Integrand-Reduction and the Color-Kinematic Duality

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Introduction

For one-loop amplitudes with a higher number of legs normal Feynman Diagram calculation becomes impossible
Reason: unphysical gauge freedom which cancels for gauge invariant objects

$\Rightarrow$ Easier to reconstruct amplitudes from their pole structure which is governed by analyticity and unitarity

- Generalized Unitarity Bern, Dixon, Dunbar, Kosower (1994)
- On-shell Techniques Britto, Cachazo, Feng, Witten (2005)
- OPP Integrand-Reduction Ossola, Papadopoulos, Pittau (2007)

Furthermore through input from algebraic geometry we improved our understanding of QFT

- Integrand-Reduction Mastrolia, Ossola (2011)
  Zhang (2012)
  Mastrolia, Mirabella, Ossola, Peraro (2012)
- On-shell Formulation of $\mathcal{N}$=4 sYM
  Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka (2012)

Introduction

Moreover revealed hidden properties of QFT

- Color-Kinematic and Gauge/Gravity Duality
  Bern, Carrasco, Johansson (2008,2010)
- Grassmanians — Arkani-Hamed, Cachazo, Cheung, Kaplan (2010)
  Mason, Skinner (2012)
- Dual conformal Symmetry — Drummond, Henn, Smirnov, Sokatchev (2007)
  Bern, Czakon, Dixon, Kosower, Smirnov (2007)

In my thesis I merged for the first time three modern techniques namely

- Integrand-Reduction
- Unitarity
- Color-Kinematic Duality

With these techniques I reproduced a known result in a new way

Five-point one and two-loop amplitudes in $\mathcal{N}=4$ super Yang-Mills
Carrasco, Johansson (2011)

Done analytically and semi-numerically

In Addition studied UV properties of the obtained amplitudes through Gauge/Gravity Duality this can be used to settle the question if $\mathcal{N}=8$ SUGRA is finite

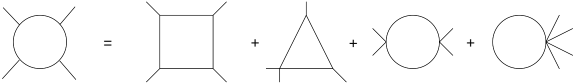
Unitarity and Integrand Decomposition at One-Loop

Ossola, Papadopoulos, Pittau (2007)

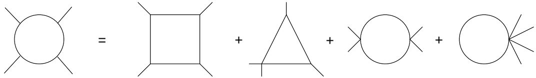
We can expand the integrand of every loop amplitude in its multipole channels e.g. at one-loop in four dimensions we have

$$I(q) = \frac{\Delta_{1234}}{D_1 D_2 D_3 D_4} + \frac{\Delta_{234}}{D_2 D_3 D_4} + \frac{\Delta_{134}}{D_1 D_3 D_4} + \frac{\Delta_{124}}{D_1 D_2 D_4} + \frac{\Delta_{123}}{D_1 D_2 D_3} + \frac{\Delta_{34}}{D_3 D_4} + .. + \frac{\Delta_{12}}{D_1 D_2} + \frac{\Delta_1}{D_1} + .. + \frac{\Delta_4}{D_4}$$

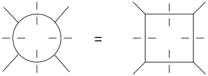
where the deltas are polynomials in q with unknown coefficients



Unitarity and Integrand Decomposition at One-Loop



Fit the coefficients at their corresponding multipole channel



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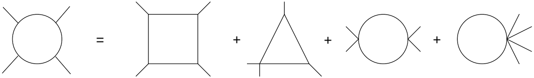

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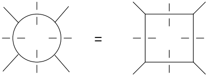

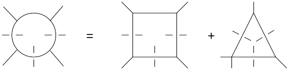

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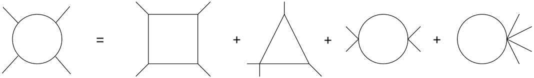

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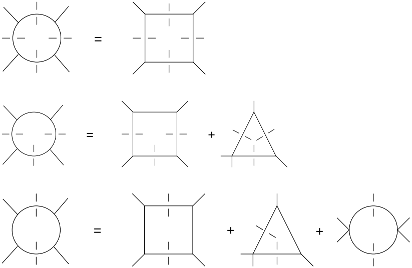

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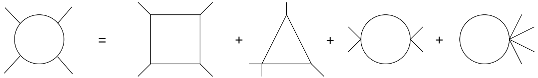

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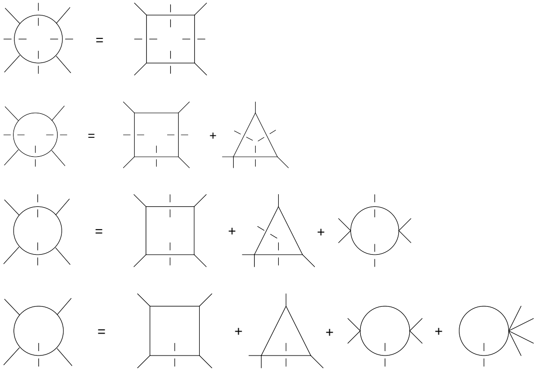

