Examination of the θ-term of QCD with tensors

Viktor Papara

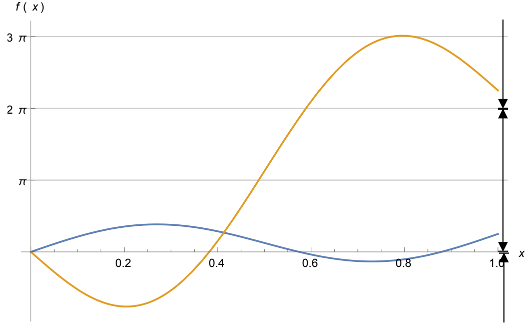
Ludwig Maximilians Universität München

November 10, 2014

Analyzing the vacuum structure of QCD
Consider only gauge fields (pure gauge):

$$A_\mu(x) = \frac{i}{g} U(x) \partial_\mu U^\dagger(x)$$

Simple example of topological classes:
$f : [0, 1] \to U(1)$



Topological classes with a winding number(integer)

$$q = \frac{g^2}{32\pi^2} \int \mathrm{d}^4 x F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}$$

where $F^a_{\mu\nu}$ is the euclidean Field strength and its dual

$$\tilde{F}^a_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} F^a_{\alpha\beta}$$

Pre-vacua: ...$|-1\rangle, |0\rangle, |1\rangle$... with non-vanishing transition amplitude

$$\langle n| e^{-H\tau} |m\rangle = \int [\mathrm{d}A_\mu]_{n-m} e^{-S_e}$$

Good combination:

$$|\theta\rangle = \sum_{n=-\infty}^{\infty} e^{in\theta} |n\rangle$$

with transition amplitude

$$\langle\theta'| \, e^{-H\tau} \, |\theta\rangle =$$

$$= \delta(\theta' - \theta) \int [dA_\mu] \exp(\int d^4x (L_e - i\theta \tfrac{g^2}{16\pi^2} Tr(F_{\mu\nu}\tilde{F}_{\mu\nu}))$$

Axion solution: adding a pseudoscalar field (the axion):

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \bar{q}_i\left(i\not{D} - m_i\right)q_i + \frac{\bar{\theta}g^2}{32\pi^2}F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu} + \frac{ag^2}{32\pi^2 f_a}F^a_{\mu\nu}\tilde{F}^{a\,\mu\nu} + \frac{1}{2}\partial^\mu a\partial_\mu a$$

Minimum of effective axion potential is at $(a/f_a + \bar{\theta}) = 0$

Terminology, differential forms and tensors

$$F(x) = F_{[\alpha\beta\gamma]}(x) dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$
$$dF(x) = \partial_{[\mu} F_{\alpha\beta\gamma]}(x) dx^\mu \wedge dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

Degrees of freedom: Number of components with an independent (Klein Gordon) equation of motion

One-form, vector field, spin-1

Massive Case

$$L = -\frac{1}{4} F^B_{\mu\nu} F^{B\,\mu\nu} + \frac{m^2}{2} B_\mu B^\mu, \quad F^B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

yields the “equation of motion” for B^0:

$$\partial_i \partial_i B_0 - m^2 B_0 = \partial_0 \partial_i B_i$$

and for the remaining spacial components

$$\partial_\mu \partial^\mu B^i + m^2 B^i = 0$$

Massless Case

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Again A^0 has no independent propagating e.o.m
Gauge invariance $A^\mu \to A^\mu + \partial^\mu \lambda$
Choosing gauge $\leftrightarrow$ completely specify λ
Coulomb gauge $\partial_i A_i = 0$:
$\rightarrow$ only two components have independent propagating e.o.m

$$\partial_\mu \partial^\mu A^j = 0$$

Massless Limit

Start with massive vector field

$$L = -\frac{1}{4} F^{B}_{\mu\nu} F^{B\,\mu\nu} + \frac{m^2}{2} B_\mu B^\mu$$

and substitute (redefine)

$$B_\mu = A_\mu + \frac{1}{m} \partial_\mu \varphi$$

to get Lagrangian and e.o.m.

