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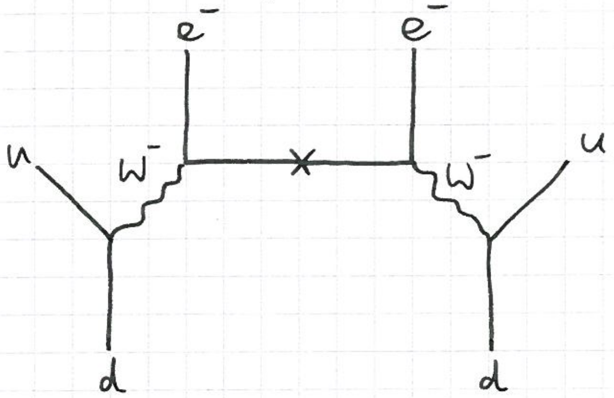
$0\nu\beta\beta$-rate: A detailed derivation

- Feynman diagram (quark-level):



$\Rightarrow$ 2nd order perturbation theory allows to write down the amplitude in terms of the S-matrix (real space Feynman rules):

$$\mathcal{M}_{0\nu\beta\beta} = \langle f | S^{(2)} | i \rangle = \underbrace{\frac{(-i)^2}{2!}}_{\text{2nd order perturbation theory}} \underbrace{\left(\frac{2G_F}{\sqrt{2}}\right)^2}_{\text{double weak vertex (4 Fermi approximation)}} \underbrace{\int d^4x_1\, d^4x_2}_{\text{"sum" over all possible space-time points for the interactions}} \cdot$$

$$\cdot \underbrace{\langle A' | T\{J_\mu(x_1) J_\nu(x_2)\} | A \rangle}_{\text{nuclear part}} \cdot \underbrace{\langle f | N\{\bar{e}_L(x_1)\gamma^\mu \nu_{eL}(x_1) \nu_{eL}^T(x_2) \gamma^{\nu T} \bar{e}_L^T(x_2)\} | i \rangle}_{\text{leptonic part}}$$

- leptonic part:
  - electron-neutrino is a superposition of (typically) 3 light mass eigenstates:

$$\nu_e = \sum_{i=1}^{3} \tilde{U}_{ei}\, \nu_i$$

↳ $\tilde{U}_{ij}$: "CKM-like" part of the PMNS-matrix

$\Rightarrow$ full PMNS-matrix U:

$$U = \tilde{U} \cdot \mathrm{diag}(e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3})$$

Majorana phases

$\Rightarrow$ here: $\nu_{eL} = P_L \nu_e = P_L \sum_{i=1}^{3} \tilde{U}_{ei} \nu_i = \sum_{i=1}^{3} \tilde{U}_{ei} P_L \nu_i$

$$\nu_{eL}^T = (P_L \nu_e)^T = \nu_e^T P_L^T = \nu_e^T P_L = \sum_{j=1}^{3} \tilde{U}_{ej} \nu_i^T P_L$$

  - Majorana condition: "fermion = antifermion"

②

$$\nu_i^c = C\bar{\nu}_i^T \overset{!}{=} e^{-i\varphi_i}\nu_i \Rightarrow \nu_i = e^{i\varphi_i} C\bar{\nu}_i^T \Rightarrow \nu_i^T = e^{i\varphi_i}\bar{\nu}_i C^T = -e^{i\varphi_i}\bar{\nu}_i C$$

· fermion-propagator: $\overbrace{\psi(x)\bar{\psi}(y)} = i S_F(x-y)$

↳ with: $$S_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \frac{\not{q}+m}{q^2-m^2+i\varepsilon} e^{-iq(x-y)}$$

