Gauge Theory and the Geometry of Elliptic Curves

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Outline

mathematical theory of elliptic curves

connection via F-theory

gauge theories

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gauge theories

1. Simple Introduction to Elliptic Fibrations and F-theory
2. Recap of Anomalies in Quantum Field Theory
3. Anomaly Cancelation in F-theory $\Leftrightarrow$ Symmetries of Elliptic Fibrations

Fibrations

Fibration (roughly):

- base space
- fiber space
- total space (fibration)

Fibration

"To each point in a base space a fixed fiber space is attached!"

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Fibration

"To each point in a base space a fixed fiber space is attached!"

- total space looks **locally** like "base x fiber"

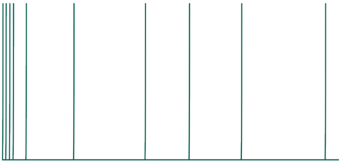
Example Trivial Fibration: Line x Line

Building blocks:

Base

Fiber

Fibration:



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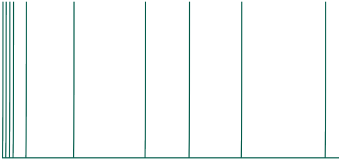


$\longrightarrow$

$\rightarrow$ trivial fibration because simple product space: line x line

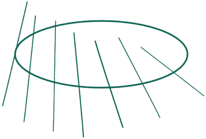
Non-Trivial Fibration: Möbius Strip

Building blocks:

Base

Fiber

Fibration:



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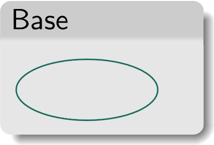

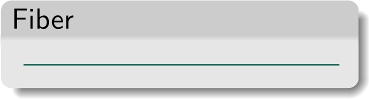

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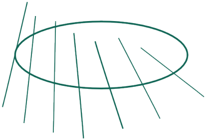

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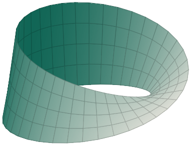


$\rightarrow$ non-trivial fibration because the fiber gets **twisted** when going around the circle

$\rightarrow$ still locally trivial: line x line

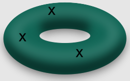
Elliptic Fibration

Building blocks:

Base

arbitrary space

Fiber



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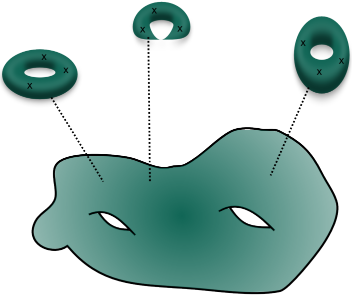

- elliptic curve: torus with special points (K-rational points)
  → later more
- shape of the torus varies over the base

Sections

Section:

- smooth map: base $\rightarrow$ total space
- point in the base $\mapsto$ point in the fiber over it

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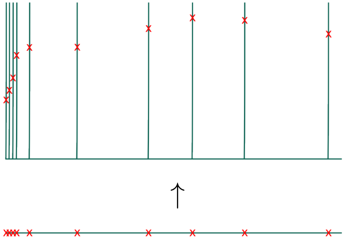

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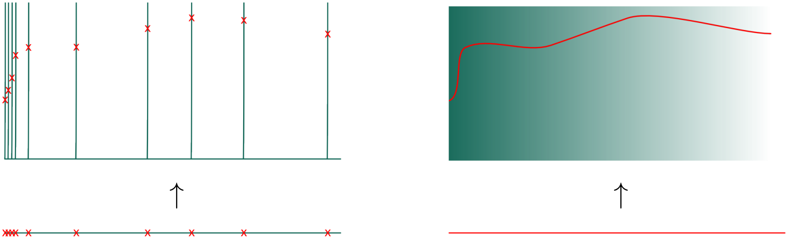
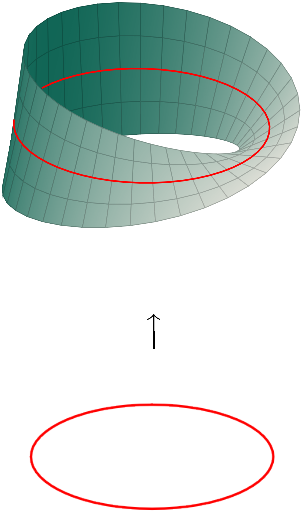


