The Role of Black Holes in the AdS/CFT Correspondence

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Part I: General Relativity and Black Holes

- Einstein Field Equations
- Lightcones and causal structure
- Conformal diagrams

Einsteins Field Equations

Spacetime curvature $\sim$ distribution of matter and energy

Einsteins Field Equations

$$G = T$$

Spacetime curvature $\sim$ distribution of matter and energy

G: Einstein tensor

T: Energy-momentum tensor (a.k.a. stress energy tensor)

Einsteins Field Equations

$$G[g] = \frac{8\pi G_N}{c^4} T[\phi, \psi, A_\nu, ...]$$

Spacetime curvature $\sim$ distribution of matter and energy

G: Einstein tensor

T: Energy-momentum tensor (a.k.a. stress energy tensor)

g: Metric tensor

$\phi, \psi, A_\nu, ...$: Matter content

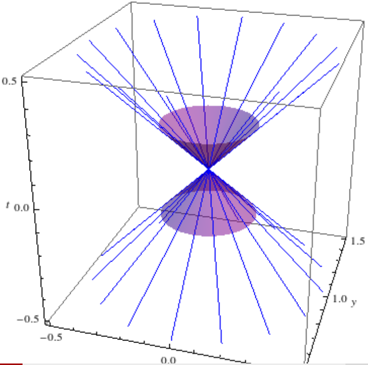
G_N: Newtons constant (often set $= 1$)

c: Speed of light (often set $= 1$)

Lightcones and causal strucutre

There are three types of curves: *timelike* (e.g. observers, realistic spaceships), *lightlike or null* (light rays) and *spacelike* (tachyons, scifi spaceships).

At every point, there is a *lightcone* generated by the light rays going through this point.



Lightcones and causal strucutre

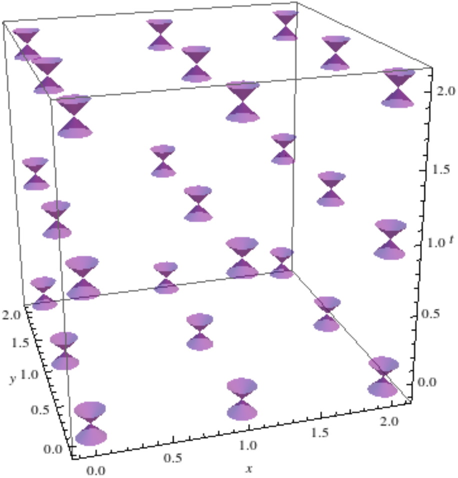


Fig.: Lightcones in flat space (Minkowski metric)

Lightcones and causal structure

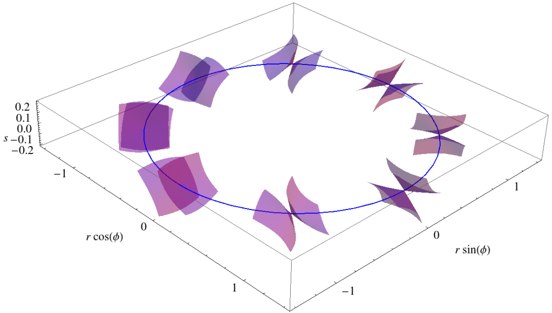


Fig.: Lightcones in Gödel spacetime: Closed causal curves!

