Sector Decomposition

Johannes Schlenk

Max Planck Institute for Physics

Particle Physics School Colloquium

November 21, 2014

Max-Planck-Institut für Physik
(Werner-Heisenberg-Institut)

Outline

- Calculation of loop amplitudes
- Feynman integrals
- Sector decomposition
- Geometric decomposition of vacuum integral
- SecDec
- Conclusions

Scattering Amplitudes

Scattering amplitudes in perturbative field theory:

$$\mathcal{A} = \alpha^n \left(A_0 + \alpha^1 A_1 + \alpha^2 A_2 + \dots \right)$$

Calculation of L-loop amplitude:

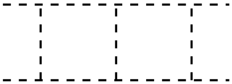
$$A_L = \sum (L\text{-loop Feynman diagrams})$$

Reduction to master integral basis:

$$A_L = \sum_i a_i \mathrm{MI}_i$$

Loop order independent: Integration by parts method Chetyrkin, Tkachov (1981)

Feynman Integrals



$$I = \int \prod_{i=1}^{L} \mathrm{d}^D k_i \frac{1}{\prod_{j=1}^{N} \left(q_j^2 - m_j^2 \right)}$$

Dimensional regularization

$$D = 4 - 2\epsilon$$

$$I = \frac{c_{-2L}}{\epsilon^{2L}} + \frac{c_{-2L+1}}{\epsilon^{2L-1}} + \dots$$

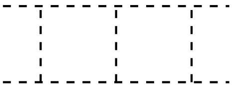
Feynman parameterization

$$I \sim \int_0^\infty \mathrm{d}^N x \, \delta(H) \frac{\mathcal{U}^{N-(L+1)D/2}}{\mathcal{F}^{N-LD/2}}$$

$\delta(H)$ projects to an arbitrary hyperplane $H = 0$

$\mathcal{U}, \mathcal{F}$: Polynomials in the integration variables x

Feynman Integrals



$$I = \int \prod_{i=1}^{L} \mathrm{d}^D k_i \frac{1}{\prod_{j=1}^{N} \left(q_j^2 - m_j^2 \right)}$$

Dimensional regularization

$$D = 4 - 2\epsilon$$

$$I = \frac{c_{-2L}}{\epsilon^{2L}} + \frac{c_{-2L+1}}{\epsilon^{2L-1}} + \dots$$

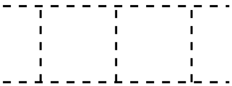
Feynman parameterization

$$I \sim \int_0^\infty \mathrm{d}^N x \, \delta(H) \frac{\mathcal{U}^{N-(L+1)D/2}}{\mathcal{F}^{N-LD/2}}$$

$\delta(H)$ projects to an arbitrary hyperplane $H = 0$

$\mathcal{U}, \mathcal{F}$: Polynomials in the integration variables x

Feynman Integrals



$$I = \int \prod_{i=1}^{L} \mathrm{d}^D k_i \frac{1}{\prod_{j=1}^{N} \left(q_j^2 - m_j^2 \right)}$$

Dimensional regularization

$$D = 4 - 2\epsilon$$

$$I = \frac{c_{-2L}}{\epsilon^{2L}} + \frac{c_{-2L+1}}{\epsilon^{2L-1}} + \dots$$

Feynman parameterization

$$I \sim \int_0^\infty \mathrm{d}^N x \, \delta(H) \frac{\mathcal{U}^{N-(L+1)D/2}}{\mathcal{F}^{N-LD/2}}$$

$\delta(H)$ projects to an arbitrary hyperplane
$H = 0$
$\mathcal{U}, \mathcal{F}$: Polynomials in the integration variables x

Sector Decomposition

Binoth, Heinrich (2000)

- Decompose integration region into sectors with simple singularity structure
- Expand in the regularization parameter ϵ
- Integrate the finite coefficients (numerically)

Decomposition strategies:

Authors	# of sectors	infinite recursion	Description
Binoth, Heinrich	small	possible	heuristic algorithm
Bogner, Weinzierl	large	no	based on Hironaka's polyhedra game
Smirnov, Tentyukov	smallish	no	hybrid of the first two
Kaneko, Ueda	small	no	Geometric strategy

Sector Decomposition

Binoth, Heinrich (2000)

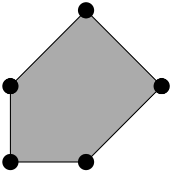
- Decompose integration region into sectors with simple singularity structure
- Expand in the regularization parameter ϵ
- Integrate the finite coefficients (numerically)

Decomposition strategies:

Authors	# of sectors	infinite recursion	Description
Binoth, Heinrich	small	possible	heuristic algorithm
Bogner, Weinzierl	large	no	based on Hironaka's polyhedra game
Smirnov, Tentyukov	smallish	no	hybrid of the first two
Kaneko, Ueda	small	no	Geometric strategy

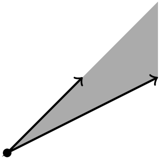
Geometric Objects

Convex polytope



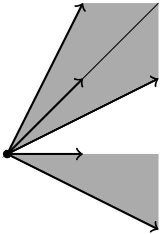
Defined by:
vertices/facets

Polyhedral cone

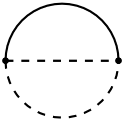


Defined by:
rays/facets

Fan



Example: Vacuum integral



$$\sim \int_0^\infty \mathrm{d}x_1 \mathrm{d}x_2 \frac{1}{P(x_1, x_2)^{1-\epsilon}}$$

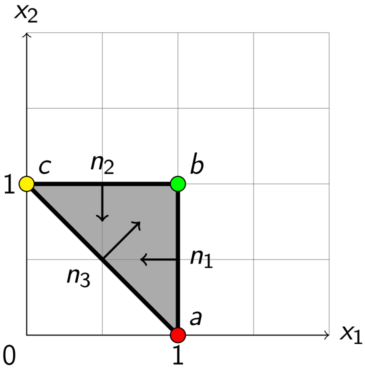
$$P(x_1, x_2) = x_1 + x_1 x_2 + x_2$$

Geometric Method - Newton Polytope

Vertices:

$$P(x_1, x_2) = x_1^1 x_2^0 + x_1^1 x_2^1 + x_1^0 x_2^1$$

$$v_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



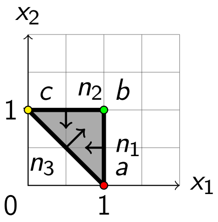
Facet normals:

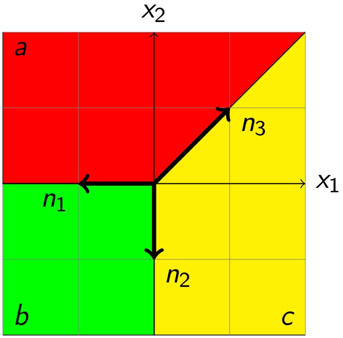
$$n_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, n_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, n_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Efficient codes for the calculation of convex polytopes are available (e.g. Normaliz Bruns, Ichim, Roemer, Soeger)

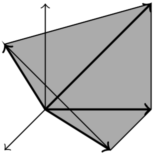
Geometric Method - Fan

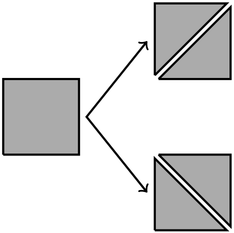
- Convex polytope defines associated fan
- Vertices map to cones
- Facets map to rays





Generalization to higher dimensions - Triangulation





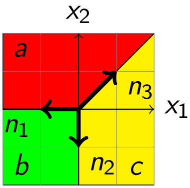
- In $d > 2$ dimensions cones can be defined by more than d rays
- Simplicial cones are defined by d rays
- Triangulation: Decomposition of a cone into simplicial cones
- Automated codes for the triangulation are available (Normaliz)

