Poincare Invariance in EFT

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Wilson coefficients in EFTs

Top-down approach: Expand the QCD Lagrangian in 1/M

$$\mathcal{L}_{NRQCD} = \phi^\dagger \mathcal{O} \phi + \chi^\dagger \mathcal{O} \chi + \cdots \tag{1}$$

- The operator $\mathcal{O}$ comes with Wilson coefficients.
- $\cdots$ includes terms other than bilinear ones.
- How do we fix or find the relations between the coefficients?

Poincare Invariance

We can find the relations by imposing the Poincare invariance of the corresponding EFTs

- Construct the Poincare generators of the EFTs.
- Impose the algebra conditions up to the interested order in $1/M$.
- Constraints yield relations between the coefficients.
- Calculations done up to $\mathcal{O}(1/M)$ in NRQCD. [Brambilla, 2003]

Outline

We can go higher orders with a simpler method, using little group formalism, rather than directly expanding the Poincare generators in $1/M$.

- Quantum Field Theory in Poincare group
- QFT in little group
- Application to NRQCD
- Outlook: Application to pNRQCD

Quantum Mechanics

Starting with a momentum eigenstate in Hilbert space

$$P^{\mu}\Psi_{p,\sigma} = p^{\mu}\Psi_{p,\sigma}, \tag{2}$$

how does it transform under Lorentz group? (i.e., $U(\Lambda)\Psi_{p,\sigma} =??$) From the fact

$$P^{\mu}[U(\Lambda)\Psi_{p,\sigma}] = \Lambda^{\mu}_{\rho}p^{\rho}U(\Lambda)\Psi_{p,\sigma} \tag{3}$$

it is natural to write

$$U(\Lambda)\Psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma',\sigma}(\Lambda, p)\Psi_{\Lambda p,\sigma'} \tag{4}$$

Little group formalism in Quantum Mechanics

One can rewrite the Lorentz transformation of the generic quantum states in terms of the fixed reference frame, k, define a standard Lorentz transformation $L(p)$ such as $L(p)k = p$, and define the generic momentum eigenstate in terms of the fixed refernce frame

$$\Psi_{p,\sigma} \equiv U(L(p))\Psi_{k,\sigma} \tag{5}$$

How is the Lorentz tranfromation of the generic eigenstate written in terms of the reference frame? Let us do some simple computations.

Little group element (1/4)

$$\begin{aligned} U(\Lambda)\Psi_{p,\sigma} &= U(\Lambda)U(L(p))\Psi_{k,\sigma} \\ &= U(L(\Lambda p))U(L^{-1}(\Lambda p)\Lambda L(p))\Psi_{k,\sigma} \\ &\equiv U(L(\Lambda p))U(W(\Lambda, p))\Psi_{k,\sigma} \end{aligned} \tag{6}$$

where $W(\Lambda, p)k = k$, and it is called *little group* element. From the general expression of the Lorentz transformation of the states, this "fixed state" transforms under the little group as

$$U(W)\Psi_{k,\sigma} = \sum_{\sigma'} D_{\sigma'\sigma}(W)\Psi_{k,\sigma'} \tag{7}$$

where $D(W)$ is a representation of the little group and hence we obtain the Lorentz transfromation of a generic states in terms of the little group

$$U(\Lambda)\Psi_{p,\sigma} = \sum_{\sigma'} D_{\sigma',\sigma}(W)\Psi_{\Lambda p,\sigma'} \tag{8}$$

Little group element (2/4)

How does the little group element look like? It is necessary to answer this question in order to obtain the generators of the Lorentz group in the EFTs. Remeber $W(\Lambda, p) = L(\Lambda p)^{-1}\Lambda L(p)$, and we have to figure out how $L(p)$ look like. From the fact that

$$L(p)k = p \quad \text{for} \quad k^2 = p^2 = M^2 \tag{9}$$

we realize $L(p)$ is a generalized rotation in the plane of $k/M \equiv w$ and $p/M \equiv v$

$$\begin{aligned} L(w,v)^{\mu}_{\nu} &= g^{\mu}_{\nu} - \frac{1}{1+v\cdot w}(w^{\mu}w_{\nu} + v^{\mu}v_{\nu}) + w^{\mu}v_{\nu} - v^{\mu}w_{\nu} \\ &\quad + \frac{v\cdot w}{1+v\cdot w}(w^{\mu}v_{\nu} + v^{\mu}w_{\nu}) \\ L(w,v)_{1/2} &= \frac{1+\not{w}\not{v}}{\sqrt{2(1+v\cdot w)}} \end{aligned} \tag{10}$$

Little group element (3/4)

Let us focus on the case when Lorentz transformation is an infinitesimal boost, $\Lambda = \mathcal{B}(\eta)$ such that

$$\mathcal{B}(\eta) v = v + \eta \tag{11}$$

for $1 = v^2 = (v+\eta)^2$; $v \cdot \eta = \mathcal{O}(\eta^2)$. Thus, the boost transformation is expressed as (vector and spinor respectively)

$$\mathcal{B}(\eta)^{\mu}_{\nu} = g^{\mu}_{\nu} - (v^{\mu}\eta_{\nu} - \eta^{\mu} v_{\nu}) + \mathcal{O}(\eta^2) \tag{12}$$

$$\mathcal{B}_{1/2}(\eta) = 1 + \frac{1}{2}\not{\eta}\not{v} + \mathcal{O}(\eta^2) \tag{13}$$

