Implications of number theory for calculations of scattering amplitudes

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Outline

1. Multiple Zeta Values (MZV)
   - Basics
   - Recent developments
   - Application in string theory
2. Multiple Polylogarithms (MPL)
   - Basics
   - Recent developments
   - Application in QFT

historical motivation

- $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty$
- $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots =$

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  Basel problem, solved by Leonard Euler in 1735
- generalisation:
  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}$

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multiple zeta values (MZV)

- $\zeta(s_1,\ldots,s_k) = \sum_{n_1>\cdots>n_k>0} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}, \quad s_i \in \mathbb{N}, \quad s_1 > 1$

$$k \text{ - length}, \quad \sum_{i=1}^{k} s_i \text{ - weight}$$

- many relations, e.g. $\zeta(2)\zeta(3) = \frac{9}{4}\zeta(5) - \frac{1}{2}\zeta(2,3)$
- sum theorem: $\sum_{s_1+\cdots+s_k=n} \zeta(s_1,\ldots,s_k) = \zeta(n)$, e.g. $\zeta(2,1)=\zeta(3)$

$$\begin{aligned}\zeta(5) &= \zeta(4,1)+\zeta(3,2)+\zeta(2,3)\\ &= \zeta(3,1,1)+\zeta(2,2,1)+\zeta(2,1,2) = \zeta(2,1,1,1)\end{aligned}$$

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- basis of MZV:

weight N	conjectured basis of $\mathcal{Z}_N$	d_N	MZV of weight N
2	$\zeta(2)$	1	1
3	$\zeta(3)$	1	2
4	$\zeta(2)^2$	1	4
5	$\zeta(5), \zeta(2)\zeta(3)$	2	8
6	$\zeta(2)^3, \zeta(3)^2$	2	16
7	$\zeta(7), \zeta(2)\zeta(5), \zeta(2)^2\zeta(3)$	3	32
8	$\zeta(2)^4, \zeta(2)\zeta(3)^2, \zeta(3)\zeta(5), \zeta(3,5)$	4	64

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New insights by A. Goncharov and F. Brown: MZV and mixed Tate motives.
Goal: Find an algebra that provides all MZV identities.

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→ There is a map ϕ from the $\zeta^m(s_1, ..., s_k)$ to polynomials in f_i, e.g.

- $\phi(\zeta^m(n)) = f_n$
- $\phi(\zeta^m(3, 5)) = -5 f_5 f_3$

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superstring amplitude

Stieberger and Schlotterer [1205.1516]

- disk-level N-point open superstring amplitude: $\mathcal{A} = F \; A$
  - vector $\mathcal{A}$ - $(N-3)!$ color ordered string subamplitudes
  - vector A - $(N-3)!$ color ordered YM subamplitudes
  - elements F_{ij} of the matrix F - generalized Euler integrals, depending on string tension α' and kinematical invariants
- α'-expansion: $F_{ij} = \delta_{ij} + \alpha'^2 p_{ij,2}\zeta(2) + \alpha'^3 p_{ij,3}\zeta(3) + \ldots$
  $p_{ij,k}$ - polynomials of degree k in the kinematical invariants, which appear at order $\mathcal{O}(\alpha'^k)$

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- decomposition of F: $\mathcal{A} = P\ Q\ M\ A$
  - even single zetas are found in $P = 1 + \sum_{n\geq 1} \alpha'^{2n}\zeta(2)^n P_{2n}$.
  - odd single zetas in $M =: \exp\left\{\sum_{n\geq 1} \alpha'^{2n+1}\zeta(2n+1)M_{2n+1}\right\}$:
  - and MZV in $Q = 1 + \sum_{n\geq 8} \alpha'^n Q_n$, e.g. $Q_8 = \frac{1}{5}\zeta(3,5)\,[M_5, M_3]$

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  $\mathcal{A}|_{\mathcal{O}(\alpha'^8)} =$
  $\left(\zeta(2)^4 P_8 + \frac{1}{2}\zeta(2)\zeta(3)^2 P_2 M_3^2 + \zeta(3)\zeta(5) M_5 M_3 + \frac{1}{5}\zeta(3,5)\,[M_5, M_3]\right) A$

superstring amplitude Stieberger and Schlotterer [1205.1516]

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motivic superstring amplitude

Use the concept of motivic MZV, e.g.

- $\phi(Q_8^m) = f_5 f_3 \, [M_3, M_5]$,
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- The complete (motivic) amplitude takes this simple form:

$$\phi(\mathcal{A}^m) = \left(\sum_{k=0}^{\infty} f_2^k P_{2k}\right) \left(\sum_{p=0}^{\infty} \sum_{\substack{i_1,\ldots,i_p \\ \in 2\mathbb{N}^+ +1}} f_{i_1} \ldots f_{i_p} M_{i_p} \ldots M_{i_1}\right) A$$

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mutliple polylogarithms (MPL)

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- multiple polylogarithm: $\mathrm{Li}_{s_1,\ldots,s_k}(x_1,\ldots,x_k) = \sum_{n_1>\cdots>n_k>0} \frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$

$$= (-1)^k G(\underbrace{0,\ldots,0}_{s_k-1}, \frac{1}{x_k}, \ldots, \underbrace{0,\ldots,0}_{s_1-1}, \frac{1}{x_1\cdots x_k})$$

- many relations, e.g.

