Differential Equations for Feynman Integrals

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based on work with M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, L. Tancredi, V. Yundin

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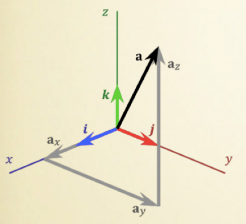
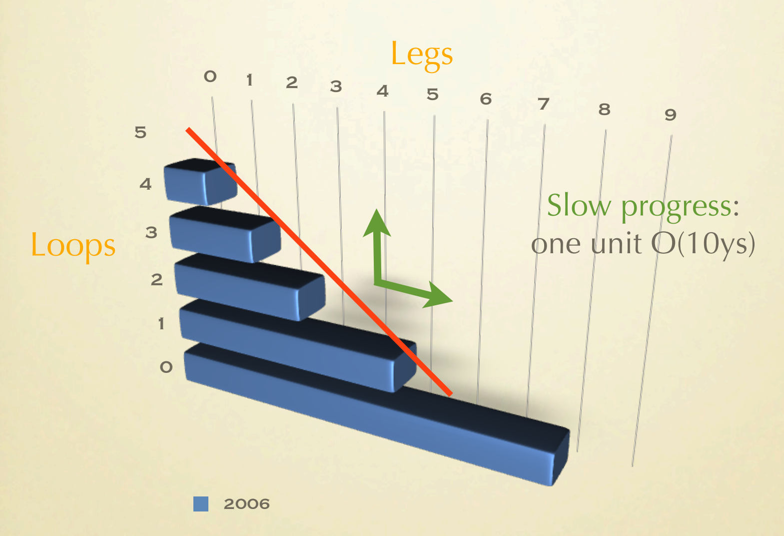


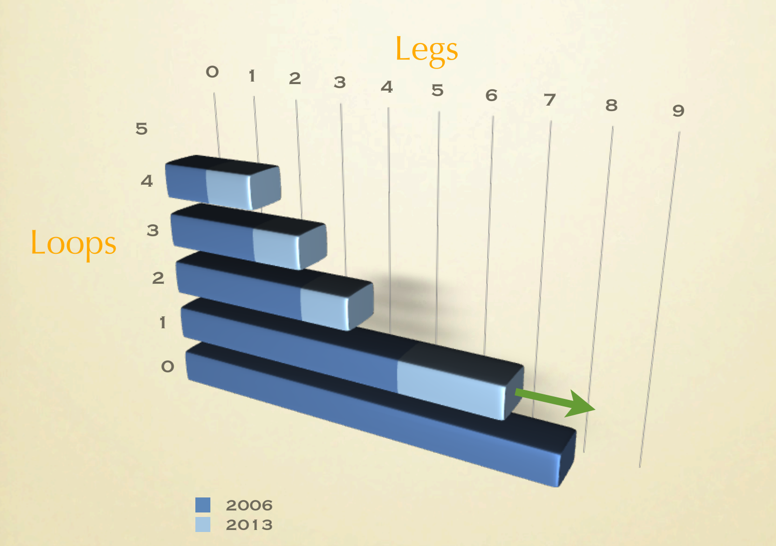
A modern multi-loop and -leg calculation is done in three steps

1) Find integral basis for the process
2) Determine coefficients of the basis elements
3) Calculate elements (integrals) of the basis

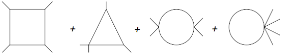
Tools for each Step

1) Integration-by-parts identities (IBP-Ids), Chetyrkin,Tkachov (1981)
   MultiLoop Integrand Reduction Mastrolia, Ossola (2011), Zhang (2012); Mastrolia, Mirabella, Ossola, Peraro (2012)
2) Generalized Unitarity, Bern, Dixon, Dunbar, Kosower (1994) Kosower, Larsen (2011)
   OPP Integrand Reduction, Ossola, Papadopoulos, Pittau (2007)
   MultiLoop Integrand Reduction
3) Differential Equations, Kotikov (1991);Remiddi (1997); Gehrmann, Remiddi (2000); Henn (2013)
   Mellin-Barnes representaion, Smirnov (1999),Tausk (1999)
   Feynman/Schwinger parameter representation