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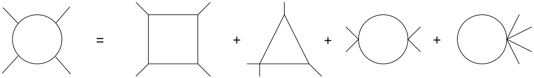

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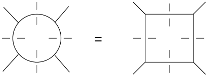

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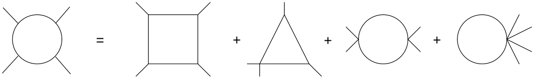

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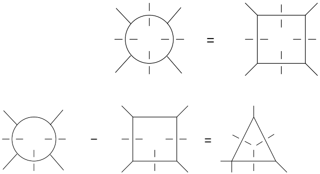

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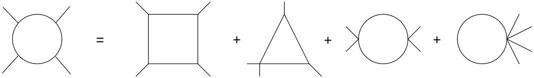

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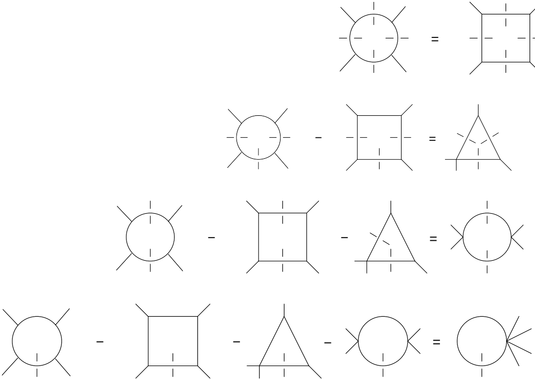

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Multivariate Polynomial Division

Mastrolia, Ossola (2011)
Zhang (2012)
Mastrolia, Mirabella, Ossola, Peraro (2012)

A general multi-loop amplitude can be written as

$$A = \int \prod_i^l \frac{d^4 q_i}{(2\pi)^4} \frac{N(q_1, .., q_l)}{D_1 D_2 .. D_n}$$

The propagators define a space of multiples called **Ideal**

$$\langle D_1, .., D_n \rangle = \left\{ \sum_k p_k D_k \, : \, p_k \in P[q] \right\}$$

The Buchbinder algorithm we constructs a **Gröbner Basis** from an Ideal

$$\langle D_1, .., D_n \rangle = \langle g_1, .., g_m \rangle$$

Multivariate Polynomial Division

With the Gröbner Basis we can perform a **multivariate polynomial division** of the integrand by an ideal

Integrand Recursion Relation

$$N(q_1,..,q_l) = \sum_k N_{1..k-1k+1..n} D_k + \Delta_{1..n}$$
$$I_{1..n}(q_1,..,q_l) = \sum_k I_{1..k-1k+1..n}(q_1,..,q_l) + \frac{\Delta_{1..n}}{D_1 D_2..D_n}$$

Can be used recursively to arrive at

$$\begin{aligned} N(q_1,..,q_l) = \Delta_{12..n} \\ &+\Delta_{23..n} D_1 + .. + \Delta_{12..n-1} D_n \\ &+\Delta_{34..n} D_1 D_2 + .. + \Delta_{12..n-2} D_{n-1} D_n \\ &+.. \\ &+\Delta_1 D_2 D_3 .. D_n + .. + \Delta_n D_1 D_2 .. D_{n-1} \end{aligned}$$

Reducibility Criterion and Maximum Cut Theorem

Starting point of the integrand-reduction fixed by the reducibility criterion

The integrand $I_{i_1..i_n}$ is reducible iff the remainder of the division modulo a Gröbner basis vanishes, i.e. iff $N_{i_1..i_n} \in \mathcal{I}_{i_1..i_n}$

An integrand $I_{i_1..i_n}$ is reducible if the cut $(i_1, .., i_n)$ leads to a system of equations with no solution.

If a maximum-cut for an l loop amplitude is defined as the $4l$-ple cut

$$D_{i_1} = D_{i_2} = .. = D_{i_{4l}} = 0$$

and has n_s solutions the maximum cut theorem states

Maximum-Cut Theorem

The residue at the maximum-cut is a polynomial parametrized by n_s coefficients, which admits a univariate representation of degree $(n_s - 1)$.