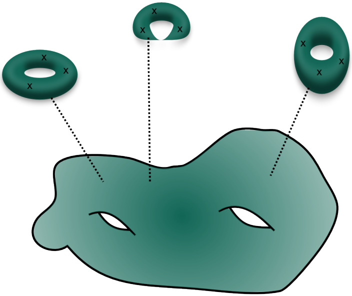
$\rightarrow$ smooth embedding of the base space into the total space

Example: a simple section of the Möbius strip



Sections of elliptic fibrations:

- elliptic curve: torus with special points (K-rational points)
- elliptic fibration: fiber elliptic curve over some base space
- a section marks a single point in each fiber
  $\Rightarrow$ the K-rational points of elliptic curves define (rational) sections of the elliptic fibration



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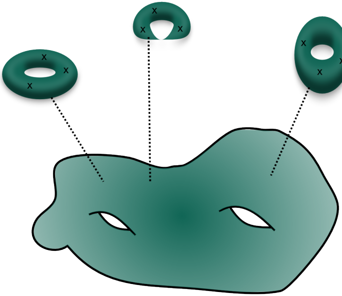

F-theory

rational sections $\Leftrightarrow$ $U(1)$ gauge symmetries

Rational points on elliptic curves

Describe spaces by polynomial equations:

Sphere (in $\mathbb{R}^3$)

$x^2 + y^2 + z^2 = 1$

Elliptic Curve/Torus (in $\mathbb{P}^2_{2,3,1}$)

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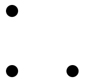
Rational points: Solutions to these equations with $(x, y, z) \in \mathbb{Q}^3$

($\rightarrow$ cf. Fermat's last theorem: integer solutions to $x^n + y^n = z^n$ for $n \geq 3$)

Suppose you have found n linear independent rational solutions $(x, y, z) \in \mathbb{Q}^3$:

- rational solutions can be “added” in a tricky way to get further rational solutions
  $\rightarrow$ group structure for rational points $\cong \mathbb{Z}^{n-1}$

Example: Group of rational points for $n = 3$: $\mathbb{Z}^2$



zero-point/origin

generators of the group of rational points

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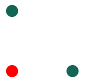

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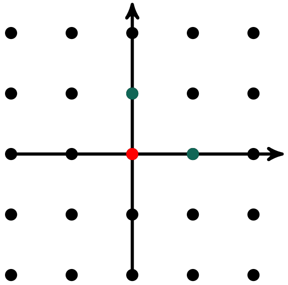

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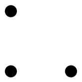

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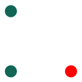

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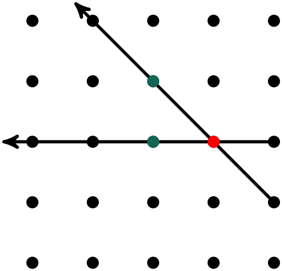

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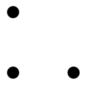

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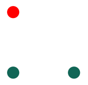

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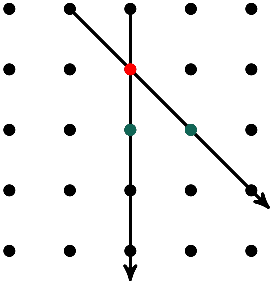

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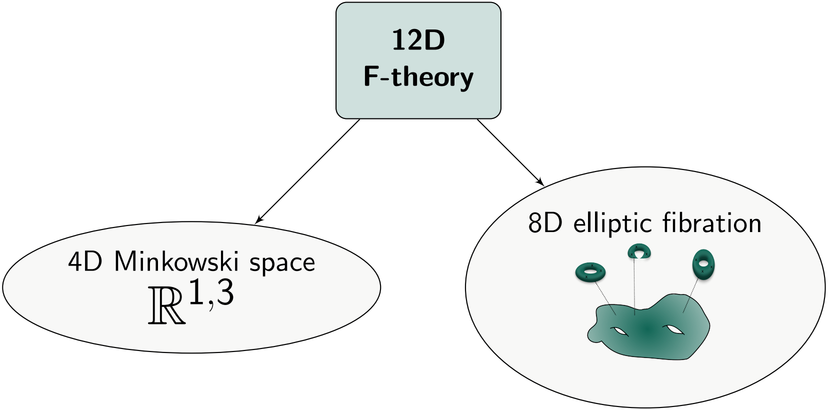
generators of the group of rational points