Lightcones and causal structure

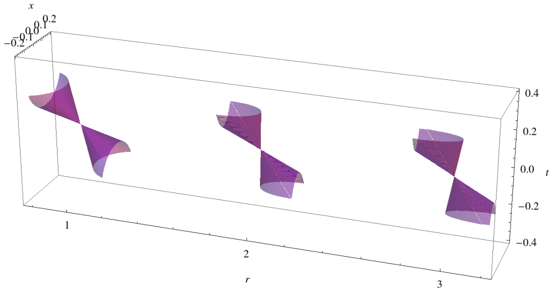


Fig.: Schwarzschild Black Hole in Gullstrand-Painlevé coordinates

Lightcones and causal structure

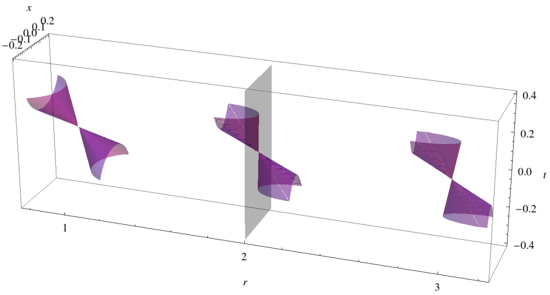


Fig.: Black Hole in Gullstrand-Painlevé coordinates: Surface of no return = event horizon!

Black Holes and Event Horizons

A region of spacetime from which no causal curve can escape outwards is called *black hole*.

Defining the precise meaning of “outward” is not as easy as it may sound.

The boundary of the black hole (“surface of no return”) is called the *event horizon*, because one cannot see events happening behind it.

Black Holes and Event Horizons

Remember:

$$G[g] = T[\phi, \psi, A_\nu, ...]$$

Spacetime curvature $\sim$ distribution of matter and energy

Known exact solutions:

- Schwarzschild, Kerr: Vacuum, i.e. $T = 0$
- Reissner-Nordström, Kerr-Newman, Vaidya: Electro-Vacuum, i.e. $T = T[A_\nu]$

Conformal Diagrams

To understand the global structure of a (spherically symmetric) black hole, we often draw *conformal diagrams*:

- Due to spherical symmetry, ignore angular coordinates $\Rightarrow$ diagram becomes two dimensional (time, radial direction).
- Use a coordinate system where lightcones are all upright and opened 90°.
- Compactify: Instead of coordinates like $x \in]-\infty, \infty[$ use e.g. $\tilde{x} = \mathrm{ArcTanh}[x], \tilde{x} \in]-1, 1[$.

You end up with a two-dimensional finite sized (i.e. drawable) diagram of the spacetime in which light rays are straight lines with slopes of $\pm 45^\circ$.

