

Diphoton plus jet production through graviton exchange at NLO

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Overview

① Introduction

② GoSam

③ Golem95C

④ Higher tensor rank support

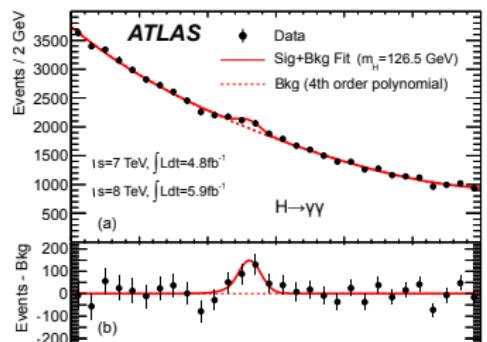
⑤ $pp \rightarrow \gamma\gamma + \text{jet}$ via Graviton decay

⑥ Summary

Introduction

Why precision calculation?

Introduction

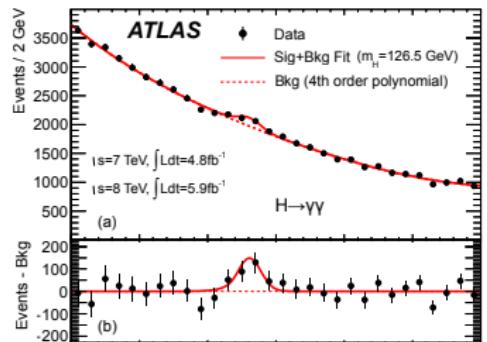


[The ATLAS Collaboration (1207.7214)]

Why precision calculation?

- Reduction of scale uncertainties
- Is the new particle the Higgs boson of the Standard Model?
- Bounds for BSM physics

Introduction

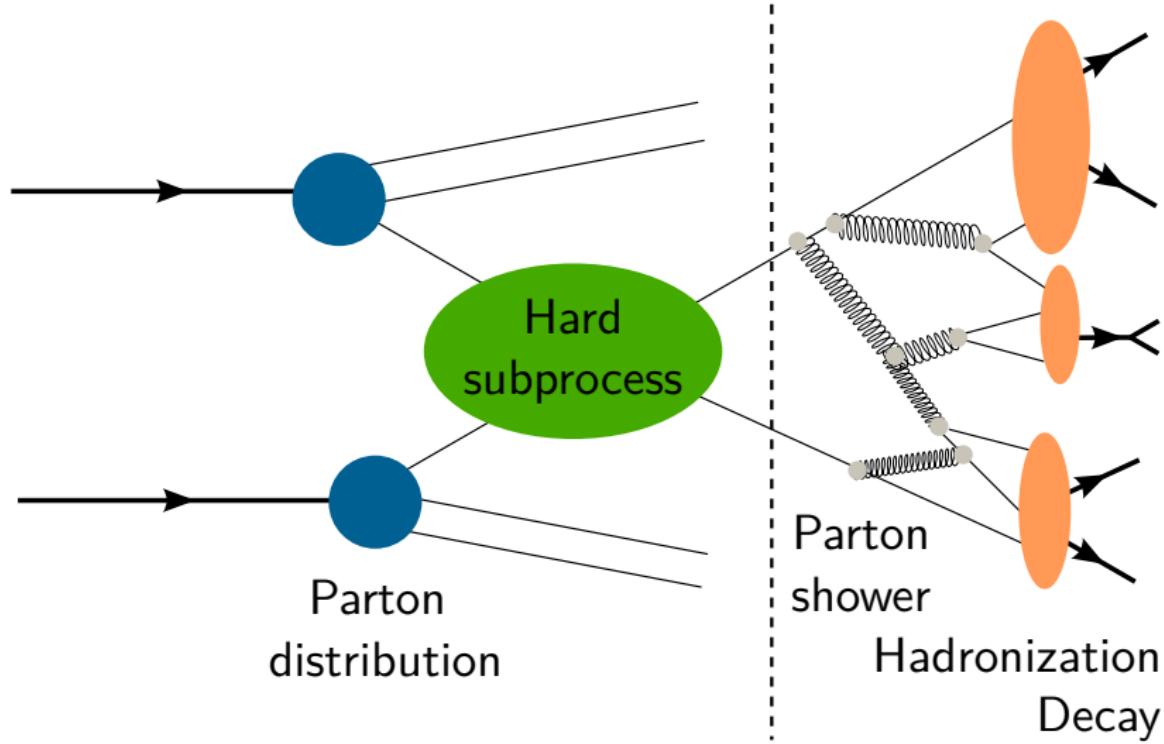


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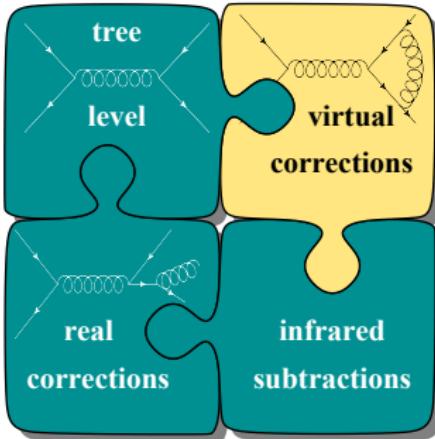
Why precision calculation?

