# Non-Linear Evolution of Cold Dark Matter and Massive Neutrino Perturbations

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# Content

1) Introduction
2) Standard Perturbation Theory for CDM
3) Boltzmann Equation and
   Perturbation Theory for Neutrinos
4) Results

# Introduction

Goal calculate density contrast:

$$\delta = \frac{\rho}{\bar{\rho}} - 1$$

$\rho \,\hat{=}\, \textit{energy density}$

$\bar{\rho} \,\hat{=}\, \textit{background density}$

e.g.: $$\delta = \frac{\Omega_{CDM}\,\delta_{CDM} + \Omega_\nu\,\delta_\nu}{\Omega_{CDM} + \Omega_\nu}$$

$$\Omega_i = \frac{8\pi G}{3}\frac{\bar{\rho}_{i,0}}{H_0^2}$$

Only correlation function predictable:

Powerspectrum: $\langle \delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\rangle = (2\pi)^3\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2)\,P(k_1)$

Bispectrum: $\langle \delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\delta(\boldsymbol{k}_3)\rangle = (2\pi)^3\,\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,B(k_1, k_2, k_3)$

Initial conditions set by inflation:

Gaussian initial conditions $P_{\delta\Phi}(k) \propto k^{-3}$ and $B_{\delta\Phi} = 0$

Linear evolution $\longrightarrow$ $P_i \neq P_{\delta\Phi}$ and $B_i = 0$

# Non-Linear Evolution of CDM Perturbations

Fluid equations:

$$\frac{\partial \varphi_a}{\partial s} + \Omega_{ab}\varphi_a = a(s)\left[\gamma_{abc}\varphi_b(\boldsymbol{k}_1)\varphi_c(\boldsymbol{k}_2)\right]_{\boldsymbol{k}}$$

CDM perturbations:

$$\varphi = \begin{pmatrix}\delta \\ \theta\end{pmatrix}$$

Peculiar velocity:

$$\theta = i\,\boldsymbol{k}\cdot\boldsymbol{u}$$

$\boldsymbol{u} \hat{=}$ *velocity field*

Super conformal time:

$$ds = \frac{d\tau}{a} = \frac{dt}{a^2}$$

$$\Omega = \begin{pmatrix} 0 & a \\ \frac{3}{2} a\,\mathcal{H}^2 \Omega_{CDM} & a\,\mathcal{H} \end{pmatrix}$$

Gravity

$$\mathcal{H} = \frac{1}{a}\frac{da}{d\tau} = aH$$

$\gamma \hat{=}$ *vertex function*

$$[\,]_{\boldsymbol{k}} = \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}(2\pi)^3\delta_D(\boldsymbol{k}-(\boldsymbol{k}_1+\boldsymbol{k}_2))$$

# Non-Linear Evolution of CDM Perturbations

Linear solution given by Green's function:

$$\frac{\partial g_{ab}}{\partial s}+\Omega_{ac}\, g_{cb}=\delta_{ab}\,\delta(s-s')$$

Integral equation equivalent to fluid equations:

$$\varphi_a=g_{ab}(s,s_i)\varphi_{i,b}+\int_{s_i}^{s} ds'\, g_{ab}(s,s')\, a(s')\left[\gamma_{bcd}\,\varphi_c(s')\,\varphi_d(s')\right]_{\boldsymbol{k}}$$

Iterative solution yields expansion in $\varphi_i$:

$$\varphi_a=g_{ab}(s,s_i)\varphi_{i,b}$$
$$+\int_{s_i}^{s} ds'\, g_{ab}(s,s')\, a(s')\left[\gamma_{bcd}\, g_{ce}(s',s_i)\varphi_{i,e}\, g_{df}(s',s_i)\varphi_{i,f}\right]_{\boldsymbol{k}}+\ldots$$

# Neutrinos

Neutrinos are massive

Earth based experiments:

$$0.056\, eV \lesssim \sum_i m_i \lesssim 6\, eV$$

Neutrino oscillations    Tritium beta decay

Lesgourgues, Pastor 2006

Cosmology:

$$\sum_i m_i = O(1\text{eV})$$

Hannestad 2006

- Neutrinos become non-relativistic at late times
- Larger neutrino mass means more neutrinos