Legs
0 1 2 3 4 5 6 7 8 9
Loops
5
4
3
2
1
0
Slow progress:
one unit O(10ys)
2006



At one-loop this was very succesful what do we need for higher-loops?

	one-loop	two-loop
graphs	only planar	planar and non-planar
integral basis	known 	determined case by case ?
integrals	known	only for certain cases
IR poles	cancellation between one-loop and tree level	cancellation between two- and one-loop and tree level
appearing functions	logs and dilogs	logs, polylogarithms, generalized polylogs, elliptic functions and more?

$\rightarrow$ a lot more exploring has to be done

1 Introduction

2 Differential Equations

Feynman integrals are functions of

- Mandelstam variables
- Internal and external masses
- Spacetime dimensions

Facts

- Not all Feynman integrals are independent
- IBP-ids connect different Feynman integrals
- We can find an integral basis called master integrals

$$\Downarrow$$

Exploit this by

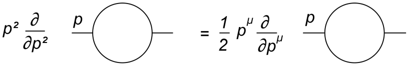
- Taking derivatives of the master integrals in respect to the kinematic invariants
- Reduce derivatives back to master integrals
- Solve the obtained first order differential equation analytically

Let's have a look at an easy example

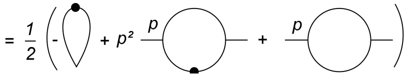
Differential Equations

The one-loop massless bubble

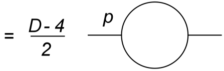
- Construct diff. Operator



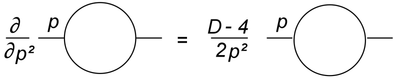
- Apply derivative



- Reduce back to Master Integrals with IBP-ids



- Gives us the differential equation



First order differential equation

$$\partial_x f(x) = A(x,\epsilon) f(x)$$

where ϵ is the dimensional regularization parameter $D = 4 - 2\epsilon$.

Conjecture: We can always find a basis such that Henn(2013)

$$\partial_x f(x) = \epsilon \tilde{A}(x) f(x)$$

Example: one-loop Bhabha scattering
Laporta basis

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{2(\epsilon-1)}{m^2\left(x^2-1\right)} & 0 & \frac{\epsilon(x-1)^2+2x}{x-x^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{2(x-1)xy(\epsilon-1)}{m^6(x+1)^3(x+y)(xy+1)} & \frac{2y(2\epsilon-1)}{m^4\left(x^2-1\right)(y-1)^2} & -\frac{2(x-1)xy(2\epsilon-1)}{m^4(x+1)^3(x+y)(xy+1)} & -\frac{2x(y+1)^2\epsilon}{m^2\left(x^2-1\right)(x+y)(xy+1)} & \frac{x^2+1}{x-x^3} - \frac{(x-1)(y-1)^2\epsilon}{(x+1)(x+y)(xy+1)} \end{pmatrix}$$

Canonical basis

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{x} & 0 & \frac{1}{x} - \frac{2}{1+x} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{x} & \frac{4y}{1+xy} - \frac{8}{1+x} + \frac{4}{x+y} & \frac{2}{x+y} - \frac{2y}{1+xy} & \frac{y}{1+xy} - \frac{2}{1+x} + \frac{1}{x+y} \end{pmatrix}$$

which simply integrates to Logarithms with arguments
$\{y, 1+y, x, 1+x, x+y, 1+xy\}$

How can we find such a basis ?