Little group element (4/4)

Therefore, the little group element for the infinitesimal boost is given by

$$\begin{aligned} W(\mathcal{B}(\eta), p) &= L(\mathcal{B}(\eta)p)^{-1}\mathcal{B}(\eta)L(p) \\ &= 1 + \frac{i}{2}\left[\frac{1}{M + v \cdot p}(\eta^\alpha p_\perp^\beta - p_\perp^\alpha \eta^\beta)\mathcal{J}_{\alpha\beta}\right] + \mathcal{O}(\eta^2) \end{aligned} \quad (14)$$

where $p_\perp^\beta \equiv p^\beta - (v \cdot p)v^\beta$ and

$$\begin{aligned} \mathcal{J}_{1/2}^{\alpha\beta} &= \frac{i}{4}[\gamma^\alpha, \gamma^\beta] \\ (\mathcal{J}^{\alpha\beta})_{\mu\nu} &= i(g_\mu^\alpha g_\nu^\beta - g_\mu^\beta g_\nu^\alpha) \end{aligned} \quad (15)$$

Quantum Field Theory

What about a generic (non-interacting) quantum field transforming under the Lorentz group?

$$\phi_a \to M(\Lambda)_{ab}\phi_b(\Lambda^{-1}x) \tag{16}$$

$M(\Lambda)$ being a representation of the Lorentz group. In the infinitesimal form

$$\delta\phi = i(a_0 h - \mathbf{a}\cdot\mathbf{p} - \boldsymbol{\theta}\cdot\mathbf{j} + \boldsymbol{\eta}\cdot\mathbf{k})\phi \tag{17}$$

and our interest lies upon the boost generator

$$\mathbf{k} = \mathbf{r}h - t\mathbf{p} \pm i\mathbf{\Sigma} \tag{18}$$

so that a generic quantum field transfroms under the spatial boost as

$$\phi_a(x) \to (e^{\mp\boldsymbol{\eta}\cdot\mathbf{\Sigma}})_{ab}\phi_b(\mathcal{B}^{-1}x) \tag{19}$$

Induced representation in EFTs [Heinonen, 2012]

What about the free massive field? We postulate the transformation of the field as

$$\phi_a(x) \to D[W(\Lambda, i\partial)]_{ab}\phi_b(\Lambda^{-1}x) \tag{20}$$

where W is a little group element as was before and in particular for the Lorentz boost,

$$\phi_a(x) \to \exp\left[\mp\boldsymbol{\eta}\cdot\left(\frac{\boldsymbol{\Sigma}\times\boldsymbol{\partial}}{M+\sqrt{M^2-\boldsymbol{\partial}^2}}\right)\right]_{ab}\phi_b(\mathcal{B}^{-1}x) \tag{21}$$

when the reference frame is chosen as $v = (1,0,0,0)$. Although we have lost the original symmetry SO(3,1), it is advantageous to work in the theory with a specific reference frame.

Non-relativistic expansion (1/3)

Up until now, we have figured out the boost transformation of the massive and relativistic field by adopting the little group formalism. Let us combine this with the non-relativistic expansion so that we can apply it later to HQET/NRQCD and pNRQCD. For the 1/M expansion of the Lagrangian, it is convenient to extract the rest mass by

$$\phi_a(x) = e^{-iMt}\phi'_a(x) \tag{22}$$

and take the non-relativisitic field normalization

$$\phi_a(x) = e^{-iMt}\left(\frac{M^2}{M^2-\partial^2}\right)^{1/4}\phi''_a(x) \tag{23}$$

How does this non-relativistic field $\phi''_a(x)$ transforms under the Lorentz boost?

Non-relativistic expansion (2/3)

From the "inverse" non-relativistic normalization

$$\phi''_a(x) = \Big(\frac{M^2}{M^2 - \boldsymbol{\partial}^2}\Big)^{-1/4} e^{iMt}\phi_a(x) \tag{24}$$

we can extract the Lorentz boost of the field

$$\begin{aligned}
\phi''_a(x) \quad &\rightarrow \quad \Big(\frac{M^2}{M^2 - \boldsymbol{\partial}^2}\Big)^{-1/4} e^{iMt} \\
&\quad \times \exp\Big[\mp\boldsymbol{\eta}\cdot\Big(\frac{\boldsymbol{\Sigma}\times\boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}\Big)\Big]_{ab}\phi_b(\mathcal{B}^{-1}x) \\
&= \quad \Big(\frac{M^2}{M^2 - \boldsymbol{\partial}^2}\Big)^{-1/4} e^{iMt}\exp\Big[\mp\boldsymbol{\eta}\cdot\Big(\frac{\boldsymbol{\Sigma}\times\boldsymbol{\partial}}{M + \sqrt{M^2 - \boldsymbol{\partial}^2}}\Big)\Big]_{ab} \\
&\quad \times e^{-iMt'}\Big(\frac{M^2}{M^2 - \boldsymbol{\partial}'^2}\Big)^{1/4}\phi''_b(x')
\end{aligned} \tag{25}$$

where $x' \equiv \mathcal{B}^{-1}x$.