⇒ ~~p~~propagator needs a definite mass

⇒ this implies for the case at hand:

$$\underline{\underline{\overbrace{\nu_{eL}(x_1)\nu_{eL}^T}(x_2)}} = \sum_{i,j=1}^{3} \tilde{U}_{ei}\tilde{U}_{ej}\cdot P_L \overbrace{\nu_i(x_1)\nu_j^T}(x_2) P_L =$$

$$= -\sum_{i,j=1}^{3} \tilde{U}_{ei}\tilde{U}_{ej}\cdot P_L e^{i\varphi_j} \underbrace{\overbrace{\nu_i(x_1)\bar{\nu}_j}(x_2)}_{\equiv iS_F^{(i)}(x_1-x_2)\delta_{ij}\text{, where}} \overbrace{C P_L}^{=P_L C} =$$

$$S_F^{(i)} \equiv S_F\big|_{m\to m_i}$$

$$= -i\sum_{i=1}^{3}\left(\tilde{U}_{ei}e^{i\varphi_i/2}\right)^2 P_L S_F^{(i)} P_L C =$$

$$P_L S_F^{(i)} P_L = \int\frac{d^4q}{(2\pi)^4}\frac{P_L(\not{q}+m_i)P_L}{q^2-m_i^2+i\varepsilon} e^{-iq(x_1-x_2)} =$$

$$= \int\frac{d^4q}{(2\pi)^4}\frac{\not{q}\overbrace{P_R P_L}^{=0} + m_i \overbrace{P_L^2}^{=P_L}}{q^2-m_i^2+i\varepsilon} e^{-iq(x_1-x_2)} =$$

$q^2 \sim \langle q^2\rangle \sim$
$\sim (100\,\mathrm{MeV})^2 \gg$
$\gg 1\,\mathrm{eV}^2 \gtrsim m_i^2$
$\Rightarrow m_i^2$ can be neglected in the denominator

$$= m_i \int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2} P_L$$

$$\underline{\underline{= -i\left(\sum_{i=1}^{3} m_i U_{ei}^2\right)\int\frac{d^4q}{(2\pi)^4}\frac{e^{-iq(x_1-x_2)}}{q^2} P_L}}$$

↳ $\equiv m_{ee} \hat{=}$ "effective neutrino mass"

· these results can be inserted into the expression for the amplitude:

3)

$$\mathcal{M}_{0\nu\beta\beta} = \frac{+i\, G_F^2\, m_{ee}}{2^4 \pi^4} \int d^4x_1\, d^4x_2\, d^4q \; \frac{e^{-iq(x_1-x_2)}}{q^2} \langle A' | T\{J_\mu(x_1) J_\nu(x_2)\} | 0\rangle \cdot$$

$$\cdot \langle f | N\{\bar{e}(x_1)\gamma^\mu C P_L \gamma^{\nu T} \bar{e}^T(x_2)\} | i\rangle$$

- one can simplify the particle physics factor further:
  - external fermionic states: $|i\rangle = |0\rangle$ (no electrons / neutrinos)

    $|f\rangle = |p_1\rangle |p_2\rangle = \hat{a}_{p_1}^{s_1 \dagger} \hat{a}_{p_2}^{s_2\dagger} |0\rangle$ (two electrons)
  - electron fields:

$$\bar{e}(x_1) = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E'}} \sum_{s'} \left(\hat{b}_{\vec{p}'}^{s'} \bar{v}^{s'}(p') e^{-ip'x_1} + \hat{a}_{\vec{p}'}^{s'\dagger} \bar{u}^{s'}(p') e^{+ip'x_1} \right)$$

$$\bar{e}^T(x_2) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E}} \sum_{s} \left(\hat{b}_{\vec{p}}^{s} \bar{v}^{sT}(p) e^{-ipx_2} + \hat{a}_{\vec{p}}^{s\dagger} \bar{u}^{sT}(p) e^{+ipx_2} \right)$$

  - normal ordering:

    $N\{\ldots\} \Rightarrow$ all daggers to the left

    $\Rightarrow$ this implies:

$$\langle f | N\{\bar{e}(x_1) \underbrace{\gamma^\mu C P_L \gamma^{\nu T}}_{\equiv \Gamma^{\mu\nu}} \bar{e}^T(x_2)\} | i\rangle =$$

$$= \langle 0 | \hat{a}_{p_1}^{s_1} \hat{a}_{p_2}^{s_2} \cdot N\{\bar{e}(x_1) \Gamma^{\mu\nu} \bar{e}^T(x_2)\} |0\rangle =$$

$$= \frac{1}{(2\pi)^6} \int d^3p\, d^3p' \frac{1}{2\sqrt{E_1 E_2}} \sum_{s,s'} \bar{u}^{s'}(p') \Gamma^{\mu\nu} \bar{u}^{sT}(p) \cdot e^{i(p'x_1 + px_2)} \cdot$$

$$\cdot \langle 0 | \hat{a}_{p_1}^{s_1} \hat{a}_{p_2}^{s_2} \{ \underbrace{0 + 0 + 0}_{\substack{\text{only terms} \\ \text{without } \hat{b}_p^s \\ \text{contribute,} \\ \text{since } \hat{b}_p^s|0\rangle = 0}} + \hat{a}_{\vec{p}'}^{s'\dagger} \hat{a}_{\vec{p}}^{s\dagger} \} |0\rangle .$$

  - anti-commutation relation: $\{\hat{a}_{\vec{p}}^r, \hat{a}_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q})$

$$\Rightarrow \text{it follows: } \langle 0 | \hat{a}_{\vec{p}_1}^{s_1} \hat{a}_{\vec{p}_2}^{s_2} \hat{a}_{\vec{p}'}^{s'\dagger} \hat{a}_{\vec{p}}^{s\dagger} |0\rangle = (2\pi)^3 \delta^{s_2 s'} \delta^{(3)}(\vec{p}' - \vec{p}_2) \langle 0 | \hat{a}_{\vec{p}_1}^{s_1} \hat{a}_{\vec{p}}^{s\dagger} |0\rangle -$$

$$- \langle 0 | \hat{a}_{\vec{p}_1}^{s_1} \hat{a}_{\vec{p}'}^{s'\dagger} \hat{a}_{\vec{p}_2}^{s_2} \hat{a}_{\vec{p}}^{s\dagger} |0\rangle =$$

(4)

$$= (2\pi)^6 \delta^{ss_1} \delta^{s's_2} \delta^{(3)}(\vec{p}-\vec{p}_1)\, \delta^{(3)}(\vec{p}\,'-\vec{p}_2) \underbrace{\langle 0|0\rangle}_{=1} -$$

$$- \underbrace{(2\pi)^3 \delta^{s's_2} \delta^{(3)}(\vec{p}\,'-\vec{p}_2) \langle 0| \hat{a}^{s\dagger}_{\vec{p}} \underbrace{\hat{a}^{s_1}_{\vec{p}_1} |0\rangle}_{=0}}_{=0} -$$

$$- (2\pi)^3 \delta^{ss_2} \delta^{(3)}(\vec{p}-\vec{p}_2) \langle 0| \hat{a}^{s_1}_{\vec{p}_1} \hat{a}^{s'\dagger}_{\vec{p}\,'} |0\rangle + \overbrace{\langle 0| \hat{a}^{s_1}_{\vec{p}_1} \hat{a}^{s'\dagger}_{\vec{p}\,'} \hat{a}^{s\dagger}_{\vec{p}} \hat{a}^{s_2}_{\vec{p}_2} |0\rangle}^{=0} =$$

$$= (2\pi)^6 \delta^{ss_1} \delta^{s's_2} \delta^{(3)}(\vec{p}-\vec{p}_1)\, \delta^{(3)}(\vec{p}\,'-\vec{p}_2) -$$
$$- (2\pi)^6 \delta^{ss_2} \delta^{s's_1} \delta^{(3)}(\vec{p}-\vec{p}_2)\, \delta^{(3)}(\vec{p}\,'-\vec{p}_1)$$