Neutrino fraction: $\dfrac{\Omega_\nu}{\Omega_{CDM}+\Omega_\nu} = O(0.1)$ $\longrightarrow$ May have significant impact

# Neutrinos as a Fluid

Fluid description follows from phase-space distribution:

$$\rho = \int d^3q\, f(q)$$

$$\rho \boldsymbol{u} = (am)^{-1} \int d^3q\, \boldsymbol{q}\, f(q)$$

Non-vanishing stress: $\boldsymbol{\sigma} = (am)^{-2} \int d^3q\, \boldsymbol{q}\,\boldsymbol{q}\, f(q) - \rho\, \boldsymbol{u}\,\boldsymbol{u}$

On large scales: $\boldsymbol{\sigma} \approx c_s^2 \delta \mathbf{1}$ Shoji&Komatsu 2010

Gravitational part:

$$\frac{3}{2} a \mathcal{H}^2 \Omega_{CDM} \delta_{CDM} + \left( \frac{3}{2} a \mathcal{H}^2 \Omega_\nu - a k^2 c_s^2 \right) \delta_\nu \longrightarrow$$

Less effective clustering on small scales

$\boldsymbol{\sigma} = ?$ $\longrightarrow$ **No fluid description valid**

# Neutrino Evolution

Split distribution function into background and perturbation: $f = \bar{f} + \delta f$

Evolution given by Boltzmann equation

$$\frac{\partial \delta f}{\partial s} + i\frac{\boldsymbol{q}}{m}\cdot\boldsymbol{k}\,\delta f - i a^2 m \frac{3}{2}\mathcal{H}^2 \frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial \bar{f}}{\partial \boldsymbol{q}}\Omega\delta = i a^2 m \frac{3}{2}\mathcal{H}^2 \left[\frac{\boldsymbol{k}_1}{k_1^2}\cdot\frac{\partial \delta f}{\partial \boldsymbol{q}}(\boldsymbol{k}_2)\Omega\delta(\boldsymbol{k}_1)\right]_{\boldsymbol{k}}$$

Straight forward expansion as for CDM in principle possible

- Much more involved due momentum dependence
- Only interested in integral over momentum

→ Try to integrate over momentum before solving

# Gilbert's Equation

At linear order distribution function determined by density contrast:

$$\delta f = g_{FS}(s, s_i)\delta f_i + \int_{s_i}^{s} ds'\, g_{FS}(s, s')\, i a^2 m \frac{3}{2}\mathcal{H}^2 \frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial \bar{f}}{\partial \boldsymbol{q}}\Omega\delta$$

Free-streaming solution: $g_{FS}(s, s') = \exp\left(-i\frac{q\cdot k}{m}(s - s')\right)$

Integrating over momentum yields Gilbert's equation:

$$\boxed{\delta = I + \int_{s_i}^{s} ds'\, K(s, s')\delta(s') \qquad \text{Solution:} \quad \delta = \int_{s_i}^{s} ds'\, G(s, s') I(s')}$$

$$I = \int d^3q\, g_{FS}(s, s_i)\delta f_i \qquad K(s, s') = -\frac{3}{2}\mathcal{H}^2 a^2 (s - s')\Omega \int d^3q\, g_{FS}(s, s')\bar{f}$$

e.g. Bertschinger 2007
Boyanovsky et al. 2008

# Gilbert's Equation

Rewrite Boltzmann equation including the non-linear term:

$$\delta f = g_{FS}(s, s_i)\delta f_i + \int_{s_i}^{s} ds'\, g_{FS}(s, s')\, i a^2 m \frac{3}{2}\mathcal{H}^2 \frac{\boldsymbol{k}}{k^2}\cdot\frac{\partial \bar{f}}{\partial \boldsymbol{q}}\Omega\delta$$

$$+\int_{s_i}^{s} ds'\, g_{FS}(s, s')\, i a^2 m \frac{3}{2}\mathcal{H}^2 \left[\frac{\boldsymbol{k}_1}{k_1^2}\cdot\frac{\partial \delta f}{\partial \boldsymbol{q}}(\boldsymbol{k}_2)\Omega\delta(\boldsymbol{k}_1)\right]_{\boldsymbol{k}}$$