This is still an open question

But it was possible for a lot of examples:
Henn(2013); Henn, Smirnov (2013); Henn Smirnov, Smirnov(2013); M. Argeri, S. Di Vita, P. Mastrolia, E. Mirabella, J. Schlenk, L. Tancredi, U.S. (2014); Henn, Melnikov, Smirnov (2014); Caron-Huot, Henn (2014); Caola, Henn, Melnikov, Smirnov (2014); Höschele, Hoff, Ueda (2014)

A first step was done by applying the **Magnus Exponential**

Ageri, Di Vita, Mastrolia, Mirabella, Schlenk, Tancredi, U.S. (2014)

Magnus Theorem

Starting from a first order differential equation

$$\partial_x f(x) = A(x,\epsilon) f(x)$$

We can write down its solution in terms of the Magnus Exponential

$$f(x) = e^{\Omega[A](x,x_0)} f_0 \equiv e^{\Omega[A](x)} f_0 \qquad \Omega(x) = \sum_{n=1}^{\infty} \Omega_n$$

$$\Omega_1(x) = \int_{x_0}^{x} d\tau_1 A(\tau_1)$$

$$\Omega_2(x) = \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 [A(\tau_1), A(\tau_2)]$$

The Magnus Theorem can aid us in finding a canonical basis and to integrate the result

Algorithm for finding a canonical basis and integrating the result

- Find a DE which is linear in ϵ (trial+ error + some experience)

$$\partial_x f(x) = (A_0(x) + \epsilon A_1(x))f(x)$$

- Basis change with Magnus $f(x) = B_0(x)g(x)$

$$\partial_x B_0(x) = A_0(x)B_0(x) \leftrightarrow B_0 = e^{\Omega[A_0](x,x_0)}$$

- Obtain a canonical system for g's

$$\partial_x g(x) = \epsilon\tilde{A}(x)g(x), \qquad \tilde{A}(x) = B_0^{-1}(x)A_1(x)B_0(x)$$

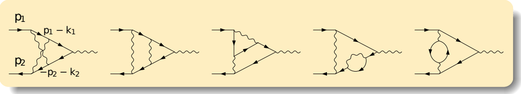
- Obtain the solution with Magnus

$$g(x) = B_1(\epsilon, x)g_0, \qquad B_1(x) = e^{\Omega[\epsilon\tilde{A}]}$$

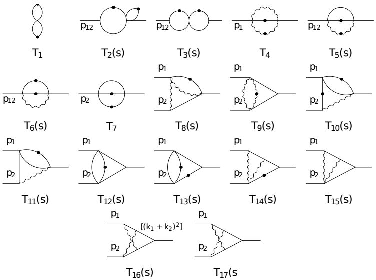
QED Vertex at two-loop

Bonciani, Mastrolia, Remiddi (2003)

17 MI's for all relevant topologies



$$-\frac{s}{m^2} = \frac{(1-x)^2}{x}$$



The f's obey an ϵ-linear DE [ADVMMSST '14]

$$f_1 = \epsilon^2 T_1 \quad f_2 = \epsilon^2 T_2 \quad f_3 = \epsilon^2 T_3 \quad f_4 = \epsilon^2 T_4 \quad f_5 = \epsilon^2 T_5$$
$$f_6 = \epsilon^2 T_6 \quad f_7 = \epsilon^2 T_7 \quad f_8 = \epsilon^3 T_8 \quad f_9 = \epsilon^3 T_9 \quad f_{10} = \epsilon^2 T_{10}$$
$$f_{11} = \epsilon^3 T_{11} \quad f_{12} = \epsilon^3 T_{12} \quad f_{13} = \epsilon^2 T_{13} \quad f_{14} = \epsilon^3 T_{14} \quad f_{15} = \epsilon^4 T_{15}$$
$$f_{16} = \epsilon^4 T_{16} \quad f_{17} = \epsilon^4 T_{17}$$