$\Rightarrow$ thus:

$$\langle f | N\{\bar{e}(x_1) \Gamma^{\mu u} \bar{e}^T(x_2)\} | i \rangle =$$

$$= \frac{1}{(2\pi)^6} \int d^3p\, d^3p' \frac{1}{2\sqrt{E_1 E_2}} \sum_{s,s'} \bar{u}^{s'}(p') \Gamma^{\mu u} \bar{u}^{sT}(p)\, e^{i(p'x_1 + px_2)} \cdot (2\pi)^6 \cdot$$

$$\cdot (-1) \cdot \left[\delta^{s's_1} \delta^{ss_2} \delta^{(3)}(\vec{p}\,'-\vec{p}_1)\, \delta^{(3)}(\vec{p}-\vec{p}_2) - (p_1 \leftrightarrow p_2,\, s_1 \leftrightarrow s_2) \right] =$$

$$= \frac{-1}{2\sqrt{E_1 E_2}} \bar{u}^{s_1}(p_1) \gamma^\mu C P_L \gamma^{uT} \bar{u}^{s_2 T}(p_2)\, e^{i(p_1 x_1 + p_2 x_2)} -$$
$$- (p_1 \leftrightarrow p_2,\, s_1 \leftrightarrow s_2)$$

$\Rightarrow$ in total:

$$\mathcal{M}_{0\nu\beta\beta} = \frac{-i G_F^2 m_{ee}}{2^5 \pi^4 \sqrt{E_1 E_2}} \int d^4x_1\, d^4x_2\, d^4q \frac{e^{-iq(x_1-x_2)}}{q^2} \langle A' | T\{J_\mu(x_1) J_u(x_2)\} | A \rangle \cdot$$

$$\cdot \bar{u}^{s_1}(p_1) \gamma^\mu C P_L \gamma^{uT} \bar{u}^{s_2 T}(p_2)\, e^{i(p_1 x_1 + p_2 x_2)} - (p_1 \leftrightarrow p_2,\, s_1 \leftrightarrow s_2)$$

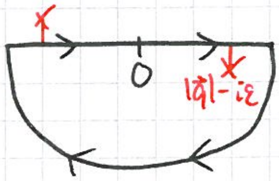
- integration over q_0:

$$\int d^4q \frac{e^{-iq(x_1-x_2)}}{q^2} = \int d^3q \int_{-\infty}^{+\infty} dq_0 \frac{e^{-iq_0(t_1-t_2)}\, e^{+i\vec{q}(\vec{x}_1-\vec{x}_2)}}{(q_0 - (|\vec{q}| - i\varepsilon))(q_0 - (-|\vec{q}| + i\varepsilon))}$$

- shifting the poles:
  - if $t_1 - t_2 > 0$: contour can be closed for $\mathrm{Im}\, q_0 < 0$, due to $e^{-iq_0(t_1-t_2)} \sim e^{-i(-i|\mathrm{Im}\, q_0|)(t_1-t_2)} = e^{-|\mathrm{Im}\, q_0|(t_1-t_2)} \to 0$ for $|\mathrm{Im}\, q_0| \to \infty$

⑤

$\Rightarrow$ for propagation forward in time, the positive energy pole should be included:

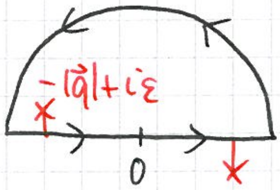


$\Rightarrow$ additional minus-sign (contour ~~an~~ in mathematically negative direction)

- if $t_1 - t_2 < 0$: contour can be closed for $\text{Im}\, q_0 > 0$, due to

$$e^{-iq_0(t_1-t_2)} \sim e^{i i \text{Im}\, q_0 (t_2 - t_1)} = e^{-\text{Im}\, q_0 (t_2-t_1)} \to 0 \text{ for } \text{Im}\, q_0 \to \infty$$

$\Rightarrow$ for propagation backward in time, the ~~po~~ negative energy pole should be included:



$\Rightarrow$ no minus-sign

- residue theorem: $\displaystyle\int f(z)\,dz = 2\pi i \cdot \sum_i \operatorname*{res}_{z_i} f(z)$

$\Rightarrow$ here: $$f(q_0) = \frac{e^{-iq_0(t_1-t_2)}}{(q_0 - (|\vec{q}| - i\varepsilon))(q_0 - (-|\vec{q}| + i\varepsilon))}$$

$\Rightarrow$ residues: ~~an~~ easy, since we only have poles of 1st order!

$$\operatorname*{res}_{+|\vec{q}|-i\varepsilon} f(q_0) = \lim_{q_0 \to |\vec{q}|-i\varepsilon} \left[(q_0 - |\vec{q}| + i\varepsilon)\, f(q_0)\right] = \frac{e^{-i|\vec{q}|(t_1-t_2)}}{2|\vec{q}|}$$ $\Rightarrow$ with minus

$$\operatorname*{res}_{-|\vec{q}|+i\varepsilon} f(q_0) = \lim_{q_0 \to -|\vec{q}|+i\varepsilon} \left[(q_0 + |\vec{q}| - i\varepsilon)\, f(q_0)\right] = -\frac{e^{+i|\vec{q}|(t_1-t_2)}}{2|\vec{q}|}$$ $\Rightarrow$ no minus

- in total, one finds:

$$\int_{-\infty}^{+\infty} dq_0 \, \frac{e^{-iq_0(t_1-t_2)}}{q^2} = \frac{-2\pi i}{2|\vec{q}|}\left[\Theta(t_1-t_2)\, e^{-i|\vec{q}|(t_1-t_2)} + \Theta(t_2-t_1)\, e^{+i|\vec{q}|(t_1-t_2)}\right]$$

$$\Rightarrow \int d^4q \, \frac{e^{-iq(x_1-x_2)}}{q^2} = \int d^3q \, \frac{-i\pi}{|\vec{q}|} \, e^{+i\vec{q}(\vec{x}_1-\vec{x}_2)} \Big[\Theta(t_1-t_2)\, e^{-i|\vec{q}|t_1} e^{+i|\vec{q}|t_2} + \Theta(t_2-t_1)\, e^{+i|\vec{q}|t_1} e^{-i|\vec{q}|t_2}\Big].$$

⑥

$\Rightarrow$ thus:

$$\mathcal{M}_{0\nu\beta\beta} = \frac{-G_F^2 m_{ee}}{2^5\pi^3\sqrt{E_1E_2}} \int d^4x_1\, d^4x_2\, d^3q\, \frac{e^{+i\vec{q}(\vec{x}_1-\vec{x}_2)}}{|\vec{q}|}\, e^{i(p_1x_1+p_2x_2)} \langle A'|T\{J_\mu(x_1)J_\nu(x_2)\}|A\rangle \cdot$$

$$\cdot\left[\Theta(t_1-t_2)e^{-i|\vec{q}|t_1}e^{+i|\vec{q}|t_2} + \Theta(t_2-t_1)e^{+i|\vec{q}|t_1}e^{-i|\vec{q}|t_2}\right]\cdot \bar{u}^{s_1}(p_1)\gamma^\mu C P_L \gamma^{\nu T}\bar{u}^{s_2 T}(p_2) -$$

$$-(p_1\leftrightarrow p_2,\, s_1\leftrightarrow s_2)$$

- nuclear physics part & related pieces:
  - one can insert a complete set of intermediate states, $\sum_n |n\rangle\langle n| = \mathbb{1}$, and extract the time-dependences ~~[illegible]~~ (non-relativistic approximation):

$$\langle A'|T\{J_\mu(x_1)J_\nu(x_2)\}|A\rangle = \Theta(t_1-t_2)\langle A'|J_\mu(x_1)\sum_n|n\rangle\langle n|J_\nu(x_2)|A\rangle +$$

$$+\Theta(t_2-t_1)\langle A'|J_\nu(x_2)\sum_n|n\rangle\langle n|J_\mu(x_1)|A\rangle = \quad J_\alpha(\vec{x}) \equiv J_\alpha(t=0,\vec{x})$$

