Results of my thesis

November 21, 2014

- Originally, I wanted to work on String theory. My advisor, Dieter Lüst initially told me, to write on graviton amplitudes in String theory and then I should compare them with articles from Dvali/Gomez on a model called “classicalization”
- However, it actually turned out that after reading Dvali/Gomez articles, I found evidence of severe problems in their approach.
- Hence in my thesis, I first collected the necessary tools in order to show the problems in the articles by Dvali/Gomez.
- My thesis also focused on certain mathematical details from topology, but that is not the topic here.

- In perturbative quantum gravity, we decompose the metric into $\overline{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$ with $g_{\mu\nu}$ as Minkowski metric and $h_{\mu\nu}$ as quantum field.
- Then follows a Taylor expansion of the Lagrange density $\mathcal{L} = \sqrt{-g}R$ in $h_{\mu\nu}$ where $\underline{\mathcal{L}}$ is linear in $h_{\mu\nu}$ and $\underline{\underline{\mathcal{L}}}$ is quadratic in $h_{\mu\nu}$.
$$\mathcal{L} = \sqrt{-g}R + \underline{\mathcal{L}} + \underline{\underline{\mathcal{L}}}$$
And we insert the perturbativly expanded action $S = \int d^4x\mathcal{L}(\phi)$ into the amplitude
$$Z = \int \mathcal{D}h_{\mu\nu}e^{iS} \tag{1}$$
- The action $S = \int \sqrt{-g}Rd^4x$ is invariant with respect to gauge transformations of the form (η^α is a displacement)
$$g_{\mu\nu} \rightarrow \overline{g}_{\mu\nu} \equiv g_{\mu\nu} + g_{\alpha\nu}\partial_\mu\eta^\alpha + g_{\mu\alpha}\partial_\nu\eta^\alpha + \eta^\alpha\partial_\alpha g_{\mu\nu} \tag{2}$$
To avoid double counting of equivalent metrics, we insert a gauge fixing term $\frac{i}{4\zeta}\int d^4xG_\alpha G^\alpha$where $G_\alpha(h_{\mu\nu}) = 0$ and a ghost term $i\int d^4x(\tilde{\eta}^\alpha)^*A_{\alpha\beta}\tilde{\eta}^\beta$ where $A_{\alpha\beta} = \frac{\delta G_\alpha(h^\eta_{\mu\nu})}{\delta\eta^\beta}$.

- The path integral then becomes

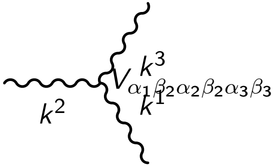
$$Z = \int \mathcal{D}h_{\mu\nu} \int \prod_{\alpha} D(\tilde{\eta}^{\alpha})^* \mathcal{D}\tilde{\eta}^{\alpha} e^{iS - \frac{i}{4\zeta}\int d^4x G_\alpha G^\alpha + i\int d^4x (\tilde{\eta}^\alpha)^* A_{\alpha\beta}\tilde{\eta}^\beta}$$

  setting the background to $g_{\mu\nu} = \eta_{\mu\nu}$, one can derive the Feynman rules of gravity.

- For example, we get a propagator

$$= D_{\mu\nu\alpha\beta}(k) =$$

$$\frac{1}{k^2 - i\epsilon}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta})$$