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⇒ *next-to-leading order calculations*



NLO calculation



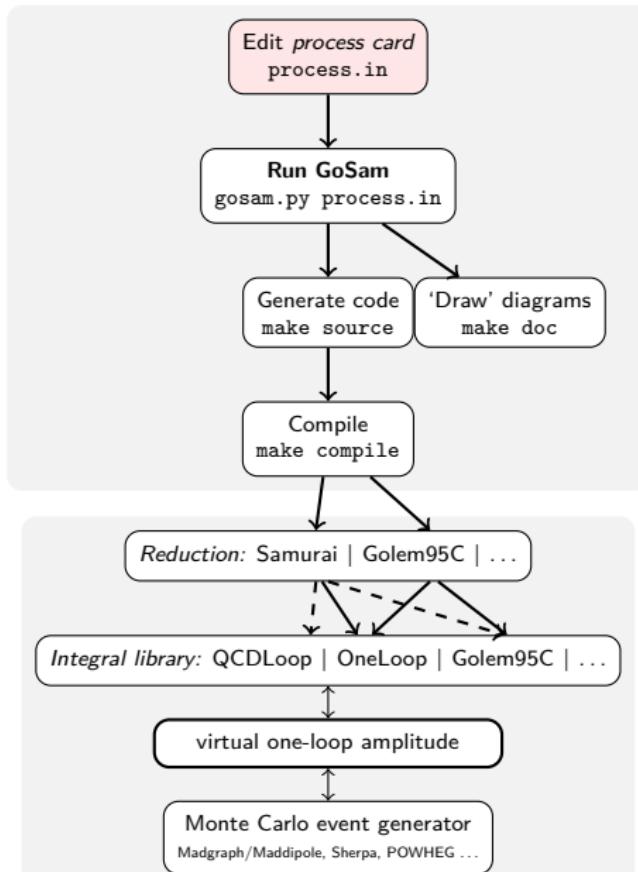
$$\sigma^{NLO} = \int_m d\sigma^B + \int_m \left[d\sigma^V + \int_1 d\sigma^A \right] + \int_{m+1} \left[d\sigma^R - d\sigma^A \right]$$

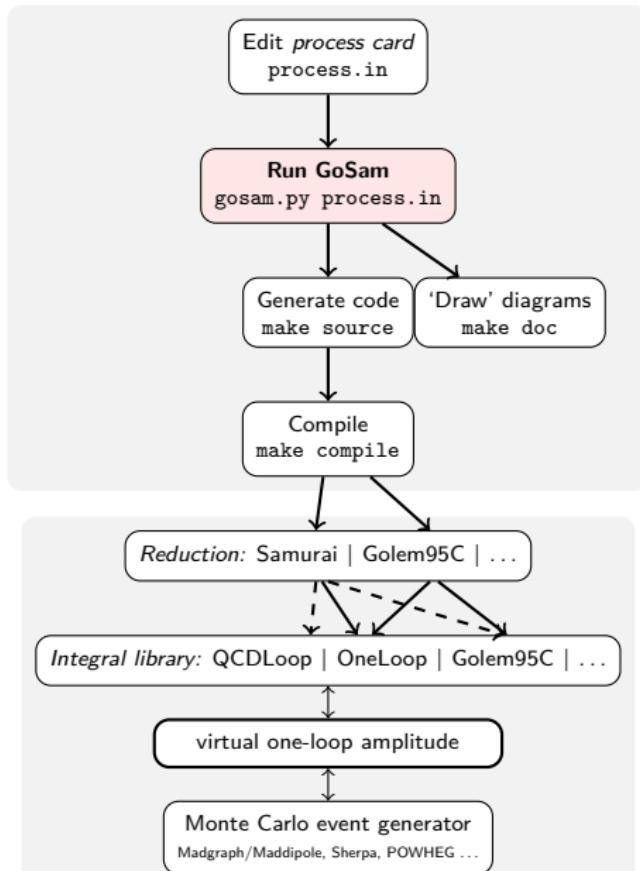
GoSam

- calculates virtual NLO corrections (one-loop multi-leg amplitudes)
- fully automated code generator
- core: integral libraries Samurai and Golem95C
- open source, uses only public tools

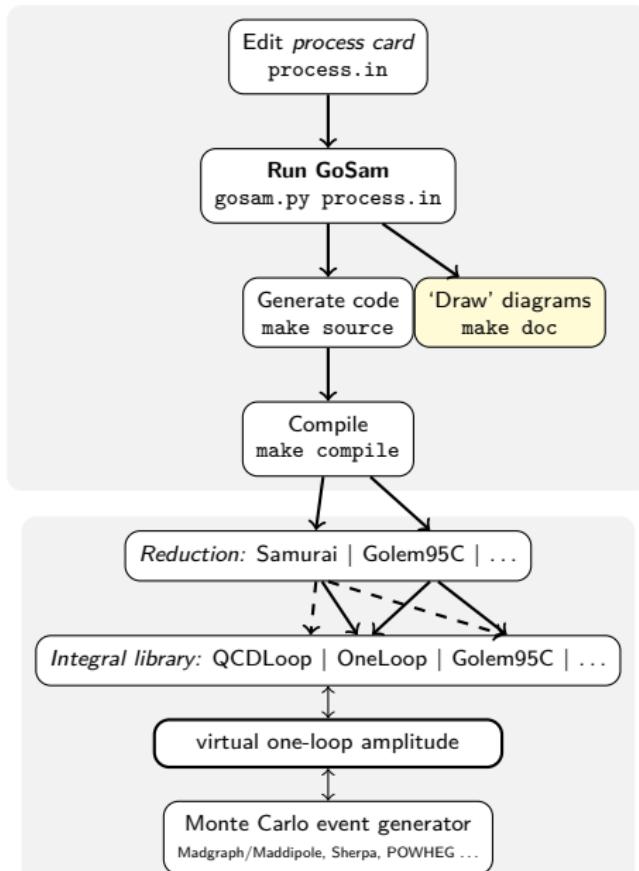
<http://projects.hepforge.org/gosam/>

[Cullen, Greiner, Heinrich, Luisoni, Mastrolia, Ossola, Reiter, Tramontano (1111.2034)]