$$= \sum_n\Big[\Theta(t_1-t_2)\langle A'|e^{iE_ft_1}J_\mu(\vec{x}_1)e^{-iE_nt_1}|n\rangle\langle n|e^{iE_nt_2}J_\nu(\vec{x}_2)e^{-iE_it_2}|A\rangle +$$

$$+\Theta(t_2-t_1)\langle A'|e^{iE_ft_2}J_\nu(\vec{x}_2)e^{-iE_nt_2}|n\rangle\langle n|e^{iE_nt_1}J_\mu(\vec{x}_1)e^{-iE_it_1}|A\rangle\Big] =$$

$$= \sum_n\Big[\Theta(t_1-t_2)e^{i(E_f-E_n)t_1}e^{i(E_n-E_i)t_2}\langle A'|J_\mu(\vec{x}_1)|n\rangle\langle n|J_\nu(\vec{x}_2)|A\rangle +$$

$$+\Theta(t_2-t_1)e^{i(E_n-E_i)t_1}e^{i(E_f-E_n)t_2}\langle A'|J_\nu(\vec{x}_2)|n\rangle\langle n|J_\mu(\vec{x}_1)|A\rangle\Big]$$

- ~~[illegible]~~ this already simplifies the amplitude, when ~~[illegible]~~ also using $\Theta^2(x)=\Theta(x)$ and $\Theta(x)\Theta(-x)=0$:

$$\mathcal{M}_{0\nu\beta\beta} = \frac{-G_F^2 m_{ee}}{2^5\pi^3\sqrt{E_1E_2}} \int d^4x_1\, d^4x_2\, d^3q\, \frac{e^{+i\vec{q}(\vec{x}_1-\vec{x}_2)}}{|\vec{q}|}\, e^{i(p_1x_1+p_2x_2)} \cdot$$

$$\cdot\sum_n\Big[\Theta(t_1-t_2)e^{i(E_f-E_n-|\vec{q}|)t_1}e^{i(E_n-E_i+|\vec{q}|)t_2}\langle A'|J_\mu(\vec{x}_1)|n\rangle\langle n|J_\nu(\vec{x}_2)|A\rangle +$$

$$+\Theta(t_2-t_1)e^{i(E_n-E_i+|\vec{q}|)t_1}e^{i(E_f-E_n-|\vec{q}|)t_2}\langle A'|J_\nu(\vec{x}_2)|n\rangle\langle n|J_\mu(\vec{x}_1)|A\rangle\Big]\cdot$$

$$\cdot\bar{u}^{s_1}(p_1)\gamma^\mu C P_L\gamma^{\nu T}\bar{u}^{s_2 T}(p_2) - (p_1\leftrightarrow p_2,\, s_1\leftrightarrow s_2)$$

⑦

· Fourier ~~transform~~ representation of the Heaviside function:

$$\Theta(t) = \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-it\omega}}{\omega+i\varepsilon}\, d\omega$$

⇒ ~~with the~~ also using the Fourier representation of the δ-function, $\delta(E) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iEt}\,dt$, one can perform the integrations over the times t_1 & t_2:

× 1st ~~integral~~ contribution:

$$\int dt_1 dt_2\, e^{i(E_1t_1+E_2t_2)}\, \frac{i}{2\pi}\int \frac{e^{-i\omega(t_1-t_2)}}{\omega+i\varepsilon}\, d\omega\; e^{i(E_f-E_n-|\vec{q}|)t_1}\, e^{i(E_n-E_i+|\vec{q}|)t_2} =$$

$$= \frac{i}{2\pi}\int\frac{d\omega}{\omega}\, \underbrace{\int e^{i(E_1-\omega+E_f-E_n-|\vec{q}|)t_1}\,dt_1}_{=2\pi\,\delta(E_1-\omega+E_f-E_n-|\vec{q}|)}\; \underbrace{\int e^{i(E_2+\omega+E_n-E_i+|\vec{q}|)t_2}\,dt_2}_{=2\pi\,\delta(E_2+\omega+E_n-E_i+|\vec{q}|)} =$$