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3-pt Generating Function

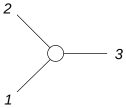
Combine all particle and helicity information of $\mathcal{N}$=4 super Yang-Mills theory into compact generating functions

Drummond, Henn, Korchemsky, Sokatchev (2008)

$$\Phi(p,\eta) = g^{+} + \eta^{a} f_{a} + \frac{1}{2}\eta^{a}\eta^{b} s_{ab} + \frac{1}{3!}\eta^{a}\eta^{b}\eta^{c}\epsilon_{abcd}\bar{f}^{d} + \frac{1}{4!}\eta^{a}\eta^{b}\eta^{c}\eta^{d}\epsilon_{abcd}\, g^{-}$$

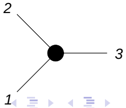
For three-point $\overline{MHV}$ amplitudes

$$A_3^{\overline{MHV}}(\eta_1,\eta_2,\eta_3) = \frac{\delta^4\left([12]\,\eta_3 + [23]\,\eta_1 + [31]\,\eta_2\right)}{[12]\,[23]\,[31]}$$



For three-point MHV amplitudes

$$A_3^{MHV}(\eta_1,\eta_2,\eta_3) = \frac{\delta^{(8)}(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3)}{\langle 12\rangle\,\langle 23\rangle\,\langle 31\rangle}$$



Merging Generating Functions

Merge two generating functions by

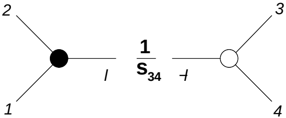
- Setting two legs equal (η,λ and $\tilde{\lambda}$)
- Integrate over the shared Grassmann variable

$$A_3^{MHV}(1,2,l) = \frac{\delta^8(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_l\eta_l)}{\langle 12\rangle \langle 2l\rangle \langle l1\rangle}$$

$$A_3^{\overline{MHV}}(3,4,-l) = \frac{\delta^4(-[34]\,\eta_l + [3l]\,\eta_4 + [l4]\,\eta_3)}{[34]\,[4l]\,[l3]}$$

$$A_4^{MHV}(1,2,3,4) = \int d^4\eta_l \frac{A_3^{MHV}(1,2,l)A_3^{\overline{MHV}}(3,4,-l)}{s_{34}}$$

$$= \frac{\delta^8(\lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3 + \lambda_4\eta_4)}{\langle 12\rangle \langle 23\rangle \langle 34\rangle \langle 41\rangle}$$



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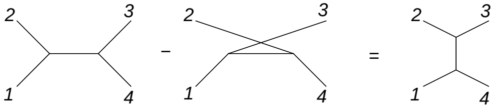
Basic Idea

Instead of stripping off color factors treat them equally with the kinematical numerators

$$\mathcal{A}_n^{tree}(1,2,..n) = \sum_i \frac{n_i c_i}{\prod_j p_j^2}$$

Especially let the kinematical numerators satisfy the Jacobi Identity



Gives us the BCJ equation for numerators Bern, Carrasco, Johansson (2008)

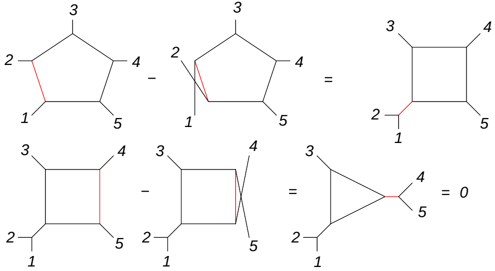
$$n_s - n_u = n_t$$

The Duality at Loop-Level

Loop-amplitude can be written as

$$\mathcal{A} = \sum_{\text{perms}} \int \left(\prod_m^L \frac{d^D l_m}{(2\pi)^D} \right) \sum_{\text{graphs}} \frac{N_i c_i}{\prod_j p_j^2}$$

Numerators must satisfy BCJ equations
e.g. at one-loop we have



Double Copy Procedure

Amplitudes in SUperGRAvity can be obtained by replacing the color factor with a copy of the kinematical factor

$$\mathcal{A}^{\text{sYM}} = \int \left(\prod_m^L \frac{d^4 l_m}{(2\pi)^4} \right) \sum_{\text{graphs}} \frac{N_i c_i}{\prod_j p_j^2}$$

$$\Rightarrow \mathcal{A}^{\text{SUGRA}} = \int \left(\prod_m^L \frac{d^D l_m}{(2\pi)^D} \right) \sum_{\text{graphs}} \frac{N_i \tilde{N}_i}{\prod_j p_j^2}$$

Is known to work for several theories

gauge numerator n	gauge numerator $\tilde{n}$	Gravity
$\mathcal{N}$=4 sYM	$\mathcal{N}$=4 sYM	$\mathcal{N}$=8 SUGRA
$\mathcal{N}$=4 sYM	$\mathcal{N}$=0 sYM	$\mathcal{N}$=4 SUGRA
$\mathcal{N}$=0 sYM	$\mathcal{N}$=0 sYM	$\mathcal{N}$=0 SUGRA

Calculation of $\mathcal{N}$=8 SUGRA loop amplitudes becomes feasible
$\Rightarrow$ investigate UV behavior of $\mathcal{N}$=8 SUGRA through direct computation