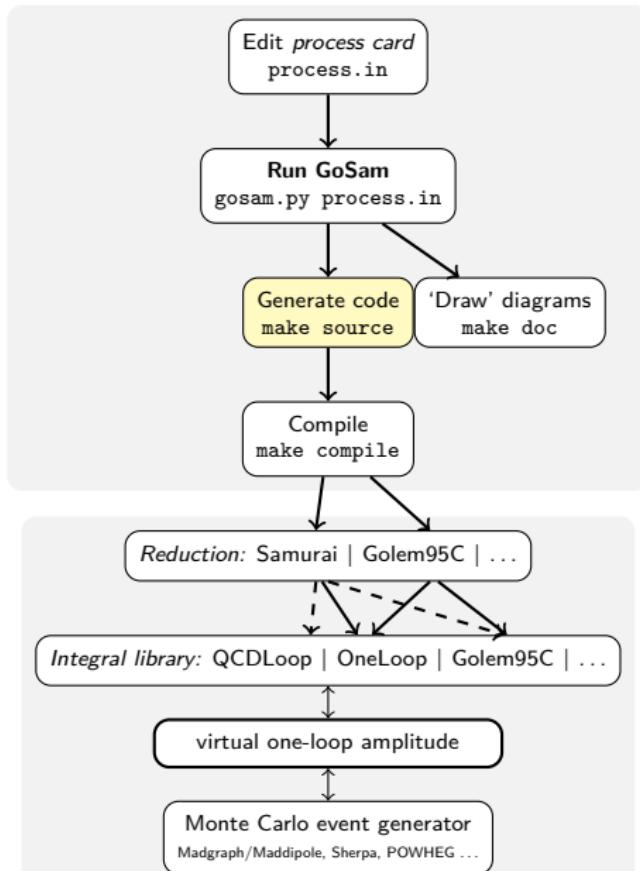




- Diagrams generated by [QGraf](#) [Nogueira]
- [L^AT_EX](#) documentation can be produced



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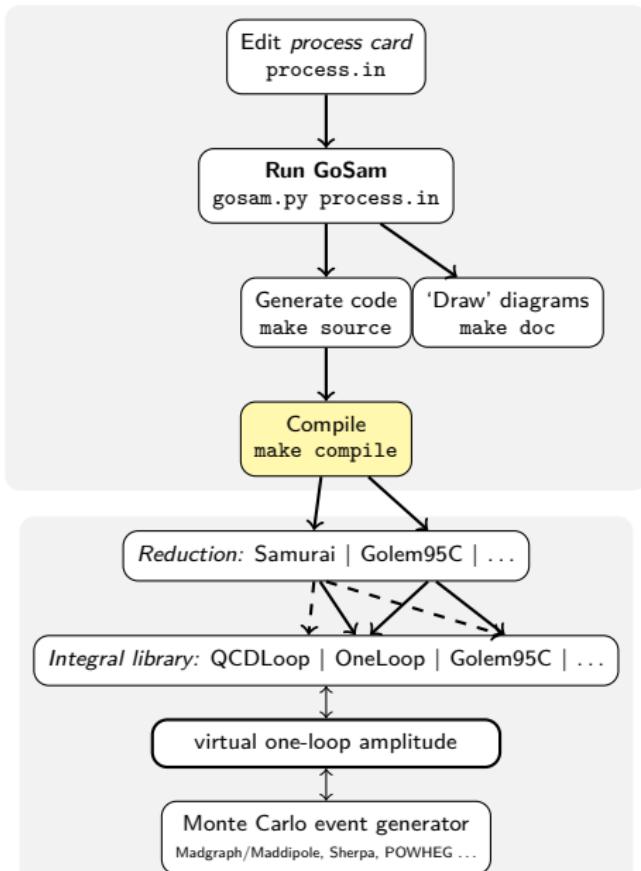
- Algebraic generation of optimized Fortran code on the fly using **FORM**

[Vermaseren (math-ph/0010025)]

- Several reduction options
 - Samurai: unitarity based
 - Golem95C: tensorial reduction

- Standard interface to Monte Carlo event generators

[Binoth et al. (1001.1307)]



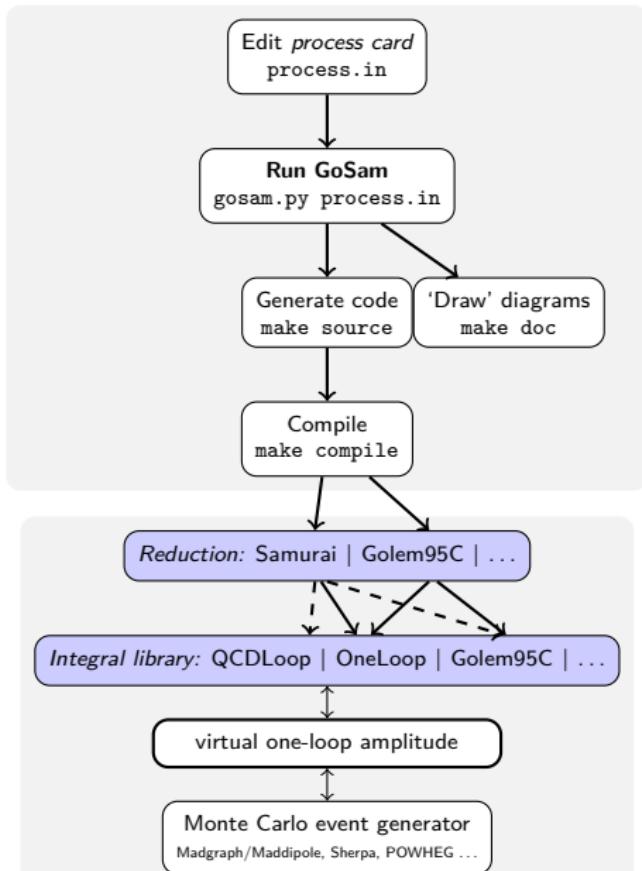
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- Algebraic generation of optimized Fortran code on the fly using FORM

