Renormalization group structure in classical nonlinear field theories

Bastian Sikora

IMPRS workshop

July 8, 2013

Confinement

- Quantum chromodynamics theory of quarks and gluons
- Observable particles bound states of quarks which are color singlets
- Confinement not proven to be consequence of quantum chromodynamics

Classical $\lambda\phi^4$ theory with negative λ and point-like external source Q (G. Dvali, C. Gomez, S. Mukhanov, arXiv:1107.0870v1 [hep-th])

- Field of single charge Q has infinite energy
- "Dipole" of Q and $-Q$ has finite energy

Massless ϕ^4 theory with external source

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{\lambda}{4}\phi^4 + 4\pi Q_0\delta(\mathbf{x})\phi$$

equation of motion for static fields $\Delta\phi = -4\pi Q_0\delta(\mathbf{x}) + \lambda\phi^3$

solution without self-interaction $\phi(r) = \dfrac{Q_0}{r}$

Iterative procedure to determine perturbative solution

$$\phi(r) = \frac{Q_0}{r} - \frac{\lambda}{r}\int_{r_0}^{r} \mathrm{d}r' r'^2\phi^3(r') - \lambda\int_{r}^{\infty} \mathrm{d}r' r'\phi^3(r')$$

First integral ultraviolet divergent $\Rightarrow$ regularize using r_0

Classical "renormalization procedure"

- "Bare" charge Q_0 screened by self-interaction of $\phi \Rightarrow Q_0$ unobvservable
- Split bare charge into observable physical charge Q and charge counterterm $\delta_Q = Q_0 - Q$
- Define physical charge Q using the field ϕ at distance r_p from the point charge

  $\Rightarrow$ renormalization condition $\phi(r_p) =: \dfrac{Q}{r_p}$

$$\delta_Q = \lambda \int_{r_0}^{r_p} \mathrm{d}r' r'^2 \phi^3(r') + \lambda r_p \int_{r_p}^{\infty} \mathrm{d}r' r' \phi^3(r')$$

$$\Rightarrow \phi(r) = \frac{Q}{r} - \frac{\lambda}{r} \int_{r_p}^{r} \mathrm{d}r' r'^2 \phi^3(r') - \lambda \int_{r}^{\infty} \mathrm{d}r' r' \phi^3(r') + \lambda \frac{r_p}{r} \int_{r_p}^{\infty} \mathrm{d}r' r' \phi^3(r')$$

Massive ϕ^4 theory with external source

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 + 4\pi Q_0\delta(\mathbf{x})\phi$$

equation of motion for static fields $(\Delta - m^2)\phi = -4\pi Q_0\delta(\mathbf{x}) + \lambda\phi^3$

solution without self-interaction $\phi(r) = \frac{Q_0}{r}e^{-mr}$

Iterative procedure to determine perturbative solution

$$\phi(r) = \frac{Q_0}{r}e^{-mr} - \frac{\lambda}{4\pi}\int \mathrm{d}^3x' \frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\phi^3(\mathbf{x}')$$

Renormalization procedure as in massless case

- Integral ultraviolet divergent
- Bare charge screened by self-interaction of ϕ, unobservable
- $Q_0 = Q + \delta_Q$
- Renormalization condition $\phi(r_p) =: \frac{Q}{r_p} e^{-mr_p}$

$$\Rightarrow \phi(r) = \frac{Q}{r} e^{-mr} - \frac{\lambda e^{-mr}}{2mr} \int_{r_p}^{r} \mathrm{d}r' r' e^{mr'} \phi^3(r') - $$

$$- \frac{\lambda e^{mr}}{2mr} \int_{r}^{\infty} \mathrm{d}r' r' e^{-mr'} \phi^3(r') + \frac{\lambda e^{2mr_p - mr}}{2mr} \int_{r_p}^{\infty} \mathrm{d}r' r' e^{-mr'} \phi^3(r')$$

Renormalization group structure - massless ϕ^4 theory

Parametrize $\phi(r) = \frac{Q_{eff}(r)}{r}$, rewrite iterative procedure as equation for $Q_{eff}(r)$

$$\frac{\mathrm{d}Q_{eff}(r)}{\mathrm{d}r} = -\lambda \int_r^\infty \mathrm{d}r' r' \phi^3(r')$$

Renormalization group equation consistent with assumption that $\phi(r)$ is independent of r_p

$$0 = \frac{\mathrm{d}\phi(r)}{\mathrm{d}r_p} = \frac{1}{r}\frac{\mathrm{d}Q}{\mathrm{d}r_p} + \frac{\lambda}{r}\int_{r_p}^\infty \mathrm{d}r' r' \phi^3(r')$$

$Q_{eff}(r)$ as a function of r is identical to Q as a function of r_p.

$$\frac{\mathrm{d}Q_{eff}(r)}{\mathrm{d}r} = -\lambda \int_r^\infty \mathrm{d}r' r' \phi^3(r')$$