and a vertex for graviton coupling to itself

$$
\begin{aligned}
&V(k^1,k^2,k^3)_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} = \\
&-\Big(k^2_{(\alpha_1} k^3_{\beta_1)} \left(2\eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2} - \eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}\right) + \\
&k^1_{(\alpha_2} k^3_{\beta_2)} \left(2\eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1} - \eta_{\alpha_1\beta_1}\eta_{\alpha_3\beta_2}\right) + \\
&k^1_{(\alpha_3} k^2_{\beta_3)} \left(2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1} - \eta_{\alpha_1\beta_1}\eta_{\alpha_2\beta_2}\right) + 2k^3_{(\alpha_2}\eta_{\beta_2)(\alpha_1}\eta_{\beta_1)(\alpha_3}k^2_{\beta_3)} + \\
&2k^1_{(\alpha_3}\eta_{\beta_3)(\alpha_2}\eta_{\beta_2)(\alpha_1}k^3_{\beta_1)} + 2k^2_{(\alpha_1}\eta_{\beta_1)(\alpha_3}\eta_{\beta_3)(\alpha_2}k^1_{\beta_2)} + \\
&k^2 k^3 \left(\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)\beta_1}\eta_{\alpha_3\beta_3} + \eta_{\alpha_1(\alpha_3}\eta_{\beta_3)\beta_1}\eta_{\alpha_2\beta_2} - 2\eta_{\alpha_1(\alpha_2}\eta_{\beta_2)(\alpha_3}\eta_{\beta_3)\beta_1}\right) + \\
&k^1 k^3 \left(\eta_{\alpha_2(\alpha_1}\eta_{\beta_1)\beta_2}\eta_{\alpha_3\beta_3} + \eta_{\alpha_2(\alpha_3}\eta_{\beta_3)\beta_2}\eta_{\alpha_1\beta_1} - 2\eta_{\alpha_2(\alpha_1}\eta_{\beta_2)(\alpha_3}\eta_{\beta_3)\beta_2}\right) + \\
&k^1 k^2 \left(\eta_{\alpha_3(\alpha_1}\eta_{\beta_1)\beta_3}\eta_{\alpha_2\beta_2} + \eta_{\alpha_3(\alpha_2}\eta_{\beta_2)\beta_3}\eta_{\alpha_1\beta_1} - 2\eta_{\alpha_3(\alpha_1}\eta_{\beta_1)(\alpha_2}\eta_{\beta_2)\beta_3}\right)\Big)
\end{aligned}
$$

- The structure of the vertex leads to non-renormalizability of the theory, as can be seen from power-counting.
- For two loops, Goroff and Sagnotti [1] found a non-zero counterterm made of terms that do not appear in the original Lagrangian. This implies non-renormalizable divergences:

$$\Delta\mathcal{L} = \frac{209}{2880(4\pi)^4}\frac{1}{\epsilon}\int d^4x\sqrt{-g}R^{\alpha\beta}{}_{\gamma\delta}R^{\gamma\delta}{}_{\epsilon\zeta}R^{\epsilon\zeta}{}_{\alpha\beta} \tag{3}$$

- This divergence of the amplitude has severe consequences. Wald [2] notes in his book "General relativity" on p. 384,

  > *The perturbation theory one obtains from this approach will, in each order, satisfy causality conditions with respect to the background metric* $g_{\mu\nu}$ *rather than the true metric* $\overline{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$. *The summed series could still satisfy appropriate causality conditions if it were to converge....*

- The divergent amplitude therefore implies that the quantum field has to be weak $|g_{\mu\nu}| > |h_{\mu\nu}|$ if a treatment with perturbative quantum gravity is possible. Otherwise, we would violate causality conditions.

- This is acknowledged by Hawking[3], who writes

  > *Attempts to quantize gravity ignoring the topological possibilities and simply drawing Feynman diagrams around flat space have not been very successful. It seems to me that the fault lies not with the pure gravity theories themselves but with the uncritical application of perturbation theory to them. In classical relativity we have found that perturbation theory has only limited range of validity. One can not describe a blackhole as a perturbation around flat space. Yet this is what writing down a string of Feynman diagrams amounts to.*

- Saying this, one should note that one can in fact derive the Schwarzschild solution approximately from three graphs.[5] Certainly, even the weak gravitational field of a golf ball creates a Schwarzschild solution. The author [5] is careful to note that he does not! derive a blackhole, but only an approximation of the Schwarzschild metric to the second order of the gravitational coupling constant.