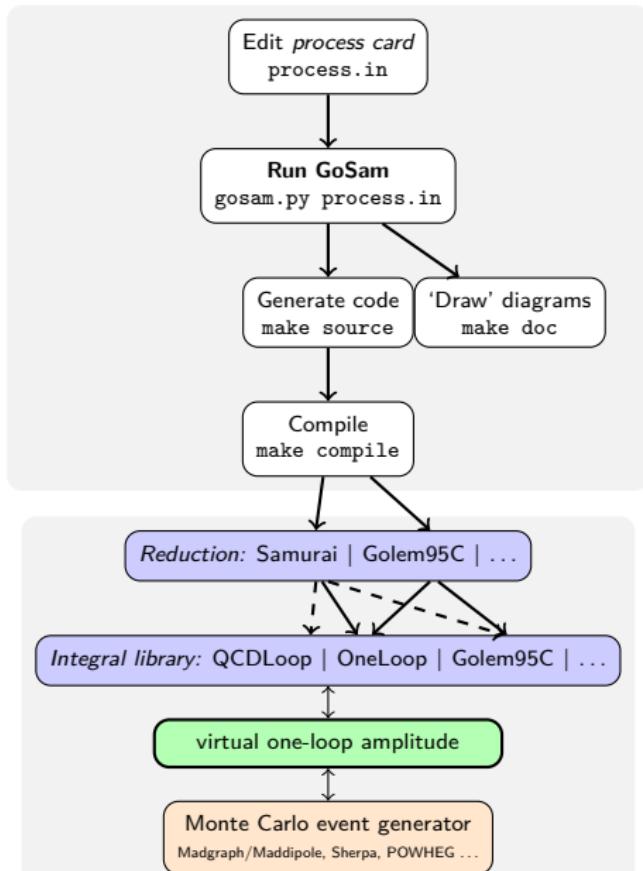
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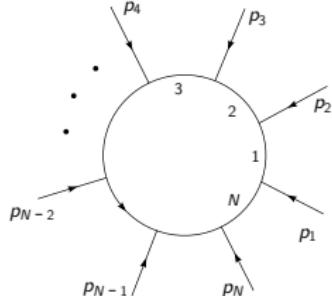
Golem95C integral library

- integral library
 - tensor coefficients up to rank 6, 6-point massive and massless integrals
 - scalar master integrals
 - support complex masses

<http://projects.hepforge.org/~golem/95/>

[Cullen, Guillet, Heinrich, Kleinschmidt, Pilon, Reiter, Rodgers (1101.5595); Binoth, Guillet, Heinrich, Pilon, Reiter (0810.0992)]

Form factor representation



$$I_N^{n,\mu_1 \dots \mu_r}(\mathcal{S}) = \int d^n k \frac{k^{\mu_1} \dots k^{\mu_r}}{\prod_{i=1}^N ((k + r_i)^2 - m_i^2 + i\delta)}$$

Form factor representation of tensor integrals:

$$I_N^{n,\mu_1, \dots, \mu_r}(\mathcal{S}) = \sum_m A^{\mu_1 \dots \mu_r}(m, r_1, \dots, r_r) \underbrace{I_N^{n+2m}(j_1, \dots, j_{r-2m}; \mathcal{S})}_{\text{Form factors}}$$

with $r_i = p_1 + \dots + p_i$

and the kinematic matrix: $\mathcal{S}_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2$

Method

- Reduction to certain set of ‘basis’ integrals
- special treatment of difficult regions \Rightarrow numerical stability
- only $r \leq N$ implemented (renormalizable theories)
 r : tensor rank, N : # of propagators

[Binoth, Guillet, Heinrich, Pilon, Schubert (hep-ph/0504267)]

Higher tensor rank extension

$$I_{\textcolor{red}{N}}^{n,\mu_1 \dots \mu_r}(S) = \int d^n k \frac{k^{\mu_1} \cdots k^{\mu_r}}{\prod_{i \in S} ((k + r_i)^2 - m_i^2 + i\delta)}$$

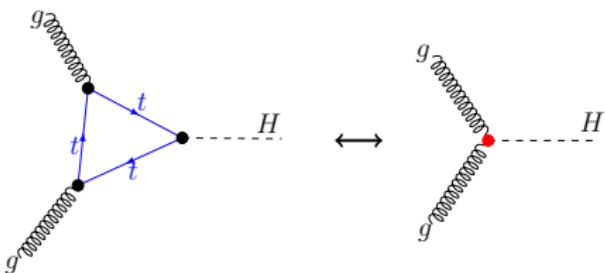
with $\boxed{r \geq N + 1}$ (especially $r = N + 1$)

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- effective couplings like ggH-vertices (top-loops integrated out)

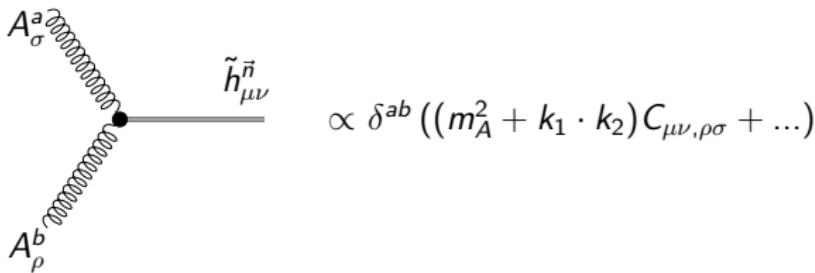


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with $\boxed{r \geq N + 1}$ (especially $r = N + 1$)

- effective couplings like ggH-vertices (top-loops integrated out)
- spin-2 particles (gravitons)



[Han, Lykken, Zhang (hep-ph/9811350)]

Implementation in Golem95C

- generic reduction formulas for arbitrary rank implemented
 - Reduction without inverse Gram matrices (Subtraction method)

- Avoid inverse Gram determinants: Subtraction method
⇒ Reduction to already implemented form factors of Golem95C

$$\begin{aligned} I_N^D(a_1, \dots, a_r; S) = & - \sum_{k=2}^r S_{a_1 a_k}^{-1} I_N^{D+2}(a_2 \dots \hat{a}_k \dots a_r; S) \\ & - b(a_1, S) \cdot (N - D - r) I_N^{D+2}(a_2, \dots, a_r; S) \\ & + \sum_{k \in S} S_{a_1 k}^{-1} I_{N-1}^D(a_2 \dots a_r; S \setminus \{k\}) \end{aligned}$$