$$= 2\pi i\int\frac{d\omega}{\omega}\, \delta(\omega-E_1-E_f+E_n+|\vec{q}|)\; \delta(\omega-(E_i-E_2-E_n-|\vec{q}|)) =$$

$$= \frac{2\pi i\,\delta(E_i-E_2-\cancel{E_n}-\cancel{|\vec{q}|}-E_1-E_f+\cancel{E_n}+\cancel{|\vec{q}|})}{E_i-E_2-E_n-|\vec{q}|} =$$

$$= \frac{2\pi i\,\delta(E_i-E_f-E_1-E_2)}{E_i-E_2-E_n-|\vec{q}|}$$

× 2nd contribution:

$$\int dt_1 dt_2\, e^{i(E_1t_1+E_2t_2)}\, \frac{i}{2\pi}\int \frac{e^{-i\omega(t_2-t_1)}}{\omega+i\varepsilon}\, d\omega \cdot e^{i(E_n-E_i+|\vec{q}|)t_1}\, e^{i(E_f-E_n-|\vec{q}|)t_2} =$$

$$= \frac{i}{2\pi}\int\frac{d\omega}{\omega}\, \underbrace{\int e^{i(E_1+\omega+E_n-E_i+|\vec{q}|)t_1}\,dt_1}_{=2\pi\,\delta(E_1+\omega+E_n-E_i+|\vec{q}|)}\; \underbrace{\int e^{i(E_2-\omega+E_f-E_n-|\vec{q}|)t_2}\,dt_2}_{=2\pi\,\delta(E_2-\omega+E_f-E_n-|\vec{q}|)} =$$

$$= 2\pi i\int\frac{d\omega}{\omega}\; \delta(\omega-(E_i-E_1-E_n-|\vec{q}|)) =$$

$$\delta(\omega-E_f-E_2+E_n+|\vec{q}|)$$

$$= \frac{2\pi i\,\delta(E_i - E_1 - \cancel{E_n} - \cancel{|\vec{q}|} - E_f - E_2 + \cancel{E_n} + \cancel{|\vec{q}|})}{E_i - E_1 - E_n - |\vec{q}|} =$$

$$= \frac{2\pi i\,\delta(E_i - E_f - E_1 - E_2)}{E_i - E_1 - E_n - |\vec{q}|}$$

$\Rightarrow$ this leads to:

$$\mathcal{M}_{0\nu\beta\beta} = \frac{-i G_F^2 m_{ee}}{2^4 \pi^2 \sqrt{E_1 E_2}} \int d^3x_1\, d^3x_2\, d^3q\, \frac{e^{+i\vec{q}(\vec{x}_1 - \vec{x}_2)}}{|\vec{q}|}\, e^{-i(\vec{p}_1\vec{x}_1 + \vec{p}_2\vec{x}_2)} \cdot$$

$$\cdot\, \delta(E_f - E_i - E_1 - E_2) \sum_n \left[\frac{\langle A'| J_\mu(\vec{x}_1)|n\rangle\langle n| J_\mu(\vec{x}_2)|A\rangle}{E_i - E_2 - E_n - |\vec{q}|} + \frac{\langle A'| J_\mu(\vec{x}_2)|n\rangle\langle n| J_\mu(\vec{x}_1)|A\rangle}{E_i - E_1 - E_n - |\vec{q}|} \right] \cdot$$

$$\cdot\, \bar{u}^{s_1}(p_1)\gamma^\mu C P_L \gamma^{\mu T} \bar{u}^{s_2 T}(p_2) - (p_1 \leftrightarrow p_2,\ s_1 \leftrightarrow s_2)$$

- closure approximation: $E_n \to \langle E_n \rangle$

  $\Rightarrow$