Transformation of variables

- Sectors defined by cones
- Map integration variables back to the unit hypercube
- Sector a:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} n_3 & n_1 \end{pmatrix} \otimes \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow \begin{array}{rcl} x_1 & = & z_1 z_2^{-1} \\ x_2 & = & z_1 \end{array}$$

$$\begin{array}{rcl} P(x_1, x_2) & = & x_1 + x_1 x_2 + x_2 \\ & \rightarrow & z_1 z_2^{-1} + z_1^2 z_2^{-1} + z_1 \\ & = & z_1 z_2^{-1} (1 + z_1 + z_2) \end{array}$$



- Include Jacobi determinant
- Result for all three sectors:

$$\int_0^1 \mathrm{d}z_1 \mathrm{d}z_2 \left(\frac{z_1^{\epsilon}}{z_2^{1+\epsilon}} + \frac{1}{z_1^{1+\epsilon} z_2^{1+\epsilon}} + \frac{z_1^{\epsilon}}{z_2^{1+\epsilon}} \right) \frac{1}{(1 + z_1 + z_2)^{1-\epsilon}}$$

Subtraction

- Expansion in dimensional regularization parameter ϵ
- Introduce subtraction terms that regulate singularities and can be integrated analytically

$$\int_0^1 \mathrm{d}x \frac{f(x)}{x^{1-\epsilon}}$$

$$= \int_0^1 \mathrm{d}x \frac{f(0)}{x^{1-\epsilon}} + \int_0^1 \mathrm{d}x \frac{f(x) - f(0)}{x^{1-\epsilon}}$$

$$= \frac{f(0)}{\epsilon} + \int_0^1 \mathrm{d}x \frac{f(x) - f(0)}{x^{1-\epsilon}}$$

- Repeat for all singular integration variables

SecDec

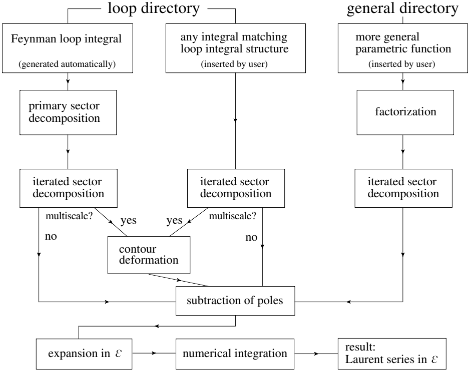
- Public implementation of the sector decomposition method

  Heinrich, Carter (2010); Borowka, Heinrich (2011)

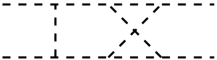
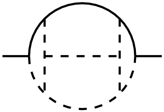
  http://secdec.hepforge.org/

SecDec3

- Improved user interface
- New decomposition strategy
- Speed improvements
- Coming soon



Comparison of decomposition strategies

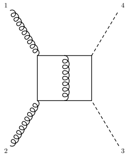
Diagram	Standard decomp.	Geometric decomp.
	282	166
	395	235
	40962 ($\sim$ 3.8hrs)	11488 ($\sim$ 2min)

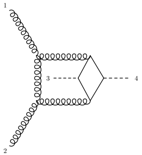
Complexity of produced functions important for the numerical integration

Conclusions and Outlook

- Sector Decomposition: Algorithm for the automated calculation of Feynman integrals
- The geometric strategy produces a small number of sectors and is guaranteed to stop
- SecDec: Public implementation of the Sector Decomposition algorithm

- Upcoming version of SecDec
- Application to the calculation of unknown processes (e.g. $gg \to HH$)



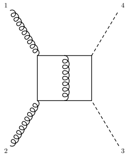


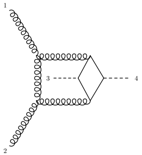
Thank you for your attention

Conclusions and Outlook

- Sector Decomposition: Algorithm for the automated calculation of Feynman integrals
- The geometric strategy produces a small number of sectors and is guaranteed to stop
- SecDec: Public implementation of the Sector Decomposition algorithm

- Upcoming version of SecDec
- Application to the calculation of unknown processes (e.g. $gg \to HH$)





Thank you for your attention