[Binoth, Guillet, Heinrich, Pilon, Schubert (hep-ph/0504267)]

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$$I_N^{D+2}(S) = \frac{1}{B(S)} \frac{1}{D - N + 1} \left(I_N^D(S) + \sum_{k \in S} b(k; S) I_{N-1}^D(S \setminus \{k\}) \right)$$

$$b_i(S) = \sum_{k \in S} S_{ki}^{-1} \quad B(S) = \sum_{i \in S} b_i(S) = \sum_{i, k \in S} S_{ki}^{-1}$$

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Implementation in Golem95C

- generic reduction formulas for arbitrary rank implemented
 - Reduction without inverse Gram matrices (Subtraction method)
 - automatic switch to Passarino-Veltman if \mathcal{S} not invertible
- only up to boxes ($N \geq 5$ in progress)
- in addition, some explicit formulas
 - arbitrary rank tadpoles
 - massless eight-dimensional triangles
 - generic (higher-rank) massive bubbles with light-like legs
- simple cache system to prevent re-computations
- applied checks
 - comparisons between both reduction methods
 - rank $N+1$ compared with LoopTools

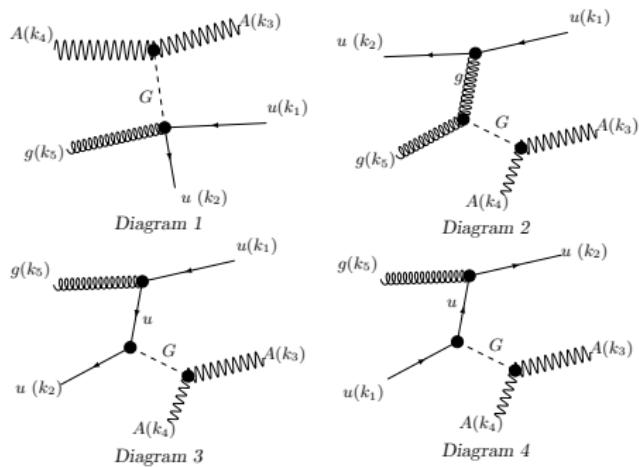
[Hahn, Perez-Victoria (hep-ph/9807565) – 2012]

Status of higher-rank support in GoSam and Samurai

- GoSam (development version)
 - support for higher rank tensor integrals
 - spin-2 particles implemented
 - arbitrary models possible
 - Higgs + up to 3 jets calculated [Deurzen, Greiner, Luisoni, Mastrolia, Mirabella et al. (1301.0493); Cullen, Deurzen, Greiner, Luisoni, Mastrolia et al. (1307.4737)]
- Samurai
 - extension to rank $N + 1$ implemented based on [Mastrolia, Mirabella, Peraro (1203.0291)]

$\gamma\gamma + \text{jet via Graviton decay}$

- Diphoton + jet production over Graviton bridge
- “automated” NLO calculation with GoSam
- ADD model [Arkani-Hamed, Dimopoulos, Dvali (hep-ph/9803315)]



ADD model

- large extra-dimensions model (LED)
- offers solution to the hierarchy problem
- 'large' means $\gg \frac{1}{M_P}$, up to mm-range
- gravitation effects at TeV scale

$$M_P^2 \sim R^\delta M_D^{\delta+2} \quad \text{in } 4 + \delta \text{ dimensions}$$

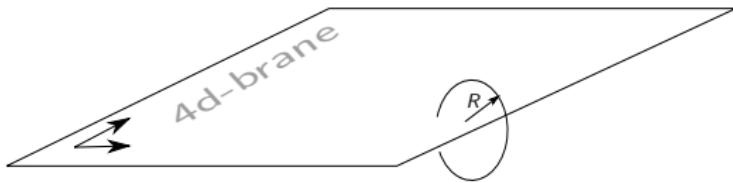
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Graviton propagator

- “Graviton” = sum over KK modes
⇒ effective propagator needed

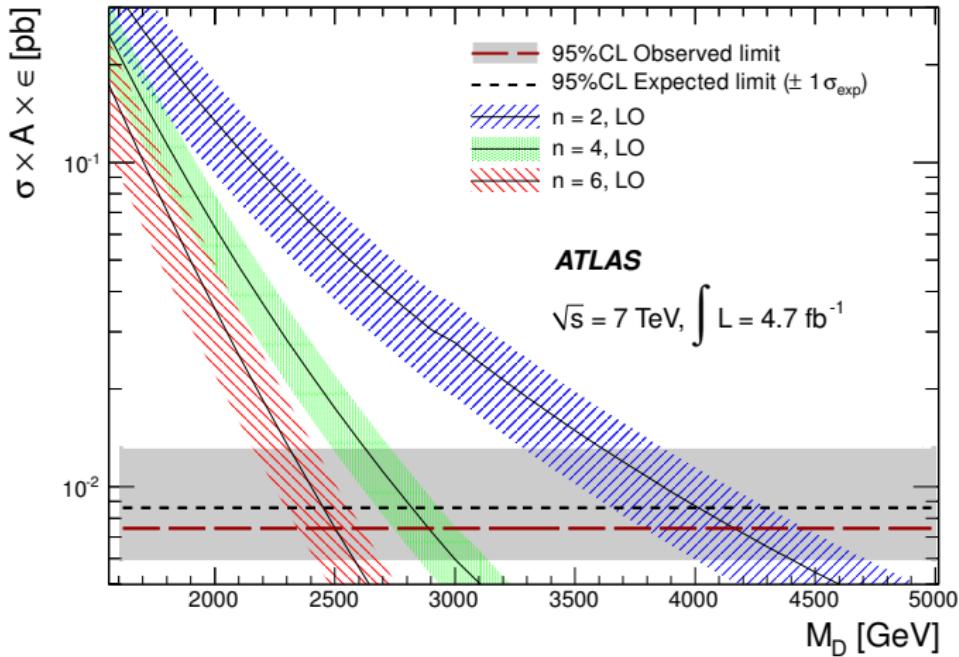
$$\sum_k \frac{i}{s - m_k^2 + i\epsilon} \approx \frac{s^{n/2-1}}{2M_s^{n+2} G_N} \left(\pi + 2i I(M_s/\sqrt{s}, n) \right)$$

where $I(x, n)$ is a smooth function.

