)$$

[Han, Lykken, Zhang (hep-ph/9811350)]

Implementation in Golem95C

- generic reduction formulas for arbitrary rank implemented
  - Reduction without inverse Gram matrices (Subtraction method)

- Avoid inverse Gram determinants: Subtraction method
  ⇒ Reduction to already implemented form factors of Golem95C

$$
\begin{aligned}
I_N^D(a_1,\ldots,a_r;S) = &-\sum_{k=2}^{r} \mathcal{S}^{-1}_{a_1 a_k} I_N^{D+2}(a_2\ldots\hat{a}_k\ldots a_r;S) \\
&- b(a_1,S)\cdot(N-D-r) I_N^{D+2}(a_2,\ldots,a_r;S) \\
&+ \sum_{k\in S} \mathcal{S}^{-1}_{a_1 k} I_{N-1}^{D}(a_2\ldots a_r;S\setminus\{k\})
\end{aligned}
$$

[Binoth, Guillet, Heinrich, Pilon, Schubert (hep-ph/0504267)]

- Avoid inverse Gram determinants: Subtraction method
  $\Rightarrow$ Reduction to already implemented form factors of Golem95C

$$I_N^D(a_1,\dots,a_r;S) = -\sum_{k=2}^{r} \mathcal{S}^{-1}_{a_1 a_k} I_N^{D+2}(a_2\dots\hat{a}_k\dots a_r;S) - b(a_1,S)\cdot(N-D-r) I_N^{D+2}(a_2,\dots,a_r;S) + \sum_{k\in S} \mathcal{S}^{-1}_{a_1 k} I_{N-1}^D(a_2\dots a_r;S\setminus\{k\})$$

$$I_N^{D+2}(S) = \frac{1}{B(S)}\frac{1}{D-N+1}\left(I_N^D(S) + \sum_{k\in S} b(k;S)\, I_{N-1}^D(S\setminus\{k\})\right)$$

$$b_i(S) = \sum_{k\in S}\mathcal{S}^{-1}_{ki} \qquad B(S) = \sum_{i\in S} b_i(S) = \sum_{i,k\in S}\mathcal{S}^{-1}_{ki}$$

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Implementation in Golem95C

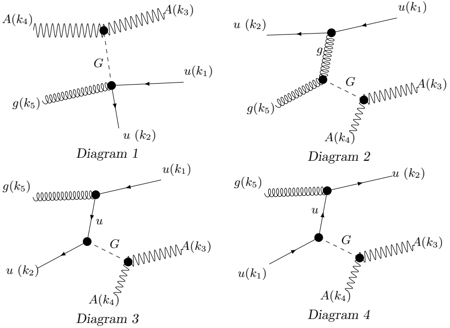
- generic reduction formulas for arbitrary rank implemented
  - Reduction without inverse Gram matrices (Subtraction method)
  - automatic switch to Passarino-Veltman if $\mathcal{S}$ not invertible
- only up to boxes ($N \geq 5$ in progress)
- in addition, some explicit formulas
  - arbitrary rank tadpoles
  - massless eight-dimensional triangles
  - generic (higher-rank) massive bubbles with light-like legs
- simple cache system to prevent re-computations
- applied checks
  - comparisons between both reduction methods
  - rank $N+1$ compared with LoopTools [Hahn, Perez-Victoria (hep-ph/9807565) – 2012]

Status of higher-rank support in GoSam and Samurai

- GoSam (SVN version)
  - support for higher rank tensor integrals
  - spin-2 particles implemented
  - arbitrary models possible
  - Higgs + 0, 1, 2 jets calculated and compared with literature
- Samurai
  - extension to rank $N + 1$ in progress
    based on [Mastrolia, Mirabella, Peraro (1203.0291)]
- both published soon

$\gamma\gamma$ + jet via Graviton decay

- Diphoton + jet production over Graviton bridge
- "automated" NLO calculation with GoSam
- ADD model [Arkani-Hamed, Dimopoulos, Dvali (hep-ph/9803315)]



ADD model

- large extra-dimensions model (LED)
- offers solution to the hierarchy problem
- 'large' means $\gg \frac{1}{M_P}$, up to mm-range
- gravitation effects at TeV scale

  $\boxed{M_P^2 \sim R^\delta M_D^{\delta+2}}$ in $4 + \delta$ dimensions
- SM particles on 4-dim brane
- Kaluza-Klein modes in full dimensions (bulk)
- changes short-range Gravitation law