Integrating over momentum adds a non-linear term to Gilbert's equation:

$$\delta = I + \int_{s_i}^{s} ds'\, K(s, s')\delta(s')$$

$$+\int_{s_i}^{s} ds' \int d^3q\, g_{FS}(s, s')\, i a^2 m \frac{3}{2}\left[\frac{\boldsymbol{k}_1}{k_1^2}\cdot\frac{\partial \delta f}{\partial \boldsymbol{q}}(\boldsymbol{k}_2)\Omega\delta(\boldsymbol{k}_1)\right]_{\boldsymbol{k}}$$

# Gilbert's Equation

Proceeding yields

$$\delta = I + \int_{s_i}^{s} ds'\, K(s,s')\,\delta(s')$$

Linear term

$$+\int_{s_i}^{s} ds' \left[\tilde{\Gamma}^{(1)} \delta(s', \boldsymbol{k}_2)\right]_{\boldsymbol{k}}$$

$$+\int_{s_i}^{s} ds_1 \int_{s_i}^{s_1} ds_2 \left[\Gamma^{(2)} \delta(s_1, \boldsymbol{k}_1)\,\delta(s_2, \boldsymbol{k}_2)\right]_{\boldsymbol{k}}$$

Second order terms

$$+\int_{s_i}^{s} ds_1 \int_{s_i}^{s_1} ds_2 \left[\tilde{\Gamma}^{(2)} \delta(s_1, \boldsymbol{k}_1)\,\delta(s_2, \boldsymbol{k}_2)\right]_{\boldsymbol{k}}$$

$$+\int_{s_i}^{s} ds_1 \int_{s_i}^{s_1} ds_2 \int_{s_i}^{s_2} ds_3 \left[\Gamma^{(3)} \delta(s_1, \boldsymbol{k}_1)\,\delta(s_2, \boldsymbol{k}_2)\,\delta(s_3, \boldsymbol{k}_3)\right]_{\boldsymbol{k}}$$

Third order terms

$$+\ldots$$

# Bispectrum

Density contrast up to second order

$$\delta(s,\boldsymbol{k};\delta f_i)=\delta^{(1)}(s,\boldsymbol{k};\delta f_i)+\delta^{(2)}(s,\boldsymbol{k};\delta f_i)+\ldots$$