- in GoSam handled by new `customspin2propagator` extension

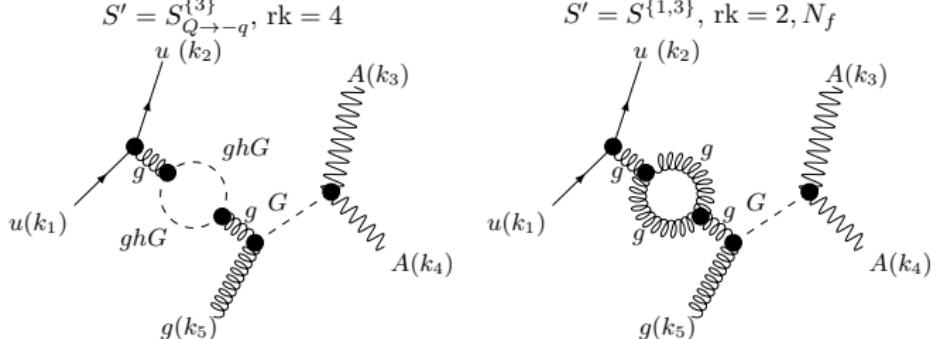
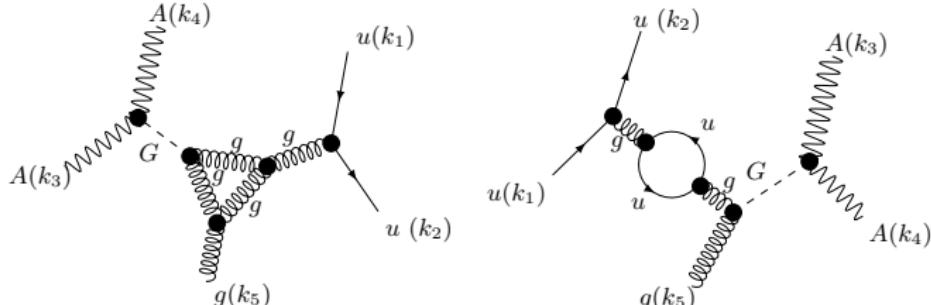
[Gleisberg, Krauss, Matchev, Schalicke, Schumann, Soff (hep-ph/0306182); Han, Lykken, Zhang (hep-ph/9811350)]

Current experimental limits



[The ATLAS Collaboration (1210.4491)]

Example diagrams produced by GoSam



Example diagrams produced by GoSam

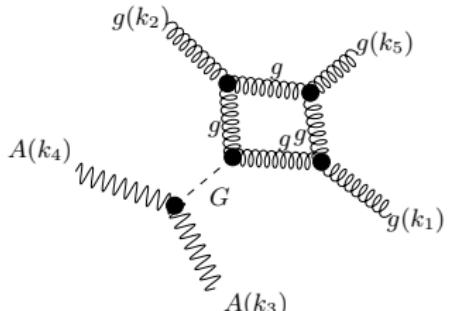


Diagram 108
 $S' = S_{Q \rightarrow q-(k_5)}$, rk = 5

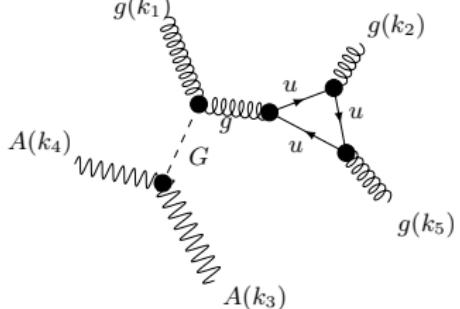


Diagram 110
 $S' = S^{\{4\}}$, rk = 3, N_f

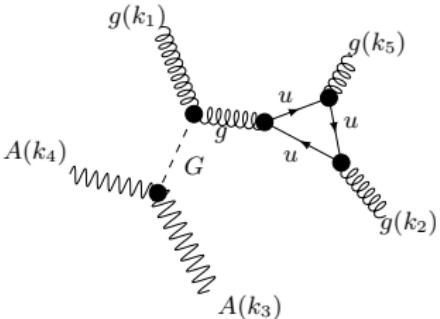


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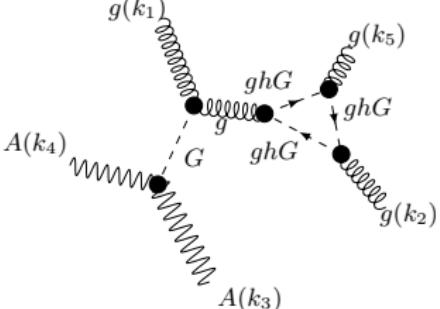
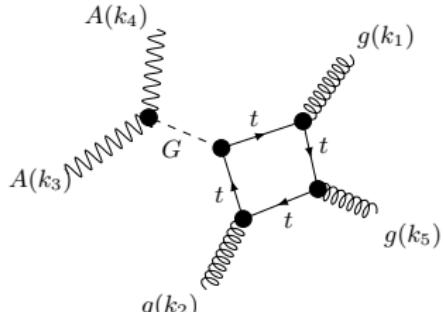