Graviton propagator

- "Graviton" = sum over KK modes
  $\Rightarrow$ effective propagator needed

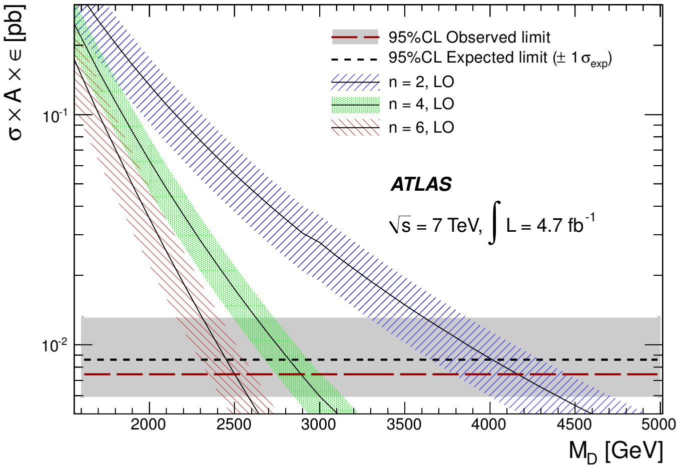
$$\sum_k \frac{i}{s - m_k^2 + i\epsilon} \approx \frac{s^{n/2-1}}{2M_s^{n+2}G_N}\left(\pi + 2i\, I(M_s/\sqrt{s}, n)\right)$$

where $I(x, n)$ is a smooth function.

- in GoSam handled by new `customspin2propagator` extension

[Gleisberg, Krauss, Matchev, Schalicke, Schumann, Soff (hep-ph/0306182); Han, Lykken, Zhang (hep-ph/9811350)]

Current experimental limits



[The ATLAS Collaboration (1210.4491)]

Example diagrams produced by GoSam

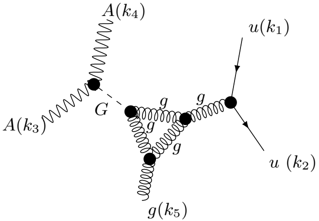
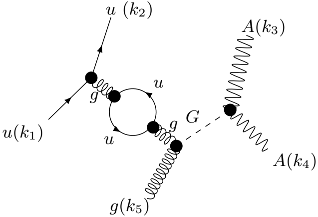
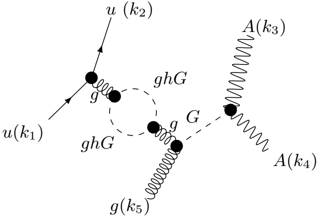


Diagram 22
$S' = S^{\{3\}}_{Q\to -q}$, $\mathrm{rk} = 4$



-Diagram 36
$S' = S^{\{1,3\}}$, $\mathrm{rk} = 2, N_f$



-Diagram 38
$S' = S^{\{1,3\}}$, $\mathrm{rk} = 2$

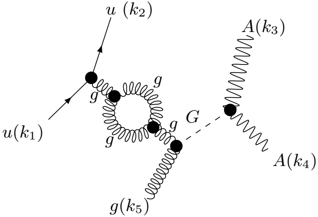


Diagram 39
$S' = S^{\{1,3\}}$, $\mathrm{rk} = 2$

Example diagrams produced by GoSam

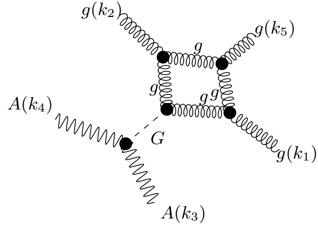
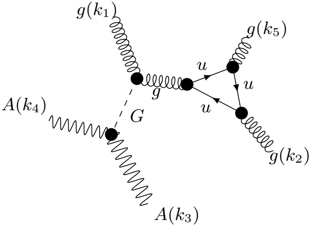
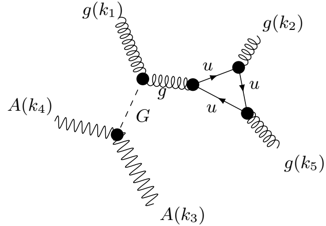


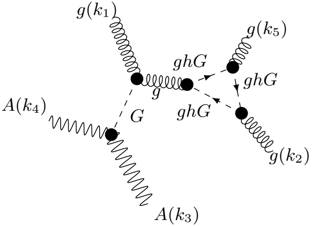
Diagram 108
$S' = S_{Q\to q-(k5)}$, $\mathrm{rk} = 5$