- A way to do non-perturbative quantum gravity is with a Hamiltonian [4].
- We decompose the metric $g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_i \beta^i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$,
- Using the Lagrangian $L = \int d^3x \sqrt{-g} R$ we get the conjugate momenta $\pi = \frac{\delta L}{\delta \partial_t N} = \pi^i = \frac{\delta L}{\delta \partial_t \beta_i} = 0$ and $\pi^{kl} = \sqrt{\gamma}(\gamma^{kl}(K - K^{kl})$ with K as extrinsic curvature.
- Then, one can derive the Hamiltonian of gravity:

$$\begin{aligned} H &= \int d^3x (\pi \partial_t N + \pi^i \partial_t \beta_i + \pi^{ij} \partial_t \gamma_{ij}) - L \\ &= \int d^3x \left(N\sqrt{\gamma}(\mathcal{G}_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{\gamma}^{(3)} R) - \beta_i 2 D_j (\gamma^{-1/2} \pi^{ij}) \right) \\ &= \int d^3x \left(N \mathcal{H}_G + \beta_i \chi^i \right) \end{aligned}$$

where $\mathcal{G}_{ijkl} = \frac{1}{2}\gamma^{-1/2}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl})$.
- One gets $\chi^i = 0$. Using $\pi = 0 \Rightarrow \partial_t \pi = 0$, Hamilton's equation implies $\{\pi, H\} = \partial_t \pi = 0 = \frac{\partial H}{\partial N} = 0 = \mathcal{H}$

- We have the Poisson brackets

$$\{\gamma_{ij}(x), \pi^{kl}(x')\} = \delta^k_{(i}\delta^l_{j)}\delta(x, x')$$

- In the quantum theory these become commutators of operators

$$[\hat{\gamma}_{ij}(x), \hat{\pi}^{kl}(x')] = \delta^k_{(i}\delta^l_{j)}\delta(x, x')$$

acting on a state functional $\psi(\gamma_{ij})$.

- These commutator rules are fulfilled if $\hat{\gamma}_{ij}\psi = \gamma_{ij}\psi$ and $\hat{\pi}^{ij}\psi = \frac{1}{i}\frac{\delta}{\delta\gamma_{ij}}\psi$
- Inserting this in the Hamiltonian constraint we get the Wheeler deWitt equation (WdW) [6]

$$\left(\mathcal{G}_{ijkl}\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}} + \sqrt{\gamma}^{(3)}R\right)\psi \quad = \quad 0$$

- Before we ask for solutions to this equation, we have to note that there is an inconsistency that emerges at high energies: Letting $x \to x'$ and using our commutator rules, we can compute

$$\left[\hat{\gamma}_{ij}(x), \hat{\pi}^{ij}(x)\right] = 6i\delta(x, x)$$

- And therefore we have with an infinitesimal displacement ζ

$$\left[6i\hbar\delta(x, x), i \int \chi_{k'} \delta\zeta^{k'} d^3x'\right] = 0. \qquad (4)$$

- On the other hand, one gets by direct calculation (,k means partial differentiation with respect to x^k)

$$\left[\hat{\gamma}_{ij}(x)\hat{\pi}^{ij}(x) - \hat{\pi}^{ij}(x)\hat{\gamma}_{ij}(x), i \int \chi_{k'} \delta\zeta^{k'} d^3x'\right] = -6i \left(\delta(x, x)\delta\zeta^{k}\right)_{,k}$$

.

- deWitt notes that the delta function, "may be taught as a limit of a sequence of successively narrower twin peaked functions, all of which vanish at point $x' = x$ in the valley between the peaks"

$$\delta(x) = \lim_{\epsilon \to 0} \delta_\epsilon = \lim_{\epsilon \to 0} \frac{1}{2\pi} \left(f_\epsilon(x - \sqrt{\epsilon}) + f_\epsilon(x + \sqrt{\epsilon}) - \frac{2 f_\epsilon(x)}{1 + \epsilon} \right) \tag{5}$$

where

$$f_\epsilon(x) = \frac{\epsilon}{x^2 + \epsilon^2}$$

- deWitt [6] writes "In an infinite world, passage to $\epsilon \to 0$ would correspond to the usual cutoff going to infinity in momentum space" and he emphasizes that the then appearing inconsistency "bears on problems of interpreting divergences". Apparently, this version of quantum gravity becomes inconsistent at high energies.