Non-relativistic expansion (3/3)

Therefore, the Lorentz transformation (boost) of the non-relativistic field in 1/M expansion is given by

$$\phi''(x) \rightarrow \Big\{1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - \frac{i\boldsymbol{\eta}\cdot\boldsymbol{\partial}}{2M} - \frac{i\boldsymbol{\eta}\cdot\boldsymbol{\partial}\boldsymbol{\partial}^2}{4M^3} + \frac{(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\boldsymbol{\partial}}{2M}\Big[1 + \frac{\boldsymbol{\partial}^2}{4M^2}\Big] + \mathcal{O}(1/M^4)\Big\}\phi''(\mathcal{B}^{-1}x) \quad (26)$$

and this is the transformation of the non-interacting and non-relativistic field. How can we implement this formalism into the interacting theory?

NR expansion in the interacting theory

It is natural to postulate the transforamtion of the field just by promoting ∂ to D

$$\phi_a(x) \quad \rightarrow \quad D[W(\Lambda, iD)]_{ab}\phi_b(\Lambda^{-1}x) \tag{27}$$

and the Lorentz boost is thereby given

$$\phi_a(x) \quad \rightarrow \quad \exp\left[\mp\boldsymbol{\eta}\cdot\left(\frac{\boldsymbol{\Sigma}\times\mathbf{D}}{M+\sqrt{M^2-\mathbf{D}^2}}\right)+\mathcal{O}(g)\right]_{ab}\phi_b(\mathcal{B}^{-1}x) \tag{28}$$

in which $\mathcal{O}(g)$ contains all the quantum corrections which vanish in the free-theory, so that the NR expansion is

$$\begin{aligned}\phi''(x) \quad \rightarrow \quad & \Big\{1 + iM\boldsymbol{\eta}\cdot\mathbf{x} - c_1\frac{i\boldsymbol{\eta}\cdot\mathbf{D}}{2M} - c_2\frac{i\boldsymbol{\eta}\cdot\mathbf{D}\mathbf{D}^2}{4M^3} \\ & + c_3\frac{(\boldsymbol{\Sigma}\times\boldsymbol{\eta})\cdot\mathbf{D}}{2M}\left[1+\frac{\mathbf{D}^2}{4M^2}\right] + \mathcal{O}(g, 1/M^4)\Big\}\phi''(\mathcal{B}^{-1}x)\end{aligned} \tag{29}$$

Non-relativistic QCD (NRQCD)

Let us apply this method to the NRQCD in particular. Since the NRQCD plays the role of underlying theory to the pNRQCD, its Wilson coefficients are to be fixed before integrating out the soft scale. Integrating out the hard scale M from the full QCD Lagrangian, one obtains

$$\mathcal{L}_{NRQCD} = \mathcal{L}_{heavy}(\phi, \chi) + \mathcal{L}_{light}(\psi) \tag{30}$$

in which the light and heavy quarks are decoupled, and we consider the heavy part of the Lagrangian, which contains the Pauli spinors of heavy quark and antiquark ϕ, χ. Apply the Lorentz boost for the NR field and we observe

$$0 = \delta\mathcal{L}_{heavy} = \frac{1}{M}\delta\mathcal{L}_1 + \frac{1}{M^2}\delta\mathcal{L}_2 + \frac{1}{M^3}\delta\mathcal{L}_3 + \cdots \tag{31}$$

NRQCD and Potential NRQCD (pNRQCD)

There exists another constraint

$$[k^i, k^j] = -i\epsilon^{ijk} J^k \tag{32}$$

which fixes the constants in the boost generator of the NR field. Our results of fixing NRQCD boost coefficients coincides with the literature [Heinonen, 2012], which simply implies that their ansatz on the interacting theory works. The reason why the ansatz works still remains to be answered.

One can do the similar analysis to the pNRQCD

$$S(t, R, r) = \chi^\dagger(t, r_1)\phi(t, r_2) \tag{33}$$

where $R = \frac{1}{2}(r_1 + r_2)$ and $r = r_1 - r_2$.

Summary/Outlook

- Little group formalism in QM and QFT
- NR expansion of the Lorentz boost
- Fixing Wilson coefficients in NRQCD
- Similar application to pNRQCD (in progress)
- Dark matter candidates with high mass

References

N. Brambilla, D. Gromes, and A. Vairo (2003)
Poincare invariance constrains on NRQCD and potential NRQCD
Phys. Lett. B 576, 314 - 327.

Johannes Heinonen, et. al (2012)
Lorentz invariance in heavy particle effective theories
Phys. Rev. D 86, 094020 (2012).