Three point function

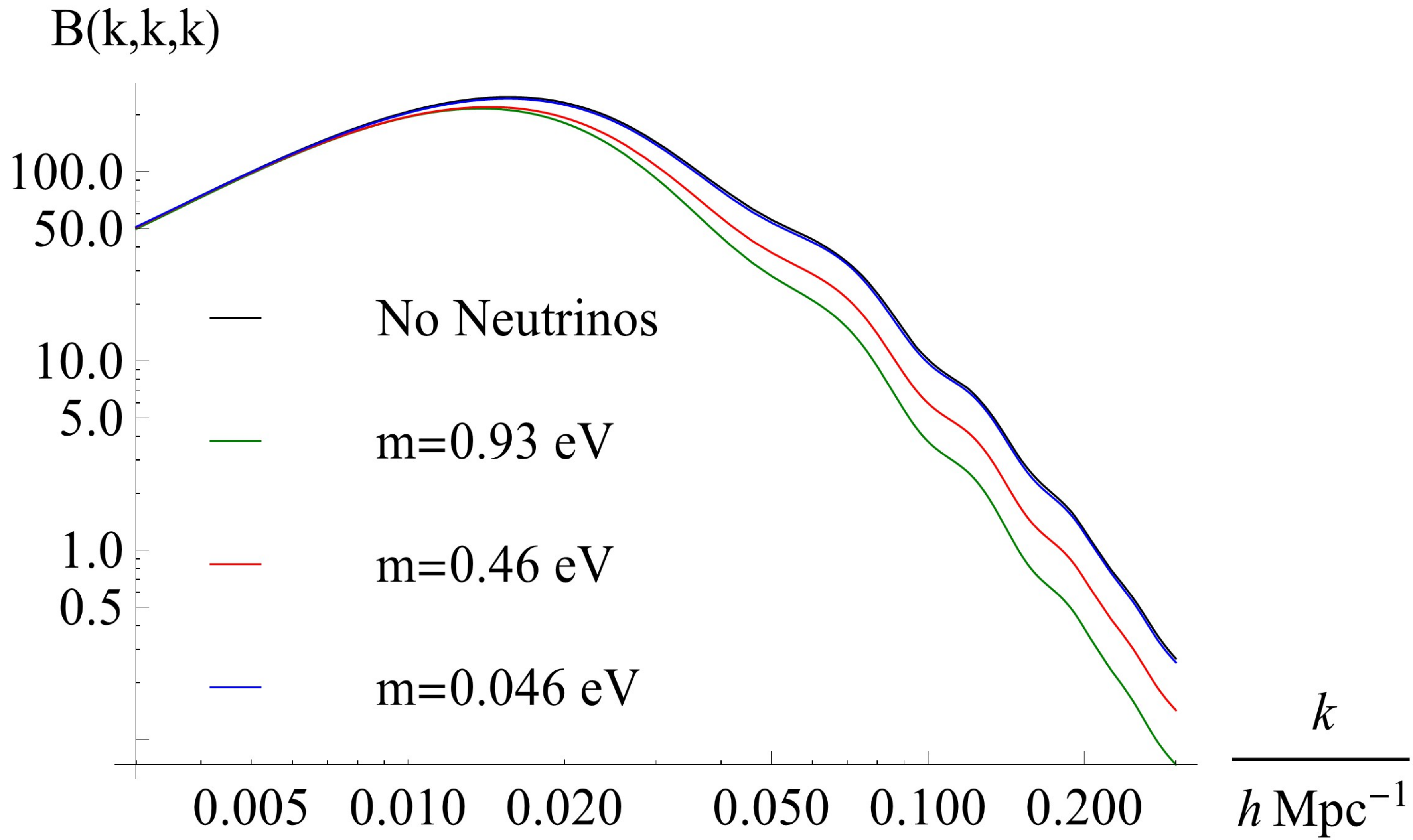
$$\left\langle \delta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2)\delta(\boldsymbol{k}_3)\right\rangle = \underbrace{\left\langle \delta^{(1)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle}_{\propto B_i = 0}$$

$$\underbrace{+\left\langle \delta^{(2)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle}_{=O(P_i^2)} + permutations + \ldots$$