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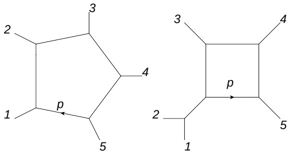
Basic Setup at One-Loop

For a one-loop five-point MHV amplitude we can find two graphs: the pentagon and the box

Carrasco, Johansson (2012)



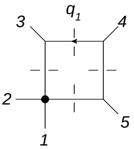
Integrand decomposition of the pentagon stops after the first step since numerators are only linear in the loop momentum

$$
\begin{aligned}
N^P(1,2,3,4,5,p) = \Delta^P_{12345}(1,2,3,4,5) + \Delta^P_{1234}(51,2,3,4)D_5 \\
+\Delta^P_{1235}(45,1,2,3)D_4 + \Delta^P_{1245}(34,5,1,2)D_3 \\
+\Delta^P_{1345}(23,4,5,1,2)D_2 + \Delta^P_{2345}(12,3,4,5)D_1
\end{aligned}
$$

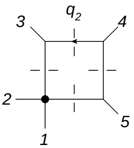
Integrand decomposition of the box

$$N^B(12,3,4,5,p) = \Delta^B_{2345}(12,3,4,5)$$

The Quadruple-Cut



$$= \frac{\Delta^{P}_{12345}(1,2,3,4,5)}{D_1(q_1)} + c_{2345}(12,3,4,5)$$



$$= \frac{\Delta^{P}_{12345}(1,2,3,4,5)}{D_1(q_2)} + c_{2345}(12,3,4,5)$$

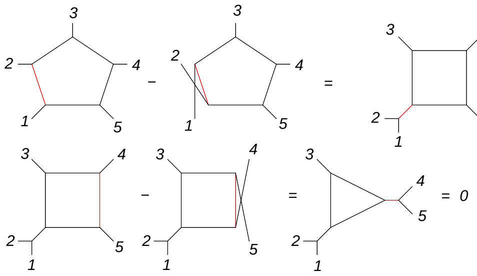
$$\Delta^{P}_{12345}(1,2,3,4,5) = \frac{\delta^{(8)}(q)\,[12]\,[23]\,[34]\,[45]\,[51]}{\langle 35\rangle\,[51]\,\langle 12\rangle\,[23] - [35]\,\langle 51\rangle\,[12]\,\langle 23\rangle} \equiv \beta_{12345}$$

$$c_{2345}(12,3,4,5) = \frac{\delta^{(8)}(q)\,[12]\,[34]\,[45]\,[35]}{\langle 21\rangle\left(\langle 35\rangle\,[51]\,\langle 12\rangle\,[23] - [35]\,\langle 51\rangle\,[12]\,\langle 23\rangle\right)} \equiv \frac{\gamma_{12345}}{s_{12}}$$

The Duality at One-Loop

At one-loop we have the following BCJ equations

$$N^P(1,2,3,4,5,q) - N^P(2,1,3,4,5,q) = N^B(12,3,4,5,q)$$
$$N^B(12,3,4,5,q) = N^B(12,4,3,5,q)$$
$$N^B(12,3,4,5,q) = N^B(12,4,5,3,q)$$



The One-Loop Result

At five-points all the external state information can be encoded in the gammas

$$\gamma_{12345} = \gamma_{12} = \delta^{(8)}(q) \frac{[12]^2 [45] [34] [35]}{\langle 12\rangle [23] \langle 35\rangle [51] - [12] \langle 23\rangle [35] \langle 51\rangle}$$

Write down an ansatz for the deltas and solve constraints

$$\left.\begin{aligned} \Delta^B_{2345}(12,3,4,5) &= \alpha_1\gamma_{12} + \alpha_2\gamma_{25} + \alpha_3\gamma_{15} + \alpha_4\gamma_{34} \\ &\quad +\alpha_5\gamma_{35} + \alpha_6\gamma_{13} \\ \Delta^P_{2345}(12,3,4,5) &= \beta_1\gamma_{12} + \beta_2\gamma_{25} + \beta_3\gamma_{15} + \beta_4\gamma_{34} + \\ &\quad \beta_5\gamma_{35} + \beta_6\gamma_{13} \end{aligned}\right\} \qquad \begin{aligned} \Delta^B_{jklm}(ij,k,l,m) &= \gamma_{ij} \\ \Delta^P_{jklm}(ij,k,l,m) &= 0 \end{aligned}$$

$$N^P(1,2,3,4,5,q) = \beta_{12345} = \frac{1}{2}\left(\gamma_{12} + \gamma_{13} + \gamma_{14} + \gamma_{23} + \gamma_{24} + \gamma_{34}\right)$$

$$N^B(12,3,4,5,q) = \gamma_{12}$$

which is in agreement with Carrasco, Johansson (2012)