-Diagram 109
$S' = S^{\{4\}}$, $\mathrm{rk} = 3, N_f$

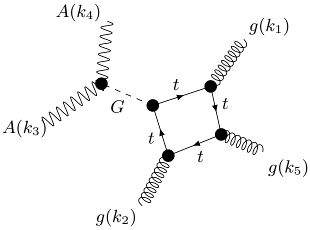


-Diagram 110
$S' = S^{\{4\}}$, $\mathrm{rk} = 3, N_f$

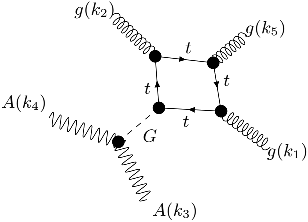


-Diagram 113
$S' = S^{\{4\}}$, $\mathrm{rk} = 3$

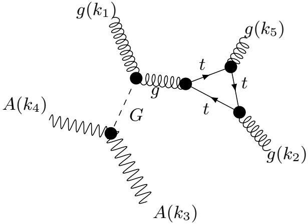
Example diagrams produced by GoSam



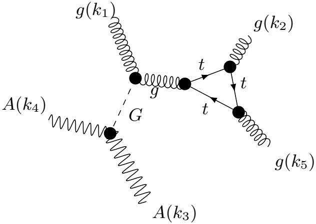
-Diagram 106
$S' = S_{Q\to q-(k5)}$, rk = 5



-Diagram 107
$S' = S_{Q\to q-(k5)}$, rk = 5

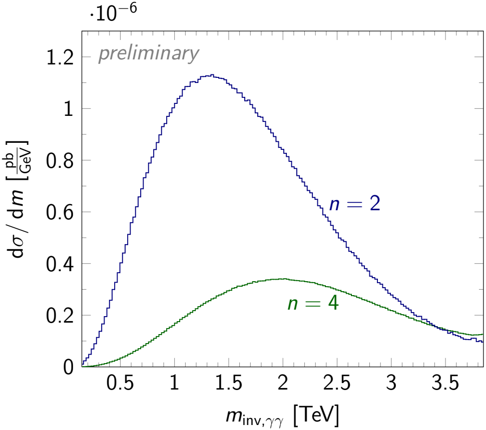


-Diagram 111
$S' = S^{\{4\}}$, rk = 3



-Diagram 112
$S' = S^{\{4\}}$, rk = 3

Preliminary results



invariant di-photon mass (LO, $M_S = 4\,\text{TeV}$)

Timings

GoSam:

- $u\bar{u}$: time to produce code: about 1 CPU day (48 diagrams $\times$ 12 helicities , max. 1.5 h per diagram)
- gg: time to produce code: about 86 CPU days (121 diagrams $\times$ 12 helicities, max. 15 h per diagram)

- Phase space point: $< 1\,\mathrm{s}$ (i7 960 3.20GHz)

Conclusion

- GoSam supports processes with higher rank loop integrals
- Golem95C can provide arbitrary rank tensor integrals
- automated NLO calculations
  - Higgs + Jets
  - BSM physics
    - Gravitons
    - generic spin-2 particles
    - ...
- phenomenological results in progress

Thank you!

References I

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Passarino-Veltman Reduction

- Idea:
  - Contract tensor integrals and form factor representation with external momenta and/or metric tensor
  - solve the equation system
- Problem: Inverse Gram determinants

Gram matrix: $Z_{ij}^{(N-1)} = 2 r_i r_j, \quad i, j = 1 \ldots (N-1)$

[Passarino, Veltman]

Passarino-Veltman Method: Example

$$I^\mu = \int \mathrm{d}\bar{k} \frac{k^\mu}{((k+r_1)^2+i\delta)((k+r_2)^2+i\delta)(k^2+i\delta)} = r_1^\mu A_1^{3,1} + r_2^\mu A_2^{3,1}$$

contract with r_1, r_2, use $r_i \cdot k_i = \frac{1}{2}\left((r_i+k)^2 - k^2 - r_i^2\right)$

$\Rightarrow$

$$Z^{(2)} \begin{pmatrix} A_1^{3,1} \\ A_2^{3,1} \end{pmatrix} = \begin{pmatrix} 2(r_1)^2 & 2r_1 \cdot r_2 \\ 2r_1 \cdot r_2 & 2(r_2)^2 \end{pmatrix} \begin{pmatrix} A_1^{3,1} \\ A_2^{3,1} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

with

$$R_1 = \underbrace{\frac{1}{((k+r_2)^2+i\delta)(k^2+i\delta)}}_{I_2(S\backslash\{r_1\})} - - \underbrace{\frac{1}{((k+r_1)^2+i\delta)((k+r_2^2+i\delta)}}_{I_2(S\backslash\{r_3\})}$$

$$- r_1^2 \cdot \underbrace{\frac{1}{((k+r_1+)^2+i\delta)((k+r_2)^2+i\delta)((k)^2+i\delta)}}_{I_3(S)}$$