Bispectrum at lowest non-trivial order

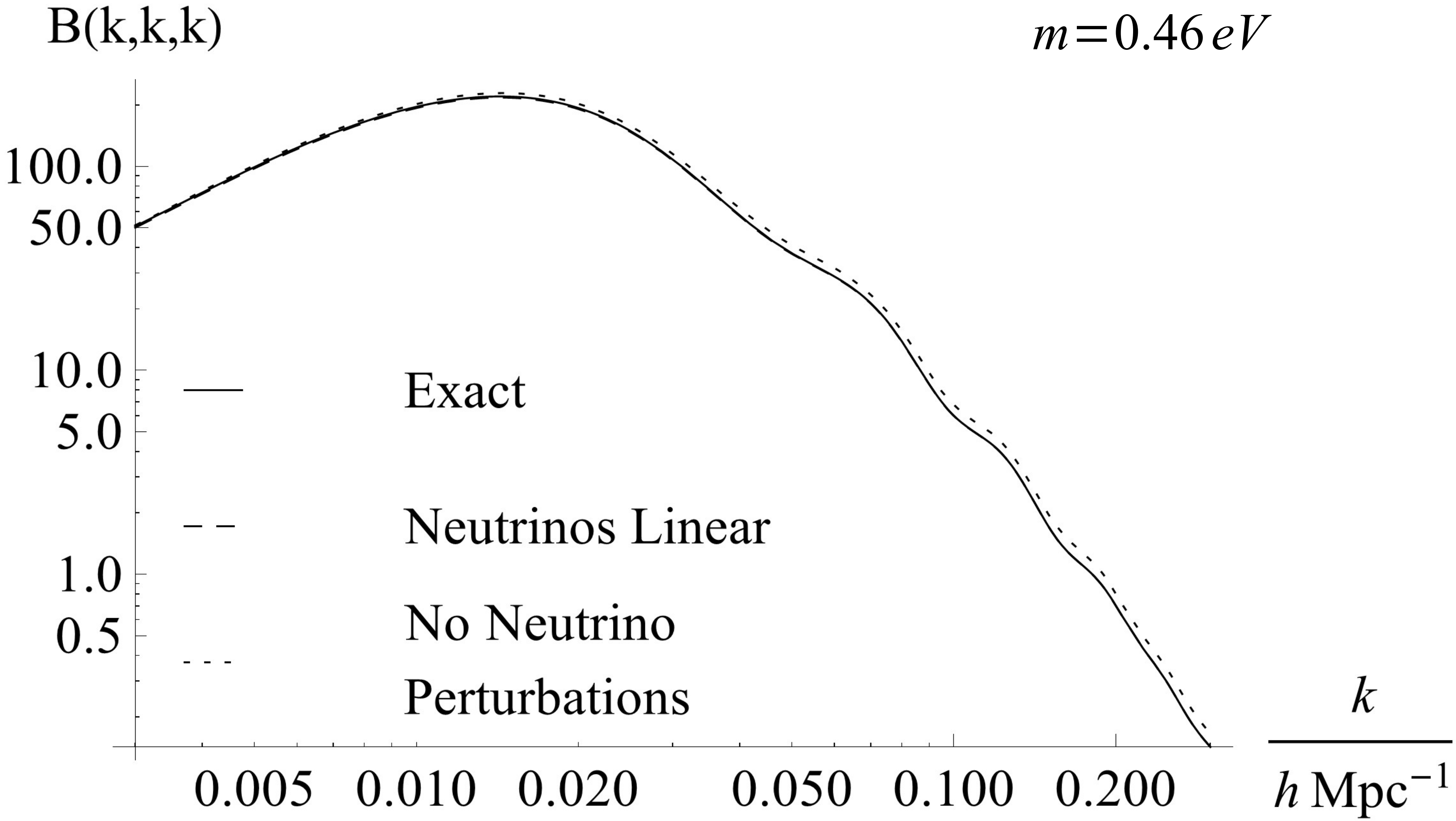
$$B(k_1,k_2,k_3)(2\pi)^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)=\left\langle \delta^{(2)}(\boldsymbol{k}_1)\delta^{(1)}(\boldsymbol{k}_2)\delta^{(1)}(\boldsymbol{k}_3)\right\rangle + permutations$$