After getting rid of A_0, the g's obey a canonical DE

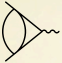
$$\partial_x g(\epsilon, x) = \epsilon \, \hat{A}_1(x) \, g(\epsilon, x) \qquad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1+x} + \frac{M_3}{1-x}$$

$$M_1 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&5&-6&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&-4&0&-2&0&-2&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&0&0&0&0&2&0&0&0&0&0&0&0&0\\
-\frac{1}{2}&0&0&0&1&-2&-3&0&0&3&3&0&0&0&0&0&0\\
0&0&0&0&1&-1&2&0&0&-2&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&-1&-1&0&0&0&0\\
0&-1&0&0&0&0&-3&0&0&0&0&3&3&0&0&0&0\\
0&-1&0&0&1&-\frac{1}{2}&0&2&2&0&0&0&0&2&2&0&0\\
0&0&0&0&0&\frac{1}{2}&0&-\frac{1}{2}&0&0&0&0&0&-1&-1&0&0\\
-\frac{1}{2}&0&0&-2&-1&0&-2&1&0&2&0&-2&0&0&-2&-2&2\\
0&0&0&0&-1&\frac{1}{2}&0&3&-2&0&-6&-2&0&0&-4&-4&4
\end{pmatrix}$$

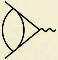
$$M_2 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-6&3&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-4&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&\frac{1}{2}&0&0&0&-4&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&-6&0&0&0&0&0&-2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-4
\end{pmatrix}$$

$$M_3 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-6&0&0&2&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-12&0&0&0&0&0&4&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&2&0&0&0&-4&0&0&-4&-2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4
\end{pmatrix}$$

Bonciani, Remiddi, **P.M.** (2013)



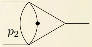
$$
\begin{aligned}
M_{-2} &= \frac{1}{2}\,, \\
M_{-1} &= \frac{5}{2} - \Big[1 - \frac{2}{(1-x)}\Big] H(0,x)\,, \\
M_0 &= \frac{19}{2} + \zeta(2) + \Big[1 - \frac{2}{(1-x)}\Big][\zeta(2) - 5H(0,x) + 2H(-1,0,x)] \\
&\quad + \frac{2}{(1-x)} H(0,0,x) + \Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big][\zeta(2) H(0,x) \\
&\quad + H(0,0,0,x)]\,.
\end{aligned}
$$



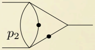
$(p_2 \cdot k_1)$

$$
\begin{aligned}
\frac{N_{-2}}{a} &= \frac{1}{8} + \frac{1}{16}\Big[x + \frac{1}{x}\Big]\,, \\
\frac{N_{-1}}{a} &= \frac{9}{32}\Big[2 + x + \frac{1}{x}\Big] - \frac{1}{8}\Big[4 + x - \frac{1}{x}\Big] H(0,x) + \frac{1}{(1-x)} H(0,x)\,, \\
\frac{N_0}{a} &= \frac{63}{32} + \frac{\zeta(2)}{2} + \frac{63}{64}\Big[\Big(1 + \frac{16}{63}\zeta(2)\Big) x + \frac{1}{x}\Big] - \frac{\zeta(2)}{(1-x)} - \frac{1}{16}\Big[32 + 9x \\
&\quad - \frac{9}{x}\Big] H(0,x) + \frac{(16+\zeta(2))}{4(1-x)} H(0,x) - \frac{\zeta(2)}{4(1+x)} H(0,x) - \frac{1}{4}\Big[2 - \frac{1}{x} \\
&\quad - \frac{4}{(1-x)}\Big] H(0,0,x) + \frac{1}{4}\Big[4 + x - \frac{1}{x} - \frac{8}{(1-x)}\Big] H(-1,0,x) \\
&\quad + \frac{1}{4}\Big[\frac{1}{(1-x)} - \frac{1}{(1+x)}\Big] H(0,0,0,x)\,.
\end{aligned}
$$

AdVMMSST (2014)