Conformal Diagrams: Flat Space

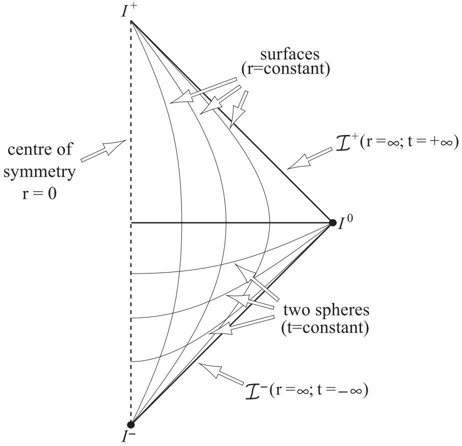


Fig.: Conformal diagram for flat (Minkowski) space

Conformal Diagrams: Asymptotically Flat Black Hole

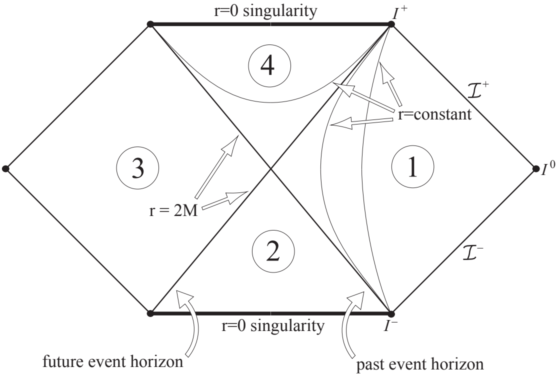


Fig.: Conformal diagram for the Schwarzschild black hole

Conformal Diagrams: Asymptotically AdS Black Hole

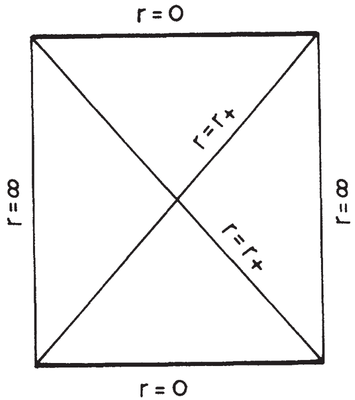


Fig.: Conformal diagram for the asymptotically AdS black hole

Part II: Properties of Black Holes

- No Hair theorem
- Entropy
- Temperature

No Hair Theorem

Kerr-Newman solution: The *most general possible* stationary asymptotically flat black hole solution with smooth event horizon in $3+1$ dimensions.

And yet: Only dependent on *three* parameters:

- Mass M
- Angular momentum J
- Electric charge Q

Black holes seem to store relatively little information about their history, i.e. they "have no hair". This theorem can be circumvented in asymptotically AdS spaces.

Black Hole Entropy

Thought experiment:

What happens when you throw a body with large entropy into a black hole?

Does entropy vanish, i.e. decrease on the global scale?

Solution:

Assign entropy to black hole.

Bekenstein-Hawking-Formula:

$\mathcal{S}_{BH} = \frac{\mathcal{A}}{4G_N}$ with horizon area $\mathcal{A}$. (Usually set $G_N = 1$)

Black Hole Temperature

If a black hole has energy E and entropy S depending on its parameters M, J and Q, then we expect quantities T, Ω, Φ to be well defined such that a first law of thermodynamics holds:

$$dE = TdS + \Omega dJ + \Phi dQ$$

T is the *Hawking-Temperature* $T = \frac{\kappa}{2\pi}$ with the *surface gravity* κ.

Due to quantum effects of matter, there is a thermal radiation associated with this temperature, the so called *Hawking radiation*. Due to this, black holes *evaporate*.

Thermodynamics of the Schwarzschild Black Hole

Schwarzschild's solution: $M \neq 0$ but $Q = J = 0$, $r_H = 2M$.
$\Rightarrow E = M,\ \mathcal{A} = 4\pi r_H^2,\ \mathcal{S} = 4\pi M^2$

Infinetesimal change in M: $dE = dM \equiv TdS = T8\pi MdM$
$\Rightarrow T = \frac{1}{8\pi M}$

What you should learn:

- Heavy (i.e. large M) Schwarzschild black holes are *large* ($r_H \sim M$) and have *much entropy* $\mathcal{S} \sim M^2$,...
- ... but they are *cooler* ($T \sim M^{-1}$) and hence evaporate *slower*.
- These black holes have *negative heat capacity*.

Summary of Black Hole Properties

- Defined by event horizon (with radius r_H).
- Parameters M, J, Q.
- Entropy $\mathcal{S} = \frac{\mathcal{A}}{4G_N}$ with area $\mathcal{A}$.
- Temperature $T = \frac{\kappa}{2\pi}$ with surface gravity κ.
- Three laws of black hole thermodynamics analogous to ordinary thermodynamics.

Part III: Black Holes in AdS/CFT

- Temperature
- Chemical Potential
- Entanglement and Temperature
- Thermalization

Temperature

- Step 1: Take AdS-Schwarzschild black hole (with metric g), and calculate the analytic continuation to *euclidean space* (g_E).
- Step 2: In order to avoid a *conical singularity* in g_E, the euclidean time τ has to be identified with $\tau \sim \tau + 2\pi\beta$ where $\beta = \frac{1}{T}$ and T is the Hawking temperature.
- Step 3: Evaluate the *euclidean (gravitational) action* $S_E[g_E]$ of this metric.
- Step 4: According to AdS/CFT: $\mathcal{Z}_{\text{CFT,therm}} \equiv \mathcal{Z}_{\text{grav}} = e^{-S_E[g_E]}$ in the *saddle point approximation*.