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \log(x)\log(1-x)$$

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- (classical) polylogarithm: $\mathrm{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$
- multiple polylogarithm: $\mathrm{Li}_{s_1,\ldots,s_k}(x_1,\ldots,x_k) = \sum_{n_1>\cdots>n_k>0} \frac{x_1^{n_1}\cdots x_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$

$$= (-1)^k G(\underbrace{0,\ldots,0}_{s_k-1}, \tfrac{1}{x_k}, \ldots, \underbrace{0,\ldots,0}_{s_1-1}, \tfrac{1}{x_1\cdots x_k})$$

- many relations, e.g.

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \log(x)\log(1-x)$$

symbol $S(f)$ of a MPL f

- definition: $S(f) = \sum_a r_a \otimes S(f_a)$, where the cut a of f goes from $r_a = 0$ to $r_a = \infty$ with discontinuity $2\pi i f_a$
- e.g.:

$$S(\text{const.}) = 0, \qquad S(\log(x)) = x,$$

$$\begin{aligned} S(\log(x)\log(y)) &= x \otimes y + y \otimes x \\ S(\mathrm{Li}_s(x)) &= -(1-x) \otimes \underbrace{x \otimes \cdots \otimes x}_{s-1} \end{aligned}$$

- Relations of MPL become simple algebraic identities, e.g.

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \log(x)\log(1-x)$$

$$\xrightarrow{S} -(1-x) \otimes x - x \otimes (1-x) = 0 - x \otimes (1-x) - (1-x) \otimes x$$

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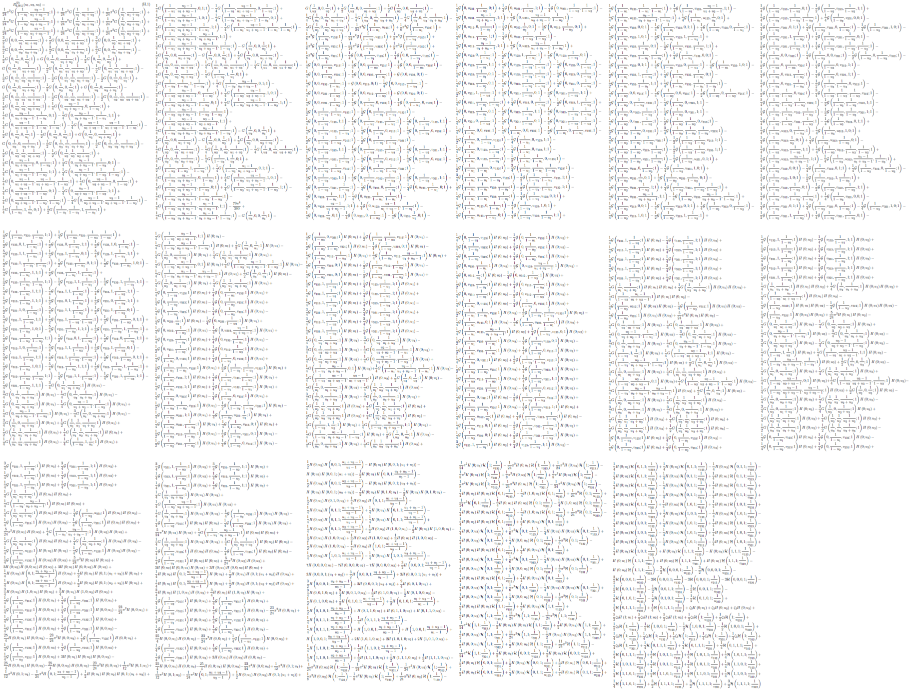

$$\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-\frac{1}{4}G\left(\frac{1}{u_1},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{\dots},0,\frac{1}{\dots},\frac{1}{\dots};1\right)-\frac{1}{4}G\left(\frac{1}{\dots},1,\frac{1}{\dots},0;1\right)+$$

simplification using the symbol

Goncharov et al. [1006.5703]

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) - \frac{1}{2}\mathrm{Li}_4(1-1/u_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2(1-1/u_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

with some abbreviations:

$$x_i^\pm = u_i x^\pm, \quad x^\pm = \frac{u_1+u_2+u_3-1\pm\sqrt{\Delta}}{2u_1u_2u_3}$$

$$\Delta = (u_1+u_2+u_3-1)^2 - 4u_1u_2u_3$$

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m(l_{4-m}(x^+) + l_{4-m}(x^-))$$

$$l_n(x) = \frac{1}{2}(\mathrm{Li}_n(x) - (-1)^n\mathrm{Li}_n(1/x))$$

$$J = \sum_{i=1}^{3}(l_1(x_i^+) - l_1(x_i^-))$$