- One can solve the WDW equation therefore only approximately, e.g. by a WKB ansatz.
- For a particle, Peskin/Schroeder show on p. 279 that the path integral $Z(x_0, x_1, T) = \int_{x_0}^{x_1} \mathcal{D}x(t) e^{iS}$, with $S(x, T) = \int_0^T dt (\frac{m}{2}\dot{x}^2 - V(x))$ fulfills the Schroedinger equation:

$$i\hbar \frac{\partial}{\partial T} Z(x_0, x_1, T) = \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_1^2} + V(x_1) \right) Z(x_0, x_1, T)$$

- Hawking [7] has shown that the amplitude of quantum gravity, $Z = \int \mathcal{D}g_{\mu\nu} e^{iS}$, without perturbation theory applied, is also a formal! (but exact) solution of the WDW equation:

$$\left(\mathcal{G}_{ijkl} \frac{\delta}{\delta \gamma_{ij}} \frac{\delta}{\delta \gamma_{kl}} + \sqrt{\gamma}^{(3)} R \right) \int \mathcal{D}g_{\mu\nu} e^{iS} = 0 \qquad (6)$$

- This is evidence that the WDW theory and the amplitude from the path integral of quantum gravity are essentially the same theory.

- Now we apply the WDW equation on blackholes [4, 8]
- For spherically symmetric spacetimes with parametrization

$$ds^2 = -N^2(r,t)dt^2 + \Lambda^2(r,t)(dr + \beta^r dt)^2 + R^2(r,t)d\Omega^2 \quad (7)$$

and an action

$$S = \int dtL = \int dt \int_0^\infty dr \left(P_\Lambda \dot{\Lambda} + P_R \dot{R} - N\mathcal{H}_G - \beta^r \mathcal{H}_r \right) \quad (8)$$

- The WDW equation reads [8]

$$\left(\frac{-\Lambda\delta^2}{2R^2\delta\Lambda^2} + \frac{1}{R}\frac{\delta^2}{\delta\Lambda\delta R} + \frac{RR''}{\Lambda} - \frac{RR'\Lambda'}{\Lambda^2} + \frac{R'^2}{2\Lambda} - \frac{\Lambda}{2} \right)\psi(\Lambda, R) = 0 \quad (9)$$

A semi-classical solution of this equation is given by a WKB ansatz

$$\psi(\Lambda, R) = C(\Lambda, R)e^{iS_0(\Lambda, R)} \quad (10)$$

- However, it turns out [4, 8] that in order to describe a spacetime with an event horizon and a flat curvature at infinity, the action must be supplied with appropriate boundary terms at infinity and at the horizon and we get

$$
\begin{aligned}
S_{total} &= \int dt \int_0^\infty dr \left(P_\Lambda \dot{\Lambda} + P_R \dot{R} - N\mathcal{H}_G - \beta^r \mathcal{H}_r \right) \\
&\quad + \int dt \frac{R_0^2}{2} \dot{\tau} - \int dt M \dot{\tau}_+
\end{aligned}
\tag{11}
$$

- In the canonical framework, the functions τ and τ_+ represent additional degrees of freedom that must be supplied with corresponding canonical momenta π_0 and π_+. These momenta can only brought consistently into the action as additional variables, if we impose the following additional constraints

$$
C_0 = \pi_0 - \frac{R_0^2}{2} = 0, \text{and } C_+ = \pi_+ + M = 0 \tag{12}
$$

- In the quantum theory, the additional momenta become $\pi = -i\frac{\delta}{\delta\tau_0}$ and $\pi_+ = -i\frac{\delta}{\delta\tau_+}$
- the momenta acting on $\psi = C(\Lambda, R)e^{iS_0(\Lambda, R.\tau_0, \tau_+)}$ with the additional degrees of freedom τ_+ and τ_0 then lead to the new quantum mechanical constraints

$$\frac{\partial_0 S_0}{\partial\tau_0} - \frac{R_0^2}{2} = 0, \text{and } \frac{\partial_0 S_0}{\partial\tau_+} + M = 0 \tag{13}$$

- This changes the phase of the solution S_0 into

$$S_0(\Lambda, R, \tau_+, \tau_0) \to S_0 + \frac{R_0^2}{2}\tau_0 - M\tau_+ \tag{14}$$