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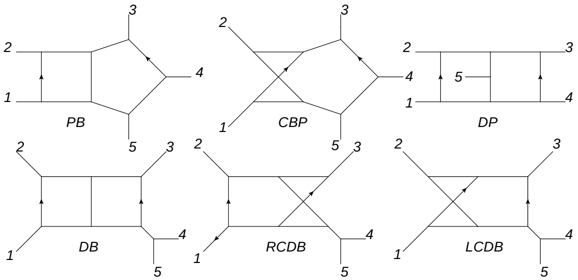
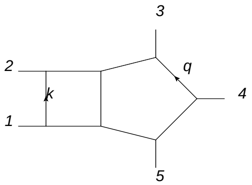
Basic Setup at Two-Loops



Carrasco, Johansson (2012),
Mastrolia, Mirabella, Ossola, Peraro (2012)

Focus on integrand decomposition of the pentabox

Basic Setup at Two-Loops



Integrand decomposition of the pentabox

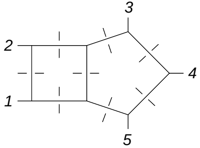
$$N^{PB}(1,2,3,4,5,q,k) = \Delta^{PB}_{12345678}(1,2,3,4,5) + \Delta^{PB}_{1235678}(1,2,3,4,5)D_4 + \Delta^{PB}_{1234678}(1,2,3,4,5)D_5 + \Delta^{PB}_{1234578}(1,2,3,4,5)D_6 + \Delta^{PB}_{1234568}(1,2,3,4,5)D_7$$

The Eightfold-Cut

The parametric form of the eightfold-cut can be found via a multivariate polynomial division

$$\Delta^{PB}_{12345678}(1,2,3,4,5) = c_0 + c_1 q\cdot p_1 + c_2 k\cdot p_4 + c_3 k\cdot p_3$$

We can fit the coefficients from unitarity cuts



$$= c_0 + c_1 q\cdot p_1 + c_2 k\cdot p_4 + c_3 k\cdot p_3$$

Solving the system gives us

$$c_0 = \delta^{(8)}(q)\frac{[12]^2\,[45]^2\,[34]\,\langle 41\rangle\,[13]}{\langle 12\rangle\,[23]\,\langle 35\rangle\,[51] - [12]\,\langle 23\rangle\,[35]\,\langle 51\rangle} \equiv \kappa_{12345}$$

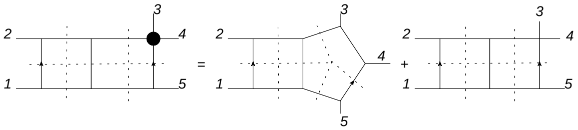
$$c_1 = 2\delta^{(8)}(q)\frac{[12]^2\,[45]\,[34]\,[35]}{\langle 12\rangle\,[23]\,\langle 35\rangle\,[51] - [12]\,\langle 23\rangle\,[35]\,\langle 51\rangle} = 2\gamma_{12345}$$

$$c_2 = 0$$

$$c_3 = 0$$

The Sevenfold-Cuts

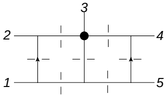
Leaving one propagator uncut we get contributions from eightfold-cut residues, which we need to subtract
e.g. leaving D_5 uncut



The parametric form of a sevenfold-cut residue has 32 coefficients
But the numerators in $\mathcal{N}=4$ sYM are only linear in the loop momentum
$\Rightarrow$ Only a constant term in the sevenfold-cut residues

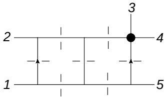
$$\Delta_{ijklmno}(1,2,3,4,5) = c_{0;ijklmno}$$

Leaving D_4 uncut



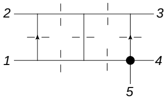
$$= c_{0;1235678} = -\frac{1}{2}\left(\gamma_{34} + \gamma_{12} + \gamma_{35} - \gamma_{45}\right)$$

Leaving D_5 uncut



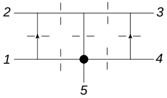
$$= c_{0;1234678} = \frac{s_{12}}{s_{34}}\gamma_{34}$$

Leaving D_6 uncut



$$= c_{0;1234578} = \frac{s_{12}}{s_{45}}\gamma_{45}$$

Leaving D_7 uncut

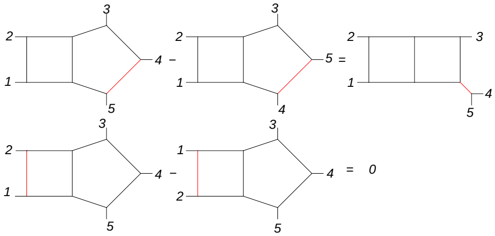


$$= c_{0;1234568} = -\frac{1}{2}\left(\gamma_{34} - \gamma_{12} + \gamma_{35} + \gamma_{45}\right)$$