Partial resummation of perturbative solution

Insert $Q_{eff}(r)$ into right hand side, compute integral, rewrite right hand side in terms of $Q_{eff}(r)$, solve differential equation for $Q_{eff}(r)$
Begin with $Q_{eff}(r) = Q$

$$\frac{\mathrm{d}Q_{eff}(r)}{\mathrm{d}\ln r} = -\lambda Q_{eff}^3(r) \Rightarrow Q_{eff}^2(r) = \frac{Q^2}{1 + 2\lambda Q^2 \ln\left(\frac{r}{r_p}\right)}$$

Corresponds to resummation of leading logarithms.
Similar approach gives partial resummation of perturbative solution of the massive ϕ^4 theory. $\left(\phi(r) = \frac{Q_{eff}(r)}{r} e^{-mr}\right)$

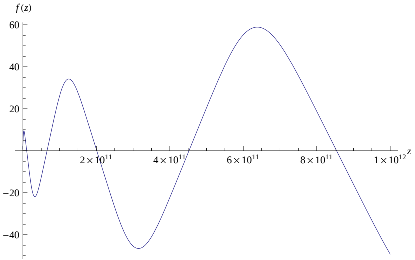
Numerical Analysis

- Parametrize fields $\phi(r) = \frac{Q}{r} f(r)$ (massless); $\phi(r) = \frac{Q}{r} e^{-mr} f(r)$ (massive)
- Rewrite equations of motion of massless and massive ϕ^4 theory as equations for $f(z)$ $\left(z := \frac{r}{r_p}, ()' := \frac{\mathrm{d}()}{\mathrm{d}z}, \alpha := \lambda Q^2\right)$

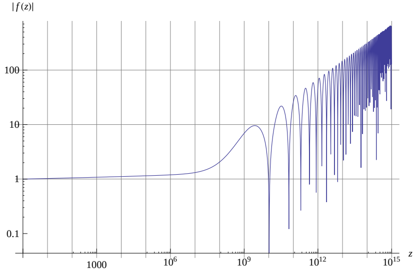
$$z^2 f''(z) = \alpha f^3(z); \quad z^2 (f''(z) - 2mr_p f'(z)) = \alpha e^{-2mr_p z} f^3(z)$$

- Choose α, mr_p& initial conditions $f(1) = 1$ ($\Leftrightarrow$ renormalization condition), $f'(1)$ according to perturbative solution $f_n'(1)$ means all terms up to order $\mathcal{O}(\alpha^n)$ taken into account.
- Solve equations numerically using Mathematica, version 8.0.4.0

Numerical solution, massless ϕ^4 theory



$\alpha = -0.01,\ f'(1) = f_6'(1)$

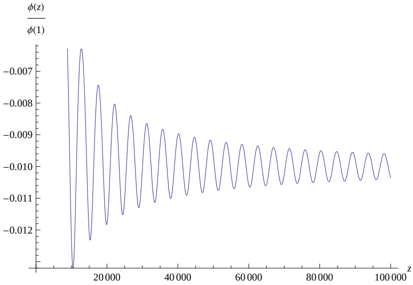


$\alpha = -0.01,\ f'(1) = f_6'(1)$

Common in all cases examined

- oscillations of $f(z)$
- amplitude of oscillations increasing $\propto z^{\frac{1}{3}}$

Numerical solution, massive ϕ^4 theory



$\alpha = -0.01$, $f'(1) = f_2'(1)$,
$mr_p = 0.001$

Plot suggests that $\phi(z)$ approaches finite non-zero value for $z \to \infty$

Energy of field of a point-like charge, massless ϕ^4 theory

$$E = \frac{2\pi Q^2}{r_p}\int_{z_0}^{Z} \mathrm{d}z \left(f'^2(z) - \frac{2}{z} f'(z) f(z) + \frac{f^2(z)}{z^2} + \frac{\lambda Q^2 f^4(z)}{2z^2} \right)$$

- Use numerical solution $f(z)$ from above
- First maximum of $f(z)$ chosen as z_0
- Energy computed for various values of Z
- Integral computed numerically using Mathemtica

Energy increasing with Z

Z	energy/$\left(\frac{2\pi Q^2}{r_p}\right)$
10^{10}	3.93829×10^{-8}
10^{11}	7.26002×10^{-8}
10^{12}	1.34952×10^{-7}
10^{13}	2.43805×10^{-7}
10^{14}	5.31086×10^{-7}

$\alpha = -0.01$, $f'(1) = f_6'(1)$, $z_0 = 2.7014 \times 10^9$

Summary

- Bare charge of external source screened by self-interaction of fields
- Classical "renormalization procedure" to determine perturbative solutions
- Renormalization group approach allows partial resummation of perturbative solution
- Numerical analysis of massless ϕ^4 theory confirms predictions by G. Dvali, C. Gomez and S. Mukhanov
- Numerical analysis of massive ϕ^4 theory suggests that $\phi(r)$ does not vanish for $r \to \infty$
- Possible future research: classical Non-Abelian gauge theory with external source