Connection to F-theory

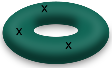
F-theory: 12D string theory → make 8 dimensions small



Properties of the elliptic fibration determine the 4D effective theory:

- singular elliptic fiber $\Leftrightarrow$ non-Abelian gauge symmetry

- n rational sections $\Leftrightarrow$ $n-1$ $U(1)$ gauge symmetries:



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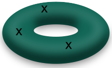

  - 1 rational section → zero-section (origin of the group of rational sections)
  - $n-1$ rational sections → $U(1)$ gauge symmetries (generators of the group of rational sections)

→ corresponds to choosing a basis for the rational points on the torus

invariance of basis choice ⇒ cancelation of Abelian gauge anomalies

Anomalies in Quantum Field Theory

Transformation of the path integral under a classical symmetry:

$$\int \mathcal{D}\Phi\, e^{iS[\Phi]} \mapsto \int \mathcal{D}\Phi\, e^{i\epsilon \int d^4x\, \mathcal{A}(x)}\, e^{iS[\Phi]}$$

- classical action $S[\Phi]$ invariant by definition
- path integral measure $\mathcal{D}\Phi$ could transform in general
  $\Rightarrow$ quantum theory has an anomaly $\mathcal{A}(x)$
- no problem for global symmetries
- disaster for gauge symmetries (gauge symmetry $\equiv$ redundancy)

Condition

$$\mathcal{A}(x) \stackrel{!}{\simeq} 0$$

Possibilities for cancelation:

- add local counterterms $\rightarrow$ irrelevant anomaly
- choose matter fields appropriately
- exploit a tree-level mechanism (Green-Schwarz mechanism)

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$\Rightarrow$ **Constraint on the matter fields:**

$U(1)$ anomaly cancelation condition

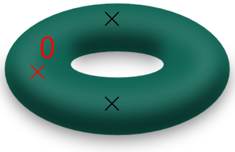
$$\sum_{\text{fermions}} q^3 \overset{!}{=} 0$$

Anomaly Cancelation in F-theory

F-theory on elliptic fibrations:

Change the choice of the zero-section:

- change choice of origin for the group structure of rational sections
- theory should be invariant under choice of zero-section



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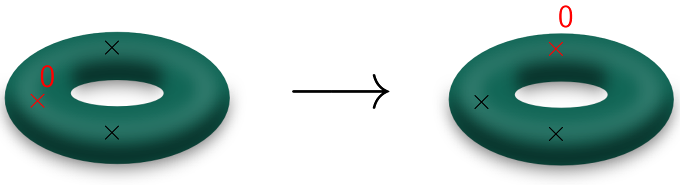

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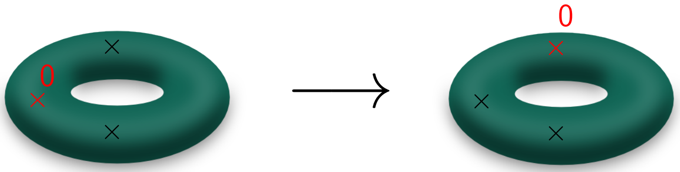

This symmetry reproduces the $U(1)$ gauge anomaly cancelation conditions.

$U(1)$ anomaly cancelation condition

$$\sum_{\text{fermions}} q^3 \stackrel{!}{=} 0$$

Conclusions

- concept of fibration, section
- elliptic curve as torus with special points, forming a group
  $\rightarrow$ elliptic fibrations with rational sections
- F-theory compactified on elliptic fibrations: $U(1)$ gauge symmetries given by rational sections
- basis invariance of rational sections $\Rightarrow$ cancelation of Abelian gauge anomalies
- similar story for non-Abelian gauge anomalies

Outlook

- map further algebraic properties of elliptic curves to corresponding expressions in gauge theory via F-theory