The Duality at Two-Loops

The pentabox has three BCJ equations which stay in the same topology

$$N^{PB}(1,2,3,4,5,q,k) - N^{PB}(1,2,4,3,5,q,k) = N^{DB}(1,2,34,5,q,k)$$
$$N^{PB}(1,2,3,4,5,q,k) - N^{PB}(1,2,3,5,4,q,k) = N^{DB}(1,2,3,45,q,k)$$
$$N^{PB}(1,2,3,4,5,q,k) - N^{PB}(2,1,3,4,5,q,k) = 0$$



The Two-Loop Result

The BCJ equations and the Unitarity cuts give us 14 constraints on the residues

Solving the constraints we find the following pentabox and doublebox numerator

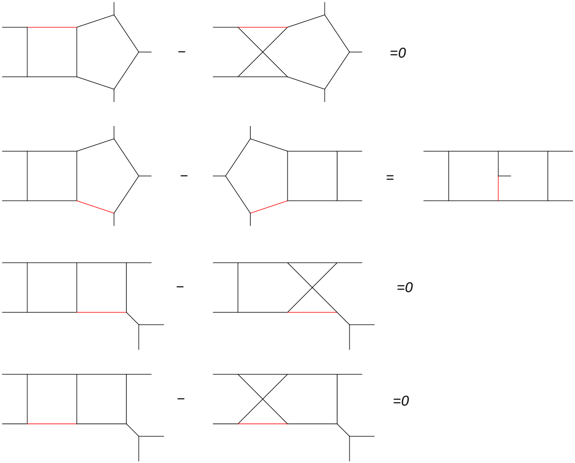
$$\begin{aligned} N^{PB}(1,2,3,4,5,q,k) &= \kappa_{12345} + 2\gamma_{12} q\cdot 1 + \frac{1}{4}\left(\gamma_{31}+\gamma_{32}-2\gamma_{12}\right) D_4 \\ &+ \frac{1}{4}\left(-\gamma_{45}+\gamma_{35}+2\gamma_{34}\right) D_5 + \frac{1}{4}\left(-\gamma_{34}+\gamma_{35}+2\gamma_{45}\right) D_6 \\ &+ \frac{1}{4}\left(\gamma_{15}+\gamma_{25}+2\gamma_{12}\right) D_7 \end{aligned}$$

$$N^{DB}(1,2,3,45,q,k) = s_{12}\gamma_{45} - \frac{s_{45}}{4}\left(-\gamma_{34}+\gamma_{35}+2\gamma_{45}\right)$$

which is in agreement with Carrasco, Johansson (2012)

The Duality between different Graphs

BCJ equations between different graphs



Obtain the other numerators from the pentabox

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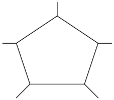
The UV poles of the One-Loop Amplitude

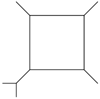
The full one-loop amplitude is

$$\mathcal{A}^{\text{1-loop}} = ig^5 \sum_{\text{all perm}} \frac{1}{10} \beta_{12345} c^P \text{Int}^P + \frac{1}{4} \frac{\gamma_{12}}{s_{12}} c^B \text{Int}^B$$

$$\text{Int}^P = \int \frac{d^D q}{(2\pi)^D} \frac{1}{D_1 D_2 D_3 D_4 D_5}$$



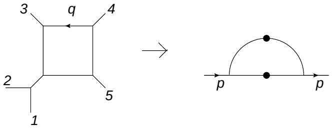
$$\text{Int}^B = \int \frac{d^D q}{(2\pi)^D} \frac{1}{D_2 D_3 D_4 D_5}$$



The leading UV divergence comes from the box integral since it has one less propagator

Small-Momentum-Injection

Reduce the graph to a two-point function by keeping a small momentum to flow through the graph



This reduces our box integral to

$$\int \frac{d^Dq}{(2\pi)^D}\frac{1}{D_2D_3D_4D_5} \to \int \frac{d^Dq}{(2\pi)^D}\frac{1}{D_1^2D_2^2}$$

We can solve this integral with the master formula

$$\int \frac{d^Dq}{(2\pi)^D}\frac{1}{D^{n_1}D^{n_2}} = i\frac{(p^2)^{D/2-n_1-n_2}}{(4\pi)^{D/2}}G(n_1,n_2)$$

and the function

$$G(n_1,n_2) = \frac{\Gamma(-D/2+n_1+n_2)\Gamma(D/2-n_1)\Gamma(D/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(D-n_1-n_2)}$$

Small-Momentum-Injection

In our case we have

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{D_1^2 D_2^2} = i \frac{(p^2)^{D/2-4}}{(4\pi)^{D/2}} G(2,2)$$

$$= i \frac{(p^2)^{D/2-4}}{(4\pi)^{D/2}} \frac{\Gamma(-D/2+4)\Gamma(D/2-2)\Gamma(D/2-2)}{\Gamma(D-4)}$$

which diverges at D=8.