# Bispectrum

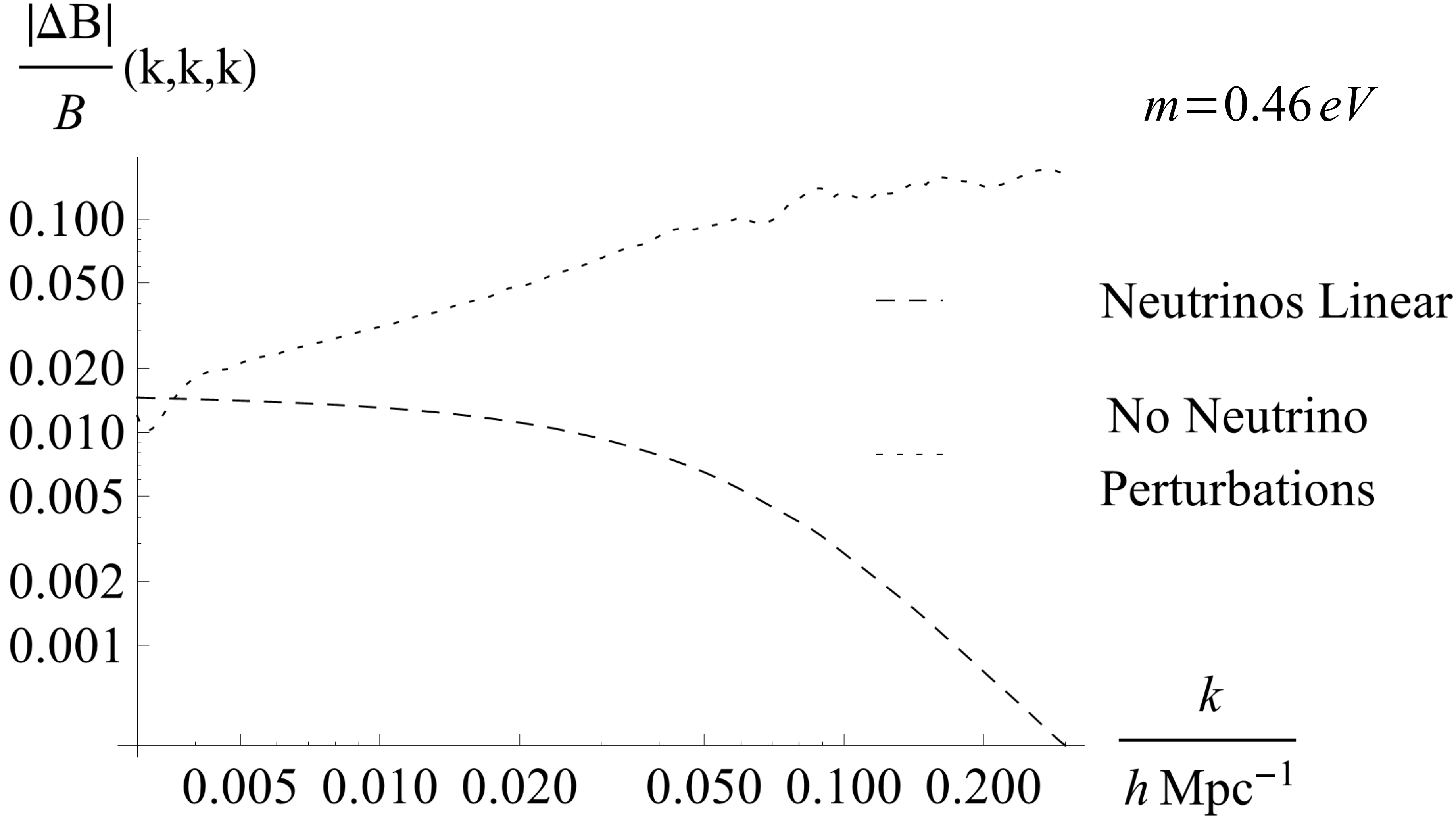


B(k,k,k)
100.0
50.0
10.0
5.0
1.0
0.5
No Neutrinos
m=0.93 eV
m=0.46 eV
m=0.046 eV
0.005
0.010
0.020
0.050
0.100
0.200
$\frac{k}{h\,\mathrm{Mpc}^{-1}}$

# Bispectrum


B(k,k,k)
m = 0.46 eV
100.0
50.0
10.0
5.0
1.0
0.5
Exact
Neutrinos Linear
No Neutrino
Perturbations
0.005
0.010
0.020
0.050
0.100
0.200
k / h Mpc^-1

# Conclusion


$\frac{|\Delta B|}{B}(k,k,k)$
$m = 0.46\, eV$
0.100
0.050
0.020
0.010
0.005
0.002
0.001
Neutrinos Linear
No Neutrino Perturbations
0.005 0.010 0.020 0.050 0.100 0.200
$\frac{k}{h\,\mathrm{Mpc}^{-1}}$

# Summary and Outlook

- Neutrinos included into perturbation theory
- Bispectrum at second order calculated

  $\rightarrow$ Neutrino perturbations cannot be ignored

  $\rightarrow$ Neglecting non-linear neutrino terms gives error up to $O\left(10^{-2}\right)$
- For higher orders $k$-integration necessary
- Use fluid description on large scales?

Questions?