- The additional degrees of freedom are in relation to the parametrization of the metric. Comparison with the euclideanized Schwarzschild metric yields $\tau_0 = 2\pi$ and one gets the correct formula the black hole entropy with ψ_0 as quantum part of the euclideanized amplitude that can be neglected and the rest as a boundary part:

$$\psi(\Lambda, R,) = \psi_0(\Lambda, R)e^{-\beta M + \frac{A}{4}} \tag{15}$$

- In their article “black holes as critical point of quantum phase transition”[9] Dvali/Gomez Gomez write on p. 2 “Black holes represent Bose-Einstein-Condensates of gravitons at the critical point of a quantum phase transition”. They start their calculation from a standard Hamiltonian for a BEC with a collective quantum state $\Psi(x)$

$$H = -\hbar L_0 \int d^3x \Psi(x) \nabla^2 \Psi(x) - g \int d^3x \Psi(x)^\dagger \Psi(x)^\dagger \Psi(x) \Psi(x)$$

- Imposing periodic boundary conditions and a plane wave expansion $\Psi = \sum_k \frac{a_k}{\sqrt{V}} e^{i\frac{\vec{k}\vec{x}}{R}}$ they get

$$\mathcal{H} = \sum_k k^2 a_k^\dagger a_k - \frac{1}{4}\alpha \sum_k a_{k+p}^\dagger a_{k'-p}^\dagger a_k a_{k'}\text{''}$$

- In contrast to the statements by Dvali/Gomez, the Hamiltonian of the WdW equation was

$$\left(\mathcal{G}_{ijkl} \frac{\delta}{\delta \gamma_{ij}} \frac{\delta}{\delta \gamma_{kl}} + \sqrt{\gamma}\, {}^{(3)}R \right) \Psi(\gamma_{ij}) = 0 \tag{16}$$

- We noted that a quantum mechanical amplitude usually satisfies the Schroedinger equation for the Hamiltonian.
- For the Gravitational amplitude $Z = \int \mathcal{D} g_{\mu\nu} e^{iS}$, this Hamiltonian had the form of the WDW equation which differs from that of Dvali/Gomez.
- So their Hamiltonian would break with the rule that the path integral of a theory solves the corresponding Schroedinger equation.

- For further comparison, we try to write the WDW equation with creation and annihilation operators. One defines the following norm:

$$\langle\Psi_1|\Psi_2\rangle = \int \prod_x d\Sigma^{ij}(x)\Psi_1^* \left(\mathcal{G}_{ijkl} \frac{\overrightarrow{\delta}}{i\delta\gamma_{kl}} - \frac{\overleftarrow{\delta}}{i\delta\gamma_{kl}} \mathcal{G}_{ijkl} \right) \Psi_2 \tag{17}$$

 where $d\Sigma^{ab}$ is the a surface element of the $6 \times \infty^3$ dimensional space spanned up by $\mathcal{G}_{ijkl}$.
- If we find a complete set of solutions Ψ_n that are orthonormal with respect to the this norm, we can write

$$\Psi(\gamma_{ij}) = \sum_k \left(a_k^{ij} \Psi_n(\gamma_{ij}) + a_k^{ij\dagger} \Psi_n^*(\gamma_{ij}) \right) \tag{18}$$

 where $a_k^{ij\dagger}$ and $a_k^{ij}$are the creation and annihilation operators for gravitons with k momentum.
- There is one article on this: McGuigan [11]writes

 > *"The presence of $\mathcal{G}_{ijkl}$ as well as $\sqrt{\gamma}\,^{(3)}R$ in the WdW eq. which are not quadratic in γ_{ij} or its derivatives will prevent us from finding such*