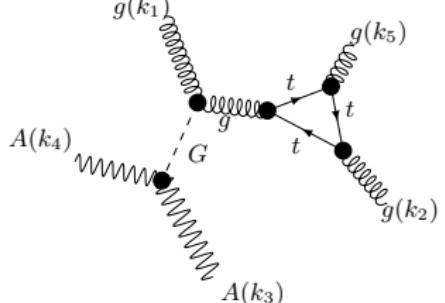
Diagram 113
 $S' = S^{\{4\}}$, rk = 3

Example diagrams produced by GoSam



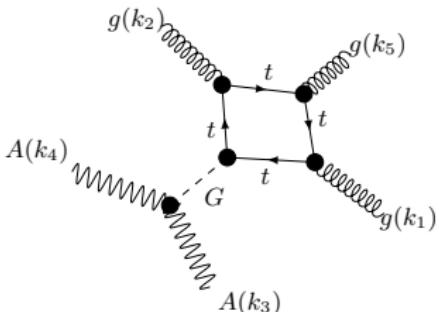
-Diagram 106

$$S' = S_{Q \rightarrow q-(k_5)}, \text{rk} = 5$$



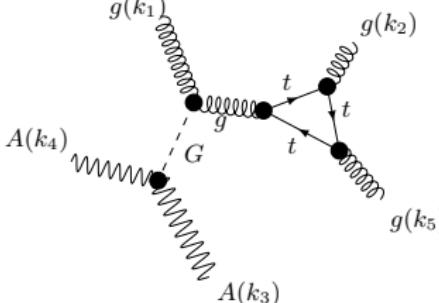
-Diagram 111

$$S' = S^{\{4\}}, \text{rk} = 3$$



-Diagram 107

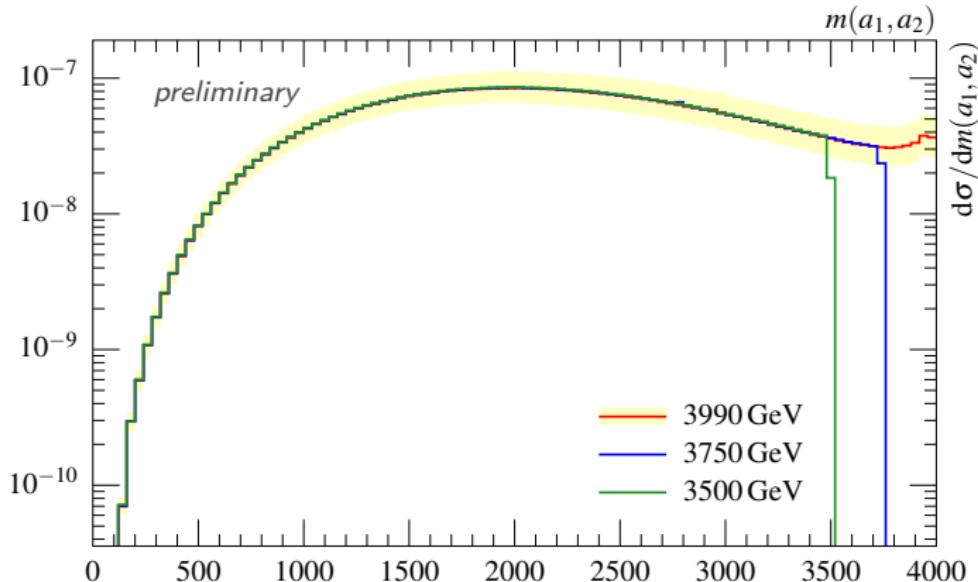
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-Diagram 112

$$S' = S^{\{4\}}, \text{rk} = 3$$

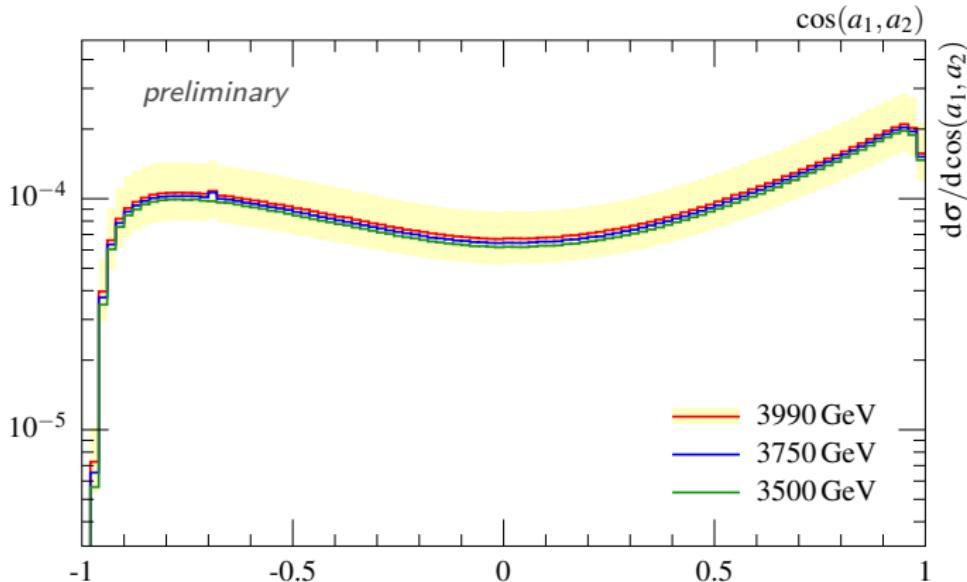
Preliminary results



invariant di-photon mass
cut-off variation (LO, $M_S = 4 \text{ TeV}$)



Preliminary results



angular distribution of photon pair
cut-off variation (LO, $M_S = 4$ TeV)



Conclusion

- GoSam supports processes with higher rank loop integrals
- Golem95C can provide arbitrary rank tensor integrals
- automated NLO calculations
 - Higgs + Jets
 - BSM physics
 - Gravitons
 - generic spin-2 particles
 - ...
- phenomenological results for di-photon plus jet production through graviton exchange in progress

Thank you!