$$\begin{aligned} L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu A^\mu + m A_\mu \partial^\mu \varphi + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi, \\ \partial_\mu F^{\mu\nu} + m^2 A^\nu + m \partial^\nu \varphi = 0 \\ m \partial_\mu A^\mu + \partial_\mu \partial^\mu \varphi = 0 \end{aligned}$$

The θ-term of QCD

The axion solution

Degrees of freedom of differential forms

Spin-1, one-form

Two-form

Three-form

Analogous CP problem and solution

Duality transformation

The strong CP problem in forms

θ-term written with three-form

Shift of $C_{\alpha\beta\gamma}$ under gauge transformations

Emergence of a gauge invariance:

$$A^{\mu} \to A^{\mu} + \partial^{\mu}\lambda, \quad \varphi \to \varphi - m\lambda$$

Limit of small mass ($m^2 = 0$):

$$\begin{aligned} L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + mA_{\mu}\partial^{\mu}\varphi + \frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi \\ \partial_{\mu}F^{\mu\nu} + m\partial^{\nu}\varphi = 0 \\ m\partial_{\mu}A^{\mu} + \partial_{\mu}\partial^{\mu}\varphi = 0 \end{aligned}$$

Now massless limit is smooth.

Massive Two-form:

$$L = \frac{3}{4}\partial_{[\mu}B_{\alpha\beta]}\partial^{[\mu}B^{\alpha\beta]} - \frac{m^2}{4}B_{\mu\nu}B^{\mu\nu}$$

$$3\partial_\alpha\partial^{[\alpha}B^{\beta\gamma]} + m^2 B^{\beta\gamma} = 0$$

Field strength is exterior derivative $F^{\alpha\beta\gamma} := \partial^{[\alpha}B^{\beta\gamma]}$
Three propagating d.o.f.
Massless Two-form:

$$L = \frac{1}{4}F^B_{\alpha\mu\nu}F^{B\,\alpha\mu\nu}, \quad \partial_\mu\partial^{[\mu}B^{\nu\alpha]} = 0$$

Gauge invariance: $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu}\lambda_{\nu]}$
One propagating d.o.f.
Stueckelberg substitution:

$$B^{\mu\nu} = A^{\mu\nu} - \frac{1}{m}\partial^{[\mu}\phi^{\nu]}$$

Massive Three-form

$$L = \partial_{[\mu} C_{\alpha\beta\gamma]} \partial^{[\mu} C^{\alpha\beta\gamma]} - m^2 C_{\alpha\beta\gamma} C^{\alpha\beta\gamma},$$
$$\partial_\mu \partial^{[\mu} C^{\alpha\beta\gamma]} + m^2 C^{\alpha\beta\gamma} = 0$$

One propagating d.o.f.

Massless Three-form

$$L = \partial_{[\alpha} C_{\beta\gamma\delta]} \partial^{[\alpha} C^{\beta\gamma\delta]}, \quad \partial_\mu C^{[\mu\alpha\beta\gamma]} = 0$$

Gauge invariance: $C^{\alpha\beta\gamma} \to C^{\alpha\beta\gamma} + \partial^{[\alpha} \lambda^{\beta\gamma]}$
Appropriate gauge → no propagating d.o.f.
Stueckelberg substitution:

$$C^{\alpha\beta\gamma} = A^{\alpha\beta\gamma} - \frac{1}{m} \partial^{[\alpha} \phi^{\beta\gamma]}$$

Analogous θ-term of a massless three-form

Euler-Lagrange equation allows $C^{\alpha\beta\gamma}$ to have a constant field strength in vacuum: $F^{\alpha\beta\gamma\delta} = \partial^{[\alpha} C^{\beta\gamma\delta]} = F_0 \epsilon^{\alpha\beta\gamma\delta}$