- The quantized Schwarzschild metric

$$\left(\frac{-\Lambda\delta^2}{2R^2\delta\Lambda^2}+\frac{1}{R}\frac{\delta^2}{\delta\Lambda\delta R}+\frac{RR''}{\Lambda}-\frac{RR'\Lambda'}{\Lambda^2}+\frac{R'^2}{2\Lambda}-\frac{\Lambda}{2}\right)\psi(\Lambda,R)=0 \tag{19}$$

  is now one of the spacetimes where the WDW equation does not have an orthonormal base, since terms like terms like $1/R^2$ or $1/\Lambda^2$occur.
- The impossibility of finding an orthonormal base does not occur for all spacetimes. For example, McGuigan[11] succeeds to construct the WDW equation with an orthonormal base of gravitons in linearized gravity for an $S^1\times S^1\times S^1$ topology.
- With a weak field WKB approximation, one can indeed find solutions of the WdW equation of Schwarzschild solutions [8]

$$\psi = Ce^{i\int_{-\infty}^{\infty} dr(RR' arcosh(\frac{R'}{\Lambda\sqrt{1-\frac{2M}{R}}}))}$$

  These are non-perturbative states but they look different than the proposals of Dvali/Gomez

- Dvali/Gomez claim that from their Hamiltonian:

  "we have reproduced the black hole evaporation law"

- Calculation of the blackhole entropy from the WDW equation shows that the entropy emerges not from the quantum part $\psi = Ce^{i\int_{-\infty}^{\infty} dr(RR'\,arcosh(\frac{R'}{\Lambda\sqrt{1-\frac{2M}{R}}}))}$ of a weak field approximation but from boundary terms at the event horizon of the blackhole. The boundary terms led to additional, non-perturbative degrees of freedom. Thereby they imply additional constraints

$$\frac{\partial_0 S_0}{\partial \tau_0} - \frac{R_0^2}{2} = 0, \text{and } \frac{\partial_0 S_0}{\partial \tau_+} + M = 0 \tag{20}$$

  which led to a modification of the phase of the blackhole quantum state from which the entropy can be derived.

$$S_0(\Lambda, R, \tau_+, \tau_0) \to S_0 + \frac{R_0^2}{2}\tau_0 - M\tau_+ \tag{21}$$

- The euclidean path integral is connected to the canonical partition sum $Z = tr(e^{-\beta H})$. Hawking computed the entropy from the classical background term $e^{-I(g_{\mu\nu})}$ of the Euclidean path integral [12]

$$Z_{eu} = e^{-I(g_{\mu\nu})} \int \mathcal{D}h_{\mu\nu} e^{-\underline{\underline{I}}(h_{\mu\nu})} \tag{22}$$

- The entropy also emerged from boundary terms in the euclidean action

$$I = -\int d^4x \sqrt{g} R - 2 \int_{\partial M} d^3x \sqrt{\gamma}(K - K^0) \tag{23}$$

The quantum field $h_{\mu\nu}$ would generate a small contribution from gravitons to the entropy. This contribution is neglected by Hawking who sets $h_{\mu\nu} = 0$ in [12]. Note also that the Euclidean section of the Schwarzschild background only covers the outer Schwarzschild solution. The entropy is therefore generated outside of the blackhole and Hawking does not even consider! the interior of the blackhole.

- The boundary terms at the event horizon are important, since a spherical star without an event horizon has no gravitational entropy. This is unlike a Bose condensate, whose entropy does not depend on an event horizon [12]
- Dvali/Gomez correctly assume that the blackhole has a collective quantum state function.
- Boundary terms depending on an event horizon that led to a change of the phase of the quantum state are absent in the Hamiltonian of Dvali/Gomez. Instead their model seems to try to model the part $\psi = Ce^{i\int_{-\infty}^{\infty} dr(RR' arcosh(\frac{R'}{\Lambda\sqrt{1-\frac{2M}{R}}}))}$ of the solution that does not generate the entropy
- Furthermore, standard quantum gravity predicts that the equation of Hamiltonian, which is solved by the state functional (the well known amplitude), has entirely different features than the Hamiltonian of Dvali/Gomez.
- Hence, whatever Dvali/Gomez postulated quantum state ψ for a blackhole is, it differs from the quantum state of a blackhole that is predicted by standard quantum gravity.