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Passarino-Veltman Reduction

- Idea:
 - Contract tensor integrals and form factor representation with external momenta and/or metric tensor
 - solve the equation system
- Problem: Inverse Gram determinants

Gram matrix: $Z_{ij}^{(N-1)} = 2r_i r_j, \quad i, j = 1 \dots (N-1)$

[Passarino, Veltman]

Passarino-Veltman Method: Example

$$I^\mu = \int d\bar{k} \frac{k^\mu}{((k+r_1)^2 + i\delta)((k+r_2)^2 + i\delta)(k^2 + i\delta)} = r_1^\mu A_1^{3,1} + r_2^\mu A_2^{3,1}$$

contract with r_1, r_2 , use $r_i \cdot k_i = \frac{1}{2} ((r_i + k)^2 - k^2 - r_i^2)$

\Rightarrow

$$Z^{(2)} \begin{pmatrix} A_1^{3,1} \\ A_2^{3,1} \end{pmatrix} = \begin{pmatrix} 2(r_1)^2 & 2r_1 \cdot r_2 \\ 2r_1 \cdot r_2 & 2(r_2)^2 \end{pmatrix} \begin{pmatrix} A_1^{3,1} \\ A_2^{3,1} \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}$$

with

$$\begin{aligned} R_1 &= \underbrace{\int d\bar{k} \frac{1}{((k+r_2)^2 + i\delta)(k^2 + i\delta)}}_{I_2(S \setminus \{r_1\})} - \underbrace{\int d\bar{k} \frac{1}{((k+r_1)^2 + i\delta)((k+r_2^2 + i\delta)}}_{I_2(S \setminus \{r_3\})} \\ &\quad - r_1^2 \cdot \underbrace{\int d\bar{k} \frac{1}{((k+r_1+)^2 + i\delta)((k+r_2)^2 + i\delta)((k)^2 + i\delta)}}_{I_3(S)} \end{aligned}$$

$$R_2 = (r_1 \text{ with } r_1 \leftrightarrow r_2)$$

Passarino-Veltman Method: General formula

Generalized formula [Denner, Dittmaier (hep-ph/0509141)]:

$$\begin{aligned}
 I_N^D(a_1, \dots, a_r; S) = & \sum_{j \in S \setminus \{N\}} (Z^{(N-1)})_{a_1 j}^{-1} \left(-I_{N-1}^{D, r-1}(a_2, \dots, a_r; S \setminus \{j\}) \bar{\delta}_{a_2 j} \cdots \bar{\delta}_{a_r j} \right. \\
 & + I_{N-1}^{D, r-1}(a_2, \dots, a_r; S \setminus \{N\}) \bar{\delta}_{a_2 N} \cdots \bar{\delta}_{a_r N} \\
 & \left. + f_j I_N^{D, r-1}(a_2, \dots, a_r; S) + \sum_{k=1}^r I_N^{D+2, r-2}(a_2 \dots \hat{a}_k \dots a_r; S) \right)
 \end{aligned}$$

$$\begin{aligned}
 I_N^{D+2, r}(a_1, \dots, a_r; S) = & \frac{2}{N - D - r + 1} \left(I_{N-1}^{D, r}(a_1, \dots, a_r; S \setminus \{N\}) \bar{\delta}_{a_1 N} \cdots \bar{\delta}_{a_r N} \right. \\
 & \left. + 2m_N I_N^{D, r}(a_1, \dots, a_r; S) - \sum_{j \in S \setminus N} f_j I_N^{D, r+1}(a_1, \dots, a_r, j; S) \right)
 \end{aligned}$$

with $f_j = r_j - m_j + m_N$ and $\bar{\delta}_{ij} = 1 - \delta_{ij}$

Reduction without inverse gram determinant

- Idea: Split into IR-finite and IR-divergent term

$$\begin{aligned}
 I_N^D(S) &= I_{\text{div}} + I_{\text{fin}} \\
 &= \sum_{i \in S} b_i(S) \underbrace{\int d\bar{k} \frac{q_i^2 - m^2}{\prod_{j \in S} (q_j^2 - m^2 + i\delta)}}_{I_{N-1}^D(S \setminus \{i\})} + \int d\bar{k} \frac{1 - \sum_{i \in S} b_i(S)(q_i^2 - m^2)}{\prod_{j \in S} (q_j^2 - m^2 + i\delta)}
 \end{aligned}$$

Kinematic matrix: $\mathcal{S}_{ij} = (r_i - r_j)^2 - m_i^2 - m_j^2$

[Binoth, Guillet, Heinrich, Pilon, Schubert (hep-ph/0504267)]

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$$\Rightarrow I_{\text{fin}} = -B(S) (N - n - 1) I_N^{D+2}(S)$$

[Binoth, Guillet, Heinrich, Pilon, Schubert (hep-ph/0504267)]

General reduction formula

$$I_N^{D+2}(S) = \frac{1}{B(S)} \frac{1}{D - N + 1} \left(I_N^D(S) + \sum_{k \in S} b(k; S) I_{N-1}^D(S \setminus \{k\}) \right)$$

For tensors:

$$\begin{aligned} I_N^D(a_1, \dots, a_r; S) &= - \sum_{k=2}^r S_{a_1 a_k}^{-1} I_N^{D+2}(a_2 \dots \hat{a}_k \dots a_r; S) \\ &\quad - b(a_1, S) \cdot (N - D - r) I_N^{D+2}(a_2, \dots, a_r; S) \\ &\quad + \sum_{k \in S} S_{a_1 k}^{-1} I_{N-1}^D(a_2 \dots a_r; S \setminus \{k\}) \end{aligned}$$

Graviton propagator

$$\sum_k \frac{i}{s - m_k^2 + i\epsilon} \approx \frac{s^{n/2-1}}{2M_s^{n+2} G_N} \left(\pi + 2i I(M_s/\sqrt{s}, n) \right)$$

$$I(x, n) = \begin{cases} -\sum_{k=1}^{n/2-1} \frac{1}{2k} x^{2k} - \frac{1}{2} \log(x^2 - 1) & \text{if } n = \text{even} \\ -\sum_{k=1}^{(n-1)/2} \frac{1}{2k-1} x^{2k-1} + \frac{1}{2} \log \left(\frac{x+1}{x-1} \right) & \text{if } n = \text{odd} \end{cases}$$

in $4 + n$ dimensions.

[Han, Lykken, Zhang (hep-ph/9811350); Gleisberg, Krauss, Matchev, Schalicke, Schumann, Soff (hep-ph/0306182)]