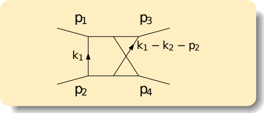
$$
\begin{aligned}
g_{12}^{(0)} &= 0\,, \\
g_{12}^{(1)} &= 0\,, \\
g_{12}^{(2)} &= 0\,, \\
g_{12}^{(3)} &= -\,\mathrm{H}(0,0,0;x) - \zeta_2 \mathrm{H}(0;x)\,, \\
g_{12}^{(4)} &= -2\,\mathrm{H}(-1,0,0,0;x) + 2\,\mathrm{H}(0,-1,0,0;x) + 2\,\mathrm{H}(0,0,-1,0;x) \\
&\quad - 3\,\mathrm{H}(0,0,0,0;x) - 4\,\mathrm{H}(0,1,0,0;x) + \zeta_2(-2\,\mathrm{H}(-1,0;x) \\
&\quad + 6\,\mathrm{H}(0,-1;x) - \mathrm{H}(0,0;x)) + 2\,\zeta_3\,\mathrm{H}(0;x) + \frac{\zeta_4}{4}\,,
\end{aligned}
$$



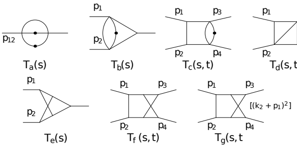
$$
\begin{aligned}
g_{13}^{(0)} &= 0\,, \\
g_{13}^{(1)} &= 0\,, \\
g_{13}^{(2)} &= \mathrm{H}(0,0;x) + \frac{3\,\zeta_2}{2}\,, \\
g_{13}^{(3)} &= -2\,\mathrm{H}(-1,0,0;x) - 2\,\mathrm{H}(0,-1,0;x) + 4\,\mathrm{H}(0,0,0;x) + 4\,\mathrm{H}(1,0,0;x) \\
&\quad + \zeta_2(-6\,\mathrm{H}(-1;x) + 2\,\mathrm{H}(0;x) - 3\log 2) - \frac{\zeta_3}{4}\,, \\
g_{13}^{(4)} &= 4\,\mathrm{H}(-1,-1,0,0;x) + 4\,\mathrm{H}(-1,0,-1,0;x) - 8\,\mathrm{H}(-1,0,0,0;x) \\
&\quad - 8\,\mathrm{H}(-1,1,0,0;x) + 4\,\mathrm{H}(0,-1,-1,0;x) - 8\,\mathrm{H}(0,-1,0,0;x) \\
&\quad - 8\,\mathrm{H}(0,0,-1,0;x) + 10\,\mathrm{H}(0,0,0,0;x) + 12\,\mathrm{H}(0,1,0,0;x) \\
&\quad - 8\,\mathrm{H}(1,-1,0,0;x) - 8\,\mathrm{H}(1,0,-1,0;x) + 16\,\mathrm{H}(1,0,0,0;x) \\
&\quad + 16\,\mathrm{H}(1,1,0,0;x) + 12\,\mathrm{Li}_4\frac{1}{2} + \frac{\log^4 2}{2} + 2\,\zeta_2\,(12\log 2\,\mathrm{H}(-1;x) \\
&\quad + 12\log 2\,\mathrm{H}(1;x) + 6\,\mathrm{H}(-1,-1;x) - 2\,\mathrm{H}(-1,0;x) - 8\,\mathrm{H}(0,-1;x) \\
&\quad + \mathrm{H}(0,0;x) - 12\,\mathrm{H}(1,-1;x) + 4\,\mathrm{H}(1,0;x) + 3\log^2 2) \\
&\quad - 2\,\zeta_3(5\,\mathrm{H}(-1;x) + 4\,\mathrm{H}(0;x) + 11\,\mathrm{H}(1;x)) - \frac{47\,\zeta_4}{4}\,,
\end{aligned}
$$

Non-planar massless box Tausk (1999); Anastasiou, Gehrmann, Oleari, Remiddi, Tausk (2000)

