Engineering Calabi-Yau manifolds

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Motivation

Our world has 4 spacetime dimensions, strings live in ten. Since

$$10 > 4$$

we need compactification manifolds.

The goal of this talk is to

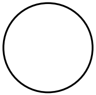
- shed some light on their construction
- discuss what geometric properties we can hope to compute

Defining manifolds

Very roughly speaking, there are two relevant ways of defining a manifold / variety. Consider the circle S^1:

- Explicitly - parametrize it by a coordinate and give its range:
  $x \sim x + 2\pi r, \quad x \in [0, 2\pi r]$
- Implicitly - embed it into a higher-dimensional, but simpler space:
  $x^2 + y^2 = r^2, \quad (x, y) \in \mathbb{R}^2$

Either way, one describes the same circle:



Defining manifolds

“Simple“ spaces like $\mathbb{R}^n$ or $\mathbb{C}^n$ are best defined explicitly. However, more complicated spaces often have no such description. Therefore we define our manifold as

$$p(x_i) = 0\,,$$

where p is a *polynomial* in the coordinates of the ambient space. $p = 0$ defines a *hypersurface*. Since p is a polynomial, we have an *algebraic* problem: $\Longrightarrow$ Use *algebraic* geometry!

Before focusing on the hypersurface, let us take a look at appropriate ambient spaces.

Ambient spaces

What properties do we want our ambient space to have?

1. Simple to describe, e.g. simple coordinate ranges
2. Under best possible mathematical control, so that we have better control over the hypersurface, too
3. Complex, i.e. parametrized by complex coordinates
4. Compact

Condition 4 forbids vector spaces. What other "simple" spaces are there?

Projective space

Projective spaces come to the rescue:

- Real projective space: $\mathbb{RP}^n = \mathbb{R}^{n+1} \backslash \{\vec{0}\} / \sim$, where $\vec{x} \sim \lambda \vec{x}$ for $\lambda \in \mathbb{R} \backslash \{0\}$. $\mathbb{RP}^n =$ space of rays in $\mathbb{R}^{n+1}$.
- Complex projective space: $\mathbb{CP}^n = \mathbb{C}^{n+1} \backslash \{\vec{0}\} / \sim$, where $\vec{x} \sim \lambda \vec{x}$ for $\lambda \in \mathbb{C} \backslash \{0\}$.

We describe $\mathbb{CP}^n$ using the redundant coordinates of $\mathbb{C}^{n+1}$ and call them *homogeneous* coordinates.

Notation: $[z_0 : z_1 : \cdots : z_n] \in \mathbb{CP}^n$.

Projective space, examples

Let's see how $\mathbb{CP}^1$ differs from $\mathbb{C}$ by studying its points $[z_0 : z_1]$:

- If $z_0 \neq 0$, rescale by $\lambda = 1/z_0$ to obtain $[1 : z_1/z_0] = [1 : z_1']$, which describes the same point. z_1' can take any value, so $[z_0 : z_1]$ with $z_0 \neq 0$ is just $\mathbb{C}$.
- If $z_0 = 0$, then $z_1 \neq 0$ and we rescale by $1/z_1$: $[0 : z_1] = [0 : 1]$, which is a single point.

Hence we find that $\mathbb{CP}^1 = \mathbb{C} + pt$. $\mathbb{CP}^1$ is the compactification of $\mathbb{C}$ obtained by adding a point at infinity and topologically $\mathbb{CP}^1 \simeq S^2$.

Projective spaces are "nicer" from a mathematical point of view than $\mathbb{C}^n$. For example, in S^2 two straight lines always intersect twice - no special case for parallel lines.

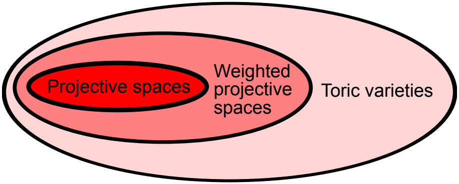
Weighted projective space: Allow e.g. $(z_0, z_1, z_2) \sim (\lambda^3 z_0, \lambda^2 z_1, \lambda z_2)$

More general spaces

Lastly, there is yet another generalization: Start with $\mathbb{C}^n$, remove some lower-dimensional piece Z and impose multiple equivalence relations of the previous kind:

$$X = \mathbb{C}^n \backslash Z / \sim$$

All such spaces are *toric varieties*. Obviously, we have:

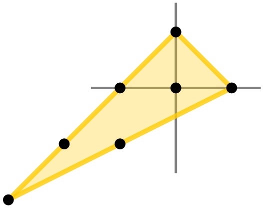


Toric varieties

Why are toric varieties interesting? Their defining data is given by discrete numbers and is hence *combinatorial*.

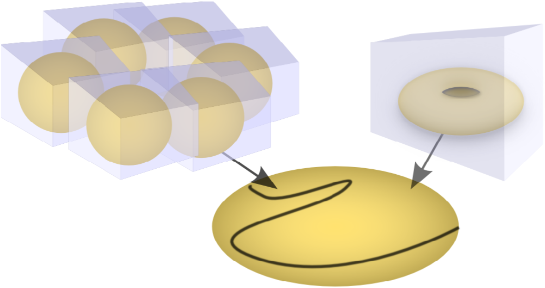
Combinatorial data can often be visualized:

- Consider the vectors $\vec{v}_1 = (1,0), \vec{v}_2 = (0,1), \vec{v}_3 = (-3,-2)$.
- They are related by $3 \cdot \vec{v}_1 + 2 \cdot \vec{v}_1 + \cdot \vec{v}_3 = \vec{0}$.
- One thus associates them with $\mathbb{C}^3 \backslash \{\vec{0}\} / \sim$ where $(z_0, z_1, z_2) \sim (\lambda^3 z_0, \lambda^2 z_1, \lambda z_2)$. This is just $\mathbb{WP}^2_{3,2,1}$.



Fibrations

Next, we would like to construct torus fibered Calabi-Yau manifold for F-theory. (cf Federico's talk)
To do so, construct a fibered ambient space as in

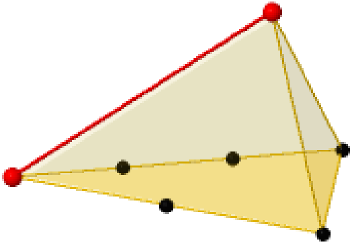


where the ambient fiber space becomes reducible over the location of the GUT branes:

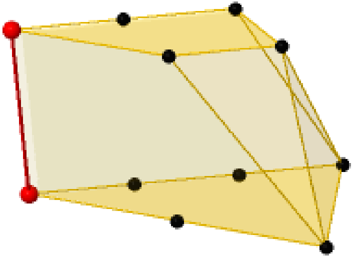
$$F \mapsto F_1 \cup F_2 \cup \cdots \cup F_n$$

Resolved fiber singularities

We can then study the ambient fiber over different places in the base and read off its form.



Fiber over generic point:
T^2 ambient space



Fiber over GUT brane:
affine Dynkin diagram of gauge group

Conclusion

In summary, the main message is:

$$\text{Constructing Calabi-Yau manifolds with many continuous parameters} \xrightarrow{\text{can be reduced to}} \text{Manipulating discrete combinatorial toric data}$$

Toric ambient spaces are hence useful for the following reasons:

- There is a large number of them and many more spaces can be embedded in them.
- Their theory is well understood and allows to compute cohomology classes and intersection numbers.
- Their data is combinatorial and can easily and efficiently be handled by a computer.

Thank you!