$$R_2 = (r_1 \,\text{with}\, r_1 \leftrightarrow r_2)$$

Passarino-Veltman Method: General formula

Generalized formula [Denner, Dittmaier (hep-ph/0509141)]:

$$I_N^D(a_1,\ldots,a_r;S) = \sum_{j\in S\setminus\{N\}} (Z^{(N-1)})^{-1}_{a_1 j}\Big(- I_{N-1}^{D,r-1}(a_2,\ldots,a_r;S\setminus\{j\})\bar{\delta}_{a_2 j}\cdots\bar{\delta}_{a_r j} + I_{N-1}^{D,r-1}(a_2,\ldots,a_r;S\setminus\{N\})\bar{\delta}_{a_2 N}\cdots\bar{\delta}_{a_r N} + f_j I_N^{D,r-1}(a_2,\ldots,a_r;S) + \sum_{k=1}^{r} I_N^{D+2,r-2}(a_2\ldots\hat{a}_k\ldots a_r;S)\Big)$$

$$I_N^{D+2,r}(a_1,\ldots,a_r;S) = \frac{2}{N-D-r+1}\Big(I_{N-1}^{D,r}(a_1,\ldots,a_r;S\setminus\{N\})\bar{\delta}_{a_1 N}\cdots\bar{\delta}_{a_r N} + 2m_N I_N^{D,r}(a_1,\ldots,a_r;S) - \sum_{j\in S\setminus N} f_j I_N^{D,r+1}(a_1,\ldots,a_r,j;S)\Big)$$

with $f_j = r_j - m_j + m_N$ and $\bar{\delta}_{ij} = 1 - \delta_{ij}$

Reduction without inverse gram determinant

- Idea: Split into IR-finite and IR-divergent term

$$I_N^D(S) = I_{\text{div}} + I_{\text{fin}}$$
$$= \sum_{i\in S} b_i(S) \underbrace{\int \mathrm{d}\bar{k} \frac{q_i^2 - m^2}{\prod_{j\in S}(q_i^2 - m^2 + i\delta)}}_{I_{N-1}^D(S\setminus\{i\})} + \int \mathrm{d}\bar{k}\, \frac{1 - \sum_{i\in S} b_i(S)(q_i^2 - m^2)}{\prod_{j\in S}(q_i^2 - m^2 + i\delta)}$$

Kinematic matrix: $\mathcal{S}_{ij} = \left(r_i - r_j\right)^2 - m_i^2 - m_j^2$

[Binoth, Guillet, Heinrich, Pilon, Schubert (hep-ph/0504267)]

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$$\Rightarrow I_{\text{fin}} = -B(S)\,(N - n - 1) I_N^{D+2}(S)$$

[Binoth, Guillet, Heinrich, Pilon, Schubert (hep-ph/0504267)]

General reduction formula

$$I_N^{D+2}(S) = \frac{1}{B(S)} \frac{1}{D-N+1} \left(I_N^D(S) + \sum_{k \in S} b(k;S)\, I_{N-1}^D(S \setminus \{k\}) \right)$$

For tensors:

$$\begin{aligned} I_N^D(a_1,\ldots,a_r;S) = &- \sum_{k=2}^{r} \mathcal{S}^{-1}_{a_1 a_k} I_N^{D+2}(a_2 \ldots \hat{a}_k \ldots a_r;S) \\ &- b(a_1,S) \cdot (N-D-r) I_N^{D+2}(a_2,\ldots,a_r;S) \\ &+ \sum_{k \in S} \mathcal{S}^{-1}_{a_1 k} I_{N-1}^D(a_2 \ldots a_r; S \setminus \{k\}) \end{aligned}$$

Graviton propagator

$$\sum_k \frac{i}{s - m_k^2 + i\epsilon} \approx \frac{s^{n/2-1}}{2M_s^{n+2} G_N} \left(\pi + 2i\, I(M_s/\sqrt{s}, n)\right)$$

$$I(x,n) = \begin{cases} -\sum_{k=1}^{n/2-1} \frac{1}{2k} x^{2k} - \frac{1}{2}\log(x^2-1) & \text{if } n = \text{even} \\ -\sum_{k=1}^{(n-1)/2} \frac{1}{2k-1} x^{2k-1} + \frac{1}{2}\log\left(\frac{x+1}{x-1}\right) & \text{if } n = \text{odd} \end{cases}$$

in $4 + n$ dimensions.

[Han, Lykken, Zhang (hep-ph/9811350); Gleisberg, Krauss, Matchev, Schalicke, Schumann, Soff (hep-ph/0306182)]