12 MI's for the crossed topology



$x = -\frac{t}{s}, \quad s > 0\,, t < 0\,, |s| > |t|$



The f's obey an ϵ-linear DE [ADVMMSST '14]

$$f_1 = \epsilon^2\, s\, T_a(s)\,, \qquad f_2 = \epsilon^2\, t\, T_a(t)\,, \qquad f_3 = \epsilon^2\, u\, T_a(u)\,,$$

$$f_4 = \epsilon^3\, s\, T_b(s)\,, \qquad f_5 = \epsilon^3\, s\, t\, T_c(s,t)\,, \qquad f_6 = \epsilon^3\, s\, u\, T_c(s,u)\,,$$

$$f_7 = \epsilon^4\, u\, T_d(s,t)\,, \qquad f_8 = \epsilon^4\, s\, T_d(t,u)\,, \qquad f_9 = \epsilon^4\, t\, T_d(u,s)\,,$$

$$f_{10} = \epsilon^4\, s^2\, T_e(s)\,,$$

$$f_{11} = \epsilon^4\, s\, t\, u\, T_f(s,t) - \frac{3}{4\, s\,(4\epsilon+1)}\Big[\epsilon^2\left(s^2\, T_a(s) + t^2\, T_a(t) + u^2\, T_a(u)\right) - 4\epsilon^4\left(u^2\, T_d(s,t) + s^2\, T_d(t,u) + t^2\, T_d(u,s)\right)\Big]\,,$$

$$f_{12} = \epsilon^4\, s\, t\, T_g(s,t) - \frac{3}{8\, u\,(4\epsilon+1)}\Big[\epsilon^2\left(s^2\, T_a(s) + t^2\, T_a(t) + u^2\, T_a(u)\right) - 4\epsilon^4\left(u^2\, T_d(s,t) + s^2\, T_d(t,u) + t^2\, T_d(u,s)\right)\Big]\,,$$

After getting rid of A_0, the g's obey a canonical DE

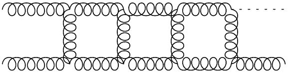
$$\partial_x g(\epsilon, x) = \epsilon\, \hat{A}_1(x)\, g(\epsilon, x) \qquad \hat{A}_1(x) = \frac{M_1}{x} + \frac{M_2}{1-x}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & -\frac{9}{2} & 0 & -4 & -2 & -18 & -12 & -12 & 1 & 1 & -2 \\ \frac{3}{4} & \frac{9}{4} & -\frac{21}{4} & 3 & 2 & -3 & 12 & -6 & -18 & 0 & 0 & -2 \end{pmatrix}$$

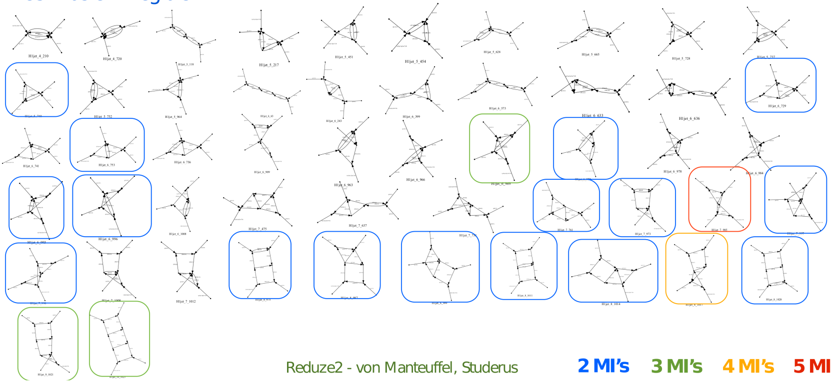
$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{2} & 0 & 3 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -6 & -6 & -\frac{9}{2} & 0 & -4 & -2 & -18 & -12 & -12 & 1 & 1 & -2 \\ -\frac{21}{4} & \frac{9}{4} & -\frac{27}{4} & -6 & 2 & -4 & 12 & -6 & -24 & 1 & -1 & 0 \end{pmatrix}$$

Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, U.S. work in progress



- 85 Master Integrals



Reduze2 - von Manteuffel, Studerus

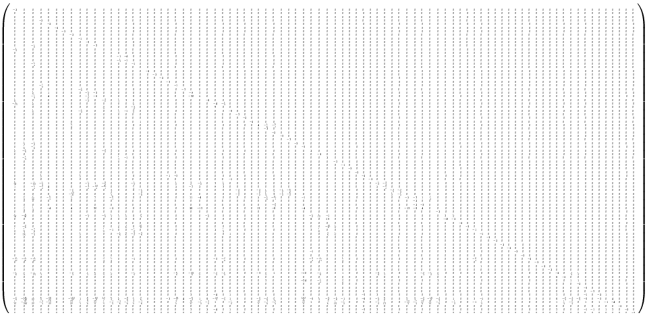
2 MI's **3 MI's** **4 MI's** **5 MI's**

taken from Pierpaolo's slides at Amplitudes 2014

Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, U.S. work in progress

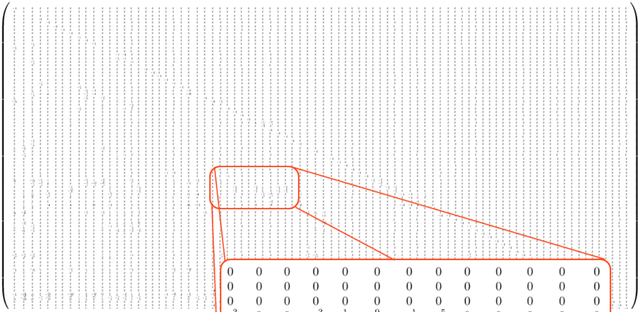
$1/x$



Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, U.S. work in progress

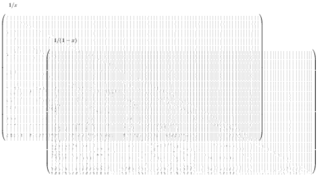
$1/x$



$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2} & 0 & 0 & -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} & -\frac{1}{2} & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{2} & 0 & 0 & -\frac{7}{2} & \frac{1}{3} & 2 & -\frac{1}{2} & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 5 & -\frac{1}{6} & -\frac{11}{4} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 12 & 0 & -6 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{12} & -1
\end{pmatrix}
$$

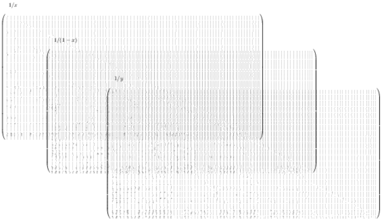
Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, U.S. work in progress



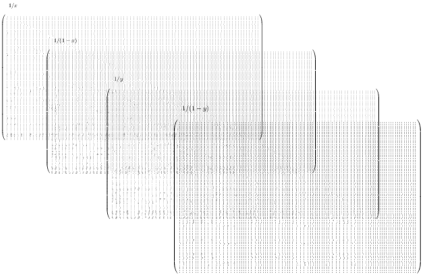
Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, U.S. work in progress



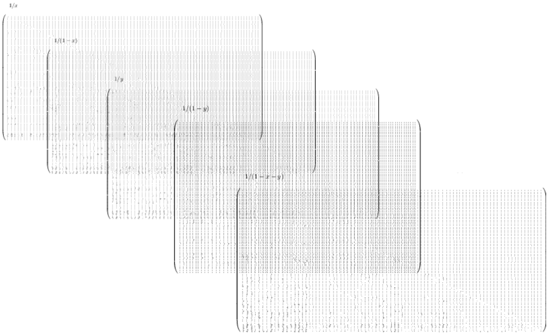
Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, U.S. work in progress



Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, U.S. work in progress



Higgs+Jet at three-loop

Di Vita, Mastrolia, Yundin, U.S. work in progress

Conclusion

- Many processes are becoming available at NLO precision, often already called the NLO revolution
- The focus slowly shifts towards NNLO precision
- In order to trigger a simillar revolution at NNLO we need a much better understanding of the structure at two-loops
- First powerful tools are emerging and are being applied
  - MultiLoop Integrand Reduction
  - Generalized Unitarity
  - Differential Equations
- New ideas stimulated the Differntial Equation method
- We provided a procedure to find a canonical basis from a system which is linear in ϵ
- Has been applied to the two-loop QED vertex and the two-loop non-planar massless box which were previously known in the "bad" basis
- New results for three loop Higgs+Jet ladder topology are on the way

Backup Slides