- In their work "Self-Completeness of Einstein Gravity" [10] Dvali/Gomez multiply the classical! Schwarzschild radius $r_s = \frac{2Gm}{c^2}$ with the reduced Compton wavelength $\bar{\lambda}_c = \frac{\lambda_c}{2\pi} = \frac{\hbar}{mc}$ which gives so called Planck length l_p

$$r_s \bar{\lambda}_c = \frac{2G\hbar}{c^3} = 2l_p^2 \geq l_p^2 \qquad (24)$$

  Since the Planck mass m_p is defined by $r_s = \lambda_c$ any attempt to generate a blackhole of mass greater than m_p will give a blackhole larger than its Compton wavelength, which should be regarded as classical.
- Putting a quantum state within a box of width L with high walls requires an energy of $E \propto \frac{1}{L^2}$. Converting this into a mass, $m = E/c^2 \propto \frac{1}{L^2c^2}$ and putting this into the formula for the Compton wavelength $\bar{\lambda}_c = \frac{L^2\hbar c}{c}$, and the Planck length, gives

$$r_s = \frac{2l_p^2}{L^2\hbar c} \qquad (25)$$

- Even though their arguments uses classical formulas ($r_s = \frac{2Gm}{c^2}$) or non-relativistic ones $E \propto \frac{1}{L^2}$, Dvali/Gomez argue from this that experiments at trans-planckian energies would imply creation of classical blackholes, whose amplitudes are finite and suppressed. They write:

  > *"We suggest that pure Einstein gravity is self complete in deep-UV. We argue that for restoring consistency no new propagating degrees of freedom are necessary at energies* $>> m_p$*"*

- They add

  > *"since our argument is fully non-perturbative, no perturbative finiteness is necessary"'*

  But they do not really evaluate the non perturbative amplitude. It's easy. It just means no perturbation in the action. We do that here...

- The path integral is usually oscillating. For an ordinary scalar field theory,

$$Z = \int \mathcal{D}x(t) e^{iS(\varphi)} \tag{26}$$

one can make the path integral convergent by making a Wick rotation:

$$Z_{eu} = \int \mathcal{D}\varphi(t) e^{-I(\varphi)} \tag{27}$$

where $I(\varphi) = -iS(\varphi)$ is the Euclidean action.
- For fields that are real on the Euclidean space, the path integral then converges.
- Note that Dvali/Gomez use the Euclidean path integral implicitly when they assert that amplitudes from classical blackholes are suppressed.

- We decompose the metric into $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, with $\Omega(x)$ as a conformal factor.
- We write the path integral as

$$Z_{eu} = \int \mathcal{D}g Y(g) \tag{28}$$

where

$$Y(g) = \int \mathcal{D}\Omega e^{-I(\Omega^2, g)} \tag{29}$$

- But the euclidean action for a metric like $\Omega^2 g_{\mu\nu}$ is [13]

$$I(\Omega^2, g) = -\int d^4x \sqrt{g} \left(\Omega^2 R + 6 g^{\mu\nu} \partial_\mu \Omega \partial_\nu \Omega\right) \tag{30}$$

- The imaginary axis has to be chosen that way to be the same as for fermions.
- Due to the derivatives of Ω, the action $I(\Omega^2, g)$ is arbitrarily negative if a rapidly varying Ω is chosen. Since we have to sum over all possible Ω, the amplitude becomes divergent.

- A summation over classical blackholes was included since we had to sum over conformal factors $\Omega^2 = 1$ and $g_{\mu\nu}$ as a Schwarzschild metric.
- Another way of doing non-perturbative quantum gravity would be the WdW equation which generates the full non-perturbative amplitude

$$\left(\mathcal{G}_{ijkl}\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}} + \sqrt{\gamma}^{(3)}R\right)\int \mathcal{D}g_{\mu\nu}e^{iS} = 0 \qquad (31)$$

- Unfortunately, this theory is only consistent for low energy states within a WKB approximation. At high energies the theory becomes inconsistent, yielding equations like

$$0 = -6i\left(\delta(x,x)\delta\zeta^k\right)_{,k}$$

- Neither the occurrence of classical blackholes render the amplitude finite, nor is the equation that generates the non perturbative amplitude consistent at high energies.
- This stands against claims that standard quantum gravity would be "self complete" and the.amplitude finite without additional assumptions.