Exampe Calculation: AdS_4 Schwarzschild Black Brane

g_E only depends on parameters M (determining r_H) and AdS-scale L. Demanding the absence of conical singularities, we find the temperature $T = \frac{3r_H}{4\pi}$.

$$
\begin{aligned}
S_E[g_E] &= \frac{-1}{\lambda} \int d^4x \sqrt{g_E} \left(R[g_E] + \frac{6}{L^2} \right) + \text{boundary terms} \\
&= -\frac{(4\pi)^3 L^2}{3^3 \lambda} V T^2
\end{aligned}
$$

where V is the spatial volume of the CFT. We find the quantities:

$$\text{Free Energy: } F = -T \ln \mathcal{Z} = T S_E[g_E] = -\frac{(4\pi)^3 L^2}{3^3 \lambda} V T^3$$

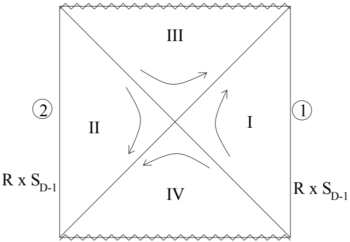
$$\text{Entropy: } S = -\frac{\partial F}{\partial T} = \frac{(4\pi)^3 L^2}{3^2 \lambda} V T^2 = \frac{\mathcal{A}}{4G} \text{ as expected using } \lambda = 16\pi G_N$$

Chemical Potential

- Step 1a: Take (charged) AdS-Reissner Nordström black hole (with metric g), and calculate the analytic continuation to *euclidean space* (g_E).
- Step 1b: The electromagnetic potential will have the form $A_\nu = A_0 \delta^0_\nu$ with near boundary expansion $A_0(z) = \mu + ...$ μ is identified as *chemical potential* and will be ralated to the charge Q of the black hole.
- Steps 2-4: Proceed as in uncharged case.

Different Approach to Temperature: Entanglement

Remember: AdS-Black hole has *two* conformal boundaries, i.e. is dual to *two* CFTs 1 and 2.



These two CFTs cannot communicate, but are entangled!

Different Approach to Temperature: Entanglement

The total Hilbertspace is $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, and the combination of the CFTs 1 and 2 is in the (pure!) state

$$|\Psi\rangle = \frac{1}{\sqrt{\mathcal{Z}(\beta)}} \sum_n e^{-\beta E_n/2} |E_n\rangle_1 |E_n\rangle_2$$

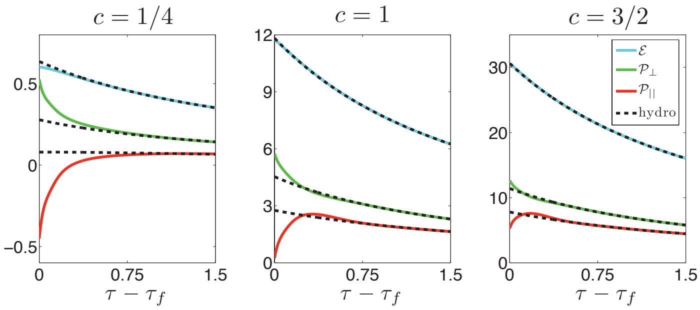
with density matrix $\rho_{\text{tot}} = |\Psi\rangle \langle\Psi|$. But the *reduced denisty matrix* of CFT 1

$$\rho_1 = \text{Tr}_2 [\rho_{\text{tot}}] = \frac{1}{\mathcal{Z}(\beta)} e^{-\beta H_1}$$

is the (mixed!) denisty matrix of a CFT at temperature $T = \frac{1}{\beta}$.

Thermalization

If a static black hole in the bulk corresponds to a CFT on the boundary with a certain temperature T, what does the *formation* of a black hole correspond to?
Answer: *Thermalization*, i.e. approach of a system towards thermal equilibrium.



Thanks a lot