In $D = 8 - 2\epsilon$ we find

$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{D_1^2 D_2^2} \overset{D=8-2\epsilon}{\longrightarrow} \frac{i}{6\epsilon(4\pi)^4}$$

The UV pole of the Full One-Loop Amplitude

Carrying out the sum over all permutations and transforming the **color factors** we arrive at the UV pole of the full amplitude

$$\mathcal{A}^{1-loop}\,|_{\mathrm{UV}} =$$

$$-g^5\frac{1}{6(4\pi)^4\epsilon}\Bigg[NTr(T^1T^2T^3T^4T^5)\left(\frac{\gamma_{12}}{s_{12}}+\frac{\gamma_{23}}{s_{23}}+\frac{\gamma_{34}}{s_{34}}+\frac{\gamma_{45}}{s_{45}}+\frac{\gamma_{51}}{s_{15}}\right)$$

$$+6Tr(T^1T^2T^3)Tr(T^4T^5)\left(\frac{\gamma_{12}}{s_{12}}+\frac{\gamma_{23}}{s_{23}}+\frac{\gamma_{31}}{s_{31}}\right)+\text{perms}\Bigg]$$

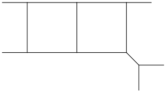
The UV Pole of the Two-Loop Amplitude

The full two-loop amplitude is

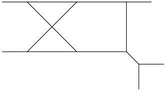
$$\mathcal{A}^{2-loop} = -g^7 \sum_{\text{all perm}} \left(\frac{1}{2} c^{BP} \text{Int}^{BP} + \frac{1}{4} c^{CBP} \text{Int}^{CBP} + \frac{1}{4} c^{DP} \text{Int}^{DP} + \frac{1}{2} c^{DB} \text{Int}^{DB} + \frac{1}{4} c^{LCDB} \text{Int}^{LCDB} + \frac{1}{4} c^{RCDB} \text{Int}^{RCDB} \right)$$

At two-loops the doubleboxes diverge first

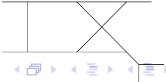
$$\text{Int}^{DB} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{N^{DB}}{s_{45} D_1 D_2 D_3 D_4 D_5 D_7 D_8}$$



$$\text{Int}^{LCDB} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{N^{LCDB}}{s_{45} D_1 D_2 D_3 D_4 D_5 D_7 D_8}$$

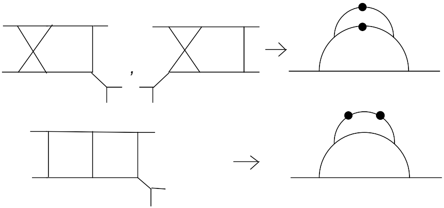


$$\text{Int}^{RCDB} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{N^{RCDB}}{s_{45} D_1 D_2 D_3 D_4 D_5 D_7 D_8}$$



Small Momentum Injection

Use small momentum injection to compute the integral



$$\xrightarrow{D=7-2\epsilon} -\frac{\pi}{30\pi^7\epsilon}$$

$$\xrightarrow{D=7-2\epsilon} -\frac{\pi}{20\pi^7\epsilon}$$

Used small-momentum-injection formula recursively

The UV pole of the Full Two-Loop Amplitude

Taking into account the sum over all permutations and transforming the color factors we find the UV divergence of the full two-loop amplitude

$$\mathcal{A}^{2-loop} = -g^7 \Big[\big(N^2 V^P + 12(V^P + V^{NP}) \big) \, Tr(T^1 T^2 T^3 T^4 T^5) \\ \Bigg(5\beta_{12345} + \frac{\gamma_{12}}{s_{12}}(s_{35} - 2s_{12}) + \frac{\gamma_{12}}{s_{12}}(s_{35} - 2s_{12}) + \frac{\gamma_{23}}{s_{23}}(s_{14} - 2s_{23}) \\ + \frac{\gamma_{34}}{s_{34}}(s_{25} - 2s_{34}) + \frac{\gamma_{45}}{s_{45}}(s_{13} - 2s_{45}) + \frac{\gamma_{51}}{s_{15}}(s_{24} - 2s_{15}) \Bigg) \\ -12N(V^P + V^{NP}) Tr(T^1 T^2 T^3) Tr(T^4 T^5) s_{45} \left(\frac{\gamma_{12}}{s_{12}} + \frac{\gamma_{23}}{s_{23}} + \frac{\gamma_{31}}{s_{31}} \right) \\ + \text{perms} \Big]$$