- However, one can try to cure the model.
- Hawking [14] summed the euclidean path integral non-perturbatively just over classical metrics and multiplied to that the perturbative one loop correction for weak quantum fields. With help of Zeta function renormalization for the quantum part and the Atiyah Singer index theorem for the non-perturbative, classical part, one can express the amplitude as a function of topological invariants. (which is a reason why every physicist should know index theory)

$$Z(\Lambda) \propto \left(\frac{\Lambda}{\Lambda_0}\right)^{-a\chi} e^{\frac{8\pi d^2 \chi}{\Lambda}}$$

- With a saddle-point approximation, one can compute that the dominant metrics have zero signature and one finds that the Euler characteristic is

$$\chi \propto hV$$

I.e we have one gravitational instanton per h^{-1}unit volumes, where h is some constant. This is a picture of a spacetime that is filled with virtual gravitational instantons.

- Unfortunately, there is a problem common in models with gravitational instantons at planck scale: The trajectories of particles that are flying in these spacetimes may be affected by the curvature of the instantons [15, 16].
- A typical amplitude for a scalar particle is

$$-\int \overline{u}(x')\overleftrightarrow{\nabla}_{\mu}G(x',y')\overleftrightarrow{\nabla}_{\nu}v(x')d\Sigma^{\mu}(x')d\Sigma^{\nu}(y') \qquad (32)$$

  where $\Sigma^{\mu}(x')$ and $\Sigma^{\nu}(y')$ are the Cauchy data for the initial and final states and $G(x',y')$ is the Green's function.
- The Green's functions of curved spacetimes are highly different from propagators in flat space. We do not even know them for all possible metric[15, 16].
- E.g. the Green's functions of CP^2 space have the form,

$$G(x',y') = \frac{1}{4\pi^2\rho'(1-L)} \qquad (33)$$

  where

$$L = \frac{(\rho' + x'y' - in^{l}_{\mu\nu}x'^{\mu}y'^{\nu})(\rho'^2 + x'y' + in^{l}_{\mu\nu}x'^{\mu}y'^{\nu})}{(\rho'^2 + x'^2)(\rho'^2 + y'^2)} \qquad (34)$$

- If the spacetime is simply connected, one has the result that the topology of manifolds up to homeomorphy is given by the signature and the Euler characteristic and all possible values of them can be reproduced by CP^2, $\overline{CP^2}$,$S^2 \times S^2$, K^3,$\overline{K^3}$. spaces (mathematical reason why that is the case is explained in my thesis).
- Hawking, Page and Pope sum over these metrics but the amplitudes they get would give large corrections for spin $s = 0$ particles[15, 16]

$$A \propto \left(\frac{k_1 k_2}{m_p} \right)^s \tag{35}$$

- Hawking et al write that this would suggest that the Higgs particle is of composite nature. In 2012, the Higgs particle has been found at the Large Hadron Collider in Genf, and the data gave evidence for the Higgs field to be a scalar.
- One may think that summing the path integral over more metrics would solve the problem but Hawking, Page and Pope write[15, 16]:

  > *"We would expect that averaging over more general metrics would change the S-matrix elements by numerical factors only but would not alter their low-energy power-law dependence on the external momenta, which is fixed by dimensional considerations."*

- Hawking does not give many details of this calculation. Interesting would be if 10 spacetime dimensions make the discrepancy go away.
- If Hawking is correct the result should create problems for all models where the gravitational path integral is summed over virtual gravitational instantons at planckscale on 4 dimensions

- I hope that in a phd thesis, I can be more constructive and that I can also write more original stuff.
- With that work, I tried to make the best of the situation and used my time to focus my thesis more mathematical details from topology or algebraic topology (which I do not touch during this lecture here).
- Some of the mathematical objects in my thesis are used in proofs of the Calabi conjecture that is important for the Calabi Yau spaces in string theory. I would be interested to do work on aspects of string theory where mathematical ideas are used.
- But first and foremost, I would like to work more "constructively" in my phd thesis than it turned out here. I am open for proposals of topics. But I probably need 1-2 weeks to decide on which proposal I want to work.

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