Solution: "Stueckelberg mechanism" introduction of massless two-form "axion"

$$L = \left(\partial_{[\alpha} B_{\beta\gamma]} - C_{\alpha\beta\gamma}\right)^2 + \partial_{[\mu} C_{\alpha\beta\gamma]} \partial^{[\mu} C^{\alpha\beta\gamma]}$$

→ $C^{\alpha\beta\gamma}$ got a gauge invariant mass (got higgsed)
→ Field strength vanishes in vacuum

Duality between two-form and pseudoscalar

Simple case:

$$L = 3\, m^2 \left(P_{\alpha\beta\gamma} - C_{\alpha\beta\gamma}\right)^2 - 12 F_{\mu\alpha\beta\gamma} F^{\mu\alpha\beta\gamma}$$

$$\text{with } P_{\alpha\beta\gamma} = \partial_{[\alpha} B_{\beta\gamma]} \text{ and } F_{\mu\alpha\beta\gamma} = \partial_{[\mu} C_{\alpha\beta\gamma]}$$

Take $P_{\alpha\beta\gamma}$ as fundamental three-form and ensure its properties with lagrange multiplier $L_a = \Lambda^2 a \varepsilon^{\mu\alpha\beta\gamma} \partial_{[\mu} P_{\alpha\beta\gamma]}$.
Integrate out $P_{\alpha\beta\gamma}$ and then $C_{\alpha\beta\gamma}$
→ effective equation of motion for a: $\partial_\mu \partial^\mu a + m^2 a = 0$

Arbitrary potential:

$$L = \Lambda^4 K\left(\frac{F}{\Lambda^2}\right) + 3m^2\left(P_{\alpha\beta\gamma} - C_{\alpha\beta\gamma}\right)^2 + \Lambda^2 a \varepsilon^{\mu\alpha\beta\gamma} \partial_\mu P_{\alpha\beta\gamma}$$

$$F = \epsilon^{\mu\alpha\beta\gamma} F_{\mu\alpha\beta\gamma} = \epsilon^{\mu\alpha\beta\gamma} \partial_{[\mu} C_{\alpha\beta\gamma]}$$

θ -term with three-form

$$L_\theta = \frac{\theta g^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a\,\mu\nu}$$

With $G^a_{\alpha\beta} = \partial_\alpha A^a_\beta - \partial_\beta A^a_\alpha + g f^{abc} A^b_\alpha A^c_\beta$ and
$\tilde{G}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} G_{\alpha\beta}$
After rearranging, L_θ takes the form

$$\frac{g^2\theta}{32\pi^2} G^a_{\alpha\beta} \tilde{G}^{a\,\alpha\beta} = \theta \varepsilon^{\alpha\beta\gamma\delta} \partial_{[\alpha} C_{\beta\gamma\delta]} = \theta F,$$

with

$$C_{\alpha\beta\gamma} = \frac{g^2}{32\pi^2} \left(A^a_{[\alpha} G^a_{\beta\gamma]} - \frac{1}{3} g f^{abc} A^a_{[\alpha} A^b_\beta A^c_{\gamma]} \right).$$

Shift of $C_{\alpha\beta\gamma}$ under gauge transformations

Infinitesimal gauge transformation:

$$A_\mu^a \to A_\mu^a + \frac{1}{g}\partial_\mu \omega^a + f^{abc} A_\mu^b \omega^c$$

Transformation of

$$C_{\alpha\beta\gamma} = \frac{g^2}{32\pi^2}\left(A^a_{[\alpha} G^a_{\beta\gamma]} - \frac{1}{3} g f^{abc} A^a_{[\alpha} A^b_\beta A^c_{\gamma]} \right)$$

$$C_{\alpha\beta\gamma} \to C_{\alpha\beta\gamma} + \partial_{[\alpha}\Omega_{\beta\gamma]},$$

$$\Omega_{\beta\gamma} = -\frac{g^2}{32\pi^2}\partial_{[\beta}\omega^a A^a_{\gamma]}$$

Questions?