Summary

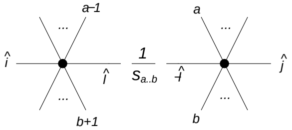
- Combined for the first time Integrand-Reduction and the Unitarity method
- Reproduced the one and two-loop amplitudes in a more systematic way
- Studied the influence of the Color-Kinematic Duality on the polynomial structure of the residues
  $\Rightarrow$ Symmetries can be used to constrain the monomials in a residue
  $\Rightarrow$ Since each monomial represents a possible master integral symmetries can reduce the number of master integrals
- Extracted the leading ultra-violet divergence for the amplitude in $\mathcal{N}$=4 sYM which can be connected through the double copy procedure to the divergence of $\mathcal{N}$=8 SUGRA
- Provided an alternative derivation of the BCJ conform numerators which only depends on geometric constraints of the amplitude (see thesis)

Backup Slides

n-Pt Generating Functions

construct higher-point generating functions via recursion relations from lower-point ones (BCFW recursion)

$$A(0) = \sum_{a,b} A_L(b+1,..,\hat{i},..,a-1,\hat{l}) \frac{1}{s_{a..b}} A_R(-\hat{l},a,..,\hat{j},..,b)$$



for MHV generating functions

$$A_n^{MHV}(\eta_1,...\eta_n) = \frac{\delta^{(8)}(\lambda_1\eta_1 + .. + \lambda_n\eta_n)}{\langle 12\rangle \langle 23\rangle .. \langle n1\rangle}$$

and for $\overline{MHV}$ amplitudes

$$A_n^{\overline{MHV}}(\eta_1,...\eta_n) = \frac{1}{[12]\,[23]\ldots[n1]} \int d^8w \prod_{i=1}^{n} \delta^4\left(\eta_i - w_{\dot\alpha}\tilde\lambda_i^{\dot\alpha}\right)$$

The Duality at One-Loop

The BCJ equations and the Unitarity cuts give us constraints on our residues

$$\Delta^P_{1345}(23,4,5,1) - \Delta^P_{1345}(13,4,5,2) + \Delta^P_{2345}(12,3,4,5) = 0$$
$$\Delta^P_{1245}(34,5,1,2) - \Delta^P_{1245}(34,5,2,1) = 0$$
$$\Delta^P_{1235}(45,1,2,3) - \Delta^P_{1235}(45,2,1,3) = 0$$
$$\Delta^P_{1234}(51,2,3,4) - \Delta^P_{1234}(52,1,3,4) + \Delta_{2345}(12,3,4,5) = 0$$
$$\Delta^B_{2345}(12,3,4,5) = \Delta^B_{2345}(12,4,3,5)$$
$$\Delta^B_{2345}(12,3,4,5) = \Delta^B_{2345}(12,3,5,4)$$
$$\frac{\Delta^B_{2345}(12,3,4,5)}{s_{12}} + \Delta^P_{2345}(12,3,4,5) = \frac{\gamma_{12}}{s_{12}}$$

Planar Two-point Integral

for the planar two-point function we find

$$V^P = i \int \frac{d^D k}{(2\pi)^D} \frac{1}{(p-k)^2} \frac{1}{(k^2)^2} \frac{(k^2)^{D/2-4}}{\pi^{D/2}} G(1,3)$$
$$= -\frac{1}{\pi^D}(k^2)^{D-7} G(1, 6-\frac{D}{2}) G(1,3)$$
$$= -\frac{1}{\pi^D}(k^2)^{D-7} \frac{\Gamma(7-D)(\Gamma(D/2-1))^2\Gamma(D-6)}{\Gamma(6-D/2)\Gamma(3/2D-7)} \frac{\Gamma(4-D/2)\Gamma(D/2-3)}{2\Gamma(D-4)}$$

which diverges at $D=7$ and gives the following pole

$$V^P \stackrel{D=7-2\epsilon}{\longrightarrow} -\frac{\pi}{20\pi^7\epsilon}$$

Non-Planar Two-point Integral

for the non-planar the first Integral is a $G(2,2)$ instead of $G(1,3)$

$$\begin{aligned} V^{NP} &= i\int \frac{d^D k}{(2\pi)^D}\frac{1}{(p-k)^2}\frac{1}{(k^2)^2}\frac{(k^2)^{D/2-4}}{(4\pi)^{D/2}}G(2,2) \\ &= -\frac{1}{\pi^D}(k^2)^{D-7}G(1,6-\frac{D}{2})G(2,2) \\ &= -\frac{1}{\pi^D}(k^2)^{D-7}\frac{\Gamma(7-D)\Gamma(D/2-1)\Gamma(D-6)}{\Gamma(6-D/2)\Gamma(3/2D-7)}\frac{\Gamma(4-D/2)(\Gamma(D/2-2))^2}{\Gamma(D-4)}. \end{aligned}$$

which also has a pole at $D=7$ giving us

$$V^{NP} \overset{D=7-2\epsilon}{\longrightarrow} -\frac{\pi}{30\pi^7\epsilon}$$