Higher Spin Conformal Correlators and the AdS / CFT Correspondence

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based on work with P.D. Johanna Erdmenger

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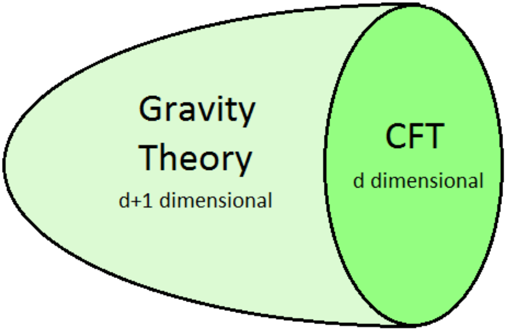
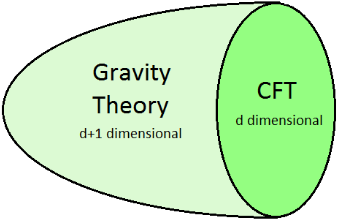
The AdS / CFT Correspondence

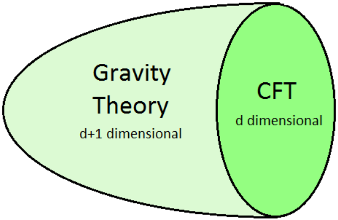


The AdS / CFT Correspondence



Gravity Theory		Boundary CFT
Gauge Field	$\Leftrightarrow$	Conserved Current
$\mathcal{A}_\mu$		$J_\mu, \quad \partial^\mu J_\mu = 0$
$g_{\mu\nu}$		$T_{\mu\nu}, \quad \partial^\mu T_{\mu\nu} = 0$

The AdS / CFT Correspondence



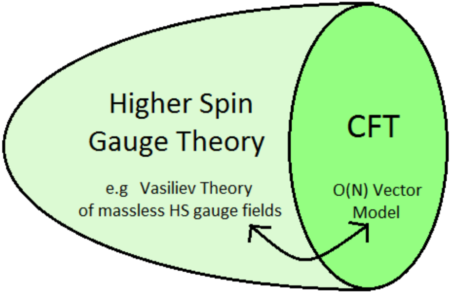
Strong $\Updownarrow$ Weak

e.g. strongly coupled CFT

$\Updownarrow$

classical non-stringy gravity

The Higher Spin Formulation



$$\mathcal{A}_{\mu_1 \ldots \mu_s} \qquad \Leftrightarrow \qquad J_{\mu_1 \ldots \mu_s}, \quad \partial^{\mu_1} J_{\mu_1 \ldots \mu_s} = 0$$

for every spin $s = 1, 2..., \infty$

$\Rightarrow$ Infinite number of constraints from Ward identities

$\Rightarrow$ A lot of control in the CFT

How can we test this correspondence?

One way is to compute three point functions of certain operators:

1. Calculate the same three point function in both theories
2. Check if the results agree

Today:

- We consider the CFT calculation of 3pt functions of the higher spin currents $J_{\mu_1 \ldots \mu_s}$
- The conformal symmetry in the CFT is very powerful, and we can use it to determine the structure of the 3pt functions

A Method of Constructing Conformal Three Point Functions

The strength of conformal symmetry fixes the structure:

$$\langle \mathcal{O}_1^{i_1}(x)\, \mathcal{O}_2^{i_2}(y)\, \mathcal{O}_3^{i_3}(z) \rangle = {F_1}^{i_1}{}_{j_1}(|x-z|)\, {F_2}^{i_2}{}_{j_2}(|y-z|)\, t^{j_1 j_2 i_3}(Z)$$

Osborn and Petkou 1993

- For general dimension d
- $Z_i = \frac{x_i - z_i}{(x-z)^2} - \frac{y_i - z_i}{(y-z)^2}$, conformal vector
- $F_1,\ F_2$ and $t(Z)$ conformally invariant up to scaling

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Osborn and Petkou 1993

- F_1 and F_2 take a form roughly universal to all 3pt functions
- $t(Z)$ contains the interesting structure
- Vital role played by conformal covariant vector Z

A Method of Constructing Conformal Three Point Functions

$$\langle \mathcal{O}_1^{i_1}(x)\,\mathcal{O}_2^{i_2}(y)\,\mathcal{O}_3^{i_3}(z)\rangle = F_1{}^{i_1}{}_{j_1}(x-z)\,F_2{}^{i_2}{}_{j_2}(y-z)\,t^{j_1 j_2 i_3}(Z)$$

$t(Z)$ can be written as a homogeneous polynomial in Z_i, schematically:

$$t(Z) = A_1 P_1(Z) + \ldots + A_k P_k(Z)$$

We're expanding t in a basis of independent conformal invariants $\{P_\alpha(Z)\}$

i.e. t has k independent forms

Imposing Current Conservation

$$t\left(Z\right)=A_1P_1\left(Z\right)+...+A_kP_k\left(Z\right)$$

In this formalism current conservation is simple to impose:

$$\mathcal{O}_1^{i_1}\left(x\right)=J_{\mu_1...\mu_s}\left(x\right), \qquad \frac{\partial}{\partial x_{\mu_1}}J_{\mu_1...\mu_s}=0$$

$$\frac{\partial}{\partial x_{\mu_1}}\langle J_{\mu_1...\mu_s}(x)\mathcal{O}_2^{i_2}\left(y\right)\mathcal{O}_3^{i_3}\left(z\right)\rangle=0 \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial Z_{\mu_1}}t_{\mu_1...\mu_s}{}^{j_2 i_3}=0}$$

$\Rightarrow$ Condition that $\mathcal{O}_1$ is conserved = set of linear equations for the A_α

#(Independent structures in t) = k - #(Independent equations for A_α)

Imposing Current Conservation

$$t(Z) = A_1 P_1(Z) + ... + A_k P_k(Z)$$

However...

- For general spins it's not so easy to find all the independent $P_\alpha(Z)$

There's a way around it:

- For currents of the same spin there is a simpler, systematic, construction
- To impose current conservation we transform back to the original method

An Adaptation for Currents of the Same Spin

We consider

$$\langle J_{i_1 \dots i_r}(x) \, J_{j_1 \dots j_r}(y) \, J_{k_1 \dots k_r}(z) \rangle$$

- $\partial_{i_1} J_{i_1 \dots i_r} = 0$ conserved current
- In a CFT higher spin currents are symmetric and traceless
- Assume $J_{i_1 \dots i_r}$ are bosonic
- Assume parity is preserved
- $J_{i_1 \dots i_r}(\lambda x) = \lambda^{r+d-2} J_{i_1 \dots i_r}(x)$

An Adaptation for Currents of the Same Spin

The trick is to make the Bose symmetry manifest:

$$\langle J_{i_1...i_r}(x)\, J_{j_1...j_r}(y)\, J_{k_1...k_r}(z)\rangle = \frac{S_{i_1...i_r j_1...j_r k_1...k_r}(x,y,z)}{(x-y)^{r+d-2}(y-z)^{r+d-2}(x-z)^{r+d-2}}$$

- S is symmetric under the interchanges:

$$(i_1...i_r, x) \quad \leftrightarrow \quad (j_1...j_r, y) \quad \leftrightarrow \quad (k_1...k_r, z)$$

- Choice of denominator $\Rightarrow$ S is conformally invariant

An Adaptation for Currents of the Same Spin

Analogous to t we expand S in a basis of conformal invariants $\{Q_\alpha\}$ built from x, y and z

$$S = B_1 Q_1 (x, y, z) + ... + B_k Q_k (x, y, z)$$

- The Q_α are also invariant under

$$(i_1 ... i_r, x) \quad \leftrightarrow \quad (j_1 ... j_r, y) \quad \leftrightarrow \quad (k_1 ... k_r, z)$$

- The Q_α can be expressed in terms of fundamental conformal invariants
- Fundamental invariants cannot be written as combinations of other conformal invariants

An Adaptation for Currents of the Same Spin

Parity even fundamental conformal invariants in $d > 3$:

$$I_{ij}(x-y) = \left(\delta_{ij} - \frac{2(x-y)_i (x-y)_j}{(x-y)^2}\right), \quad I(x-z), \quad I(y-z)$$

$$\frac{X_{i_1}}{|X|}, \quad \frac{Y_{j_1}}{|Y|}, \quad \frac{Z_{k_1}}{|Z|}$$

With $X_i = \frac{y_i - x_i}{(y-x)^2} - \frac{z_i - x_i}{(z-x)^2}$, Y and Z defined similarly

Have to be careful in $d = 3$, when there is a relation between them

An Adaptation for Currents of the Same Spin

Parity even fundamental conformal invariants in $d > 3$:

$$I_{ij}\left(x-y\right)=\left(\delta_{ij}-\frac{2\left(x-y\right)_i\left(x-y\right)_j}{\left(x-y\right)^2}\right),\quad I\left(x-z\right),\quad I\left(y-z\right)$$

$$\frac{X_{i_1}}{|X|},\quad \frac{Y_{j_1}}{|Y|},\quad \frac{Z_{k_1}}{|Z|}$$

With $X_i = \frac{y_i - x_i}{(y-x)^2} - \frac{z_i - x_i}{(z-x)^2}$, Y and Z defined similarly

$$\Rightarrow S = B_1 Q_1\left(I, X, Y, Z\right) + ... + B_k Q_k\left(I, X, Y, Z\right)$$

This time we can use the Bose symmetry to find Q_α

Simplest Example: Spin 1

$$\langle J^{a}_{i_1}(x)\, J^{b}_{j_1}(y)\, J^{c}_{k_1}(z)\rangle = \frac{S^{abc}_{i_1 j_1 k_1}(x,y,z)}{(x-y)^{d-1}(y-z)^{d-1}(x-z)^{d-1}}$$

What independent conformal invariants can be made with three indices?

$$(1) \qquad I_{i_1 j_1}(x-y)\frac{Z_{k_1}}{|Z|}, \quad I_{j_1 k_1}(y-z)\frac{X_{i_1}}{|X|}, \quad I_{i_1 k_1}(x-z)\frac{Y_{j_1}}{|Y|}$$

$$(2) \qquad \frac{X_{i_1} Y_{j_1} Z_{k_1}}{|X|\,|Y|\,|Z|}$$

- $Z \to -Z$, $X \to -Y$ and $Y \to -X$ under $x \leftrightarrow y$
- $I(x-y)$ invariant

Simplest Example: Spin 1

$$\langle J^{a}_{i_1}(x)\, J^{b}_{j_1}(y)\, J^{c}_{k_1}(z)\rangle = \frac{S^{abc}_{i_1 j_1 k_1}(x,y,z)}{(x-y)^{d-1}(y-z)^{d-1}(x-z)^{d-1}}$$

What independent conformal invariants can be made with three indices?

(1) $$f^{abc} I_{i_1 j_1}(x-y)\frac{Z_{k_1}}{|Z|}, \quad f^{abc} I_{j_1 k_1}(y-z)\frac{X_{i_1}}{|X|}, \quad f^{abc} I_{i_1 k_1}(x-z)\frac{Y_{j_1}}{|Y|}$$

(2) $$f^{abc}\frac{X_{i_1} Y_{j_1} Z_{k_1}}{|X|\,|Y|\,|Z|}$$

- f^{abc} anti symmetric group structure constant

Simplest Example: Spin 1

$$\langle J^{a}_{i_1}(x)\, J^{b}_{j_1}(y)\, J^{c}_{k_1}(z)\rangle = \frac{S^{abc}_{i_1 j_1 k_1}(x,y,z)}{(x-y)^{d-1}(y-z)^{d-1}(x-z)^{d-1}}$$

$$S^{abc}_{i_1 j_1 k_1} = f^{abc} B_1 \left(I_{i_1 j_1}(x-y)\frac{Z_{k_1}}{|Z|} + I_{j_1 k_1}(y-z)\frac{X_{i_1}}{|X|} + I_{i_1 k_1}(x-z)\frac{Y_{j_1}}{|Y|} \right)$$

$$+ f^{abc} B_2 \frac{X_{i_1} Y_{j_1} Z_{k_1}}{|X|\,|Y|\,|Z|}$$

- For current conservation we transform S into $t(Z)$
- There we obtain linear equations for B_1 and B_2

Procedure for General Spin r

1. Find all independent products of the fundamental conformal invariants with the required index structure

2. To obtain the basis elements $\{Q_\alpha\}$, we find all Bose symmetric combinations of the above

3. Transform back to the original framework to impose current conservation - a set of linear equations on the B_α

4. This is well suited to computer generation

A bit more complicated: Spin 3

$$\langle J^a_{i_1 i_2 i_3}(x)\, J^b_{j_1 j_2 j_3}(y)\, J^c_{k_1 k_2 k_3}(z)\rangle = \frac{S^{abc}_{i_1 i_2 i_3 j_1 j_2 j_3 k_1 k_2 k_3}(x,y,z)}{(x-y)^{d+1}(y-z)^{d+1}(x-z)^{d+1}}$$

$$\begin{aligned} S^{abc}_{i_1 i_2 i_3 j_1 j_2 j_3 k_1 k_2 k_3} = P\, f^{abc} \Bigg(& B_1 \left(I^{xy}_{i_1 j_1} I^{xz}_{i_2 k_3} I^{yz}_{j_2 k_1} I^{yz}_{j_3 k_2} \frac{X_{i_3}}{|X|} + I^{xy}_{i_1 j_1} I^{yz}_{j_2 k_1} I^{xz}_{i_2 k_2} I^{xy}_{i_3 k_3} \frac{Y_{j_3}}{|Y|} + I^{yz}_{j_3 k_1} I^{xz}_{i_3 k_2} I^{xy}_{i_1 j_1} I^{xy}_{i_2 j_2} \frac{Z_{k_3}}{|Z|} \right) \\ & + B_2 \left(I^{yz}_{j_1 k_1} I^{yz}_{j_2 k_2} I^{yz}_{j_3 k_3} \frac{X_{i_1} X_{i_2} X_{i_3}}{X^3} + I^{xz}_{i_1 k_1} I^{xz}_{i_2 k_2} I^{xz}_{i_3 k_3} \frac{Y_{j_1} Y_{j_2} Y_{j_3}}{Y^3} + I^{xy}_{i_1 j_1} I^{xy}_{i_2 j_2} I^{xy}_{i_3 j_3} \frac{Z_{k_1} Z_{k_2} Z_{k_3}}{Z^3} \right) \\ & + B_3 \Bigg(I^{xz}_{i_1 k_3} I^{yz}_{j_1 k_1} I^{yz}_{j_2 k_2} \frac{X_{i_2} X_{i_3} Y_{j_3}}{X^2 Y} + I^{xy}_{i_1 j_1} I^{yz}_{j_2 k_1} I^{yz}_{j_3 k_2} \frac{X_{i_2} X_{i_3} Z_{k_3}}{X^2 Z} + I^{xy}_{i_1 j_1} I^{xz}_{i_2 k_1} I^{xz}_{i_3 k_2} \frac{Y_{j_2} Y_{j_3} Z_{k_3}}{Y^2 Z} \\ & + I^{xy}_{i_1 j_1} I^{xy}_{i_2 j_2} I^{yz}_{j_3 k_1} \frac{Z_{j_2} Z_{j_3} X_{i_3}}{Z^2 X} + I^{xy}_{i_1 j_1} I^{xy}_{i_2 j_2} I^{xz}_{i_3 k_1} \frac{Z_{k_2} Z_{k_3} Y_{j_3}}{Z^2 Y} \Bigg) \\ & \vdots \\ & + B_8 \frac{Z_{k_1} Z_{k_2} Z_{k_3} X_{i_1} X_{i_2} X_{i_3} Y_{j_1} Y_{j_2} Y_{j_3}}{Z^3 Y^3 X^3} \Bigg) \end{aligned}$$

P projects onto fields symmetric and traceless in $i_1 \ldots i_r$, $j_1 \ldots j_r$ and $k_1 \ldots k_r$

Number of Independent Structures

$$\#(B_\alpha) \quad - \quad \#(\text{Indep. equations for } B_\alpha)$$

Spin 3: $= 8 \ - \ 4 \ = \ 4$

Existing formula

$$\#(\text{Independent Structures}) = \text{r} + 1$$

Costa, Penedones, Poland, Rychkov 2011

- Our method agrees for currents of spin 1, 2 and 3
- Yet to implement this for spins greater than 3 (Mathematica heavey)

Summary

- We can test a simplified form of the AdS / CFT by computing higher spin current three point functions
- Shown how the strength of conformal symmetry in a CFT determines the structure of its 3pt functions
- For higher spin currents of equal spin we introduced a systematic and computational construction

Outlook

- Compute the corresponding gravity theory three point functions in general dimensions
- This method already agrees with existing gravity calculations in $d = 3$ (Giombi & Yin 2010)
- This work lays down a strong, computational framework for future investigations based on higher spin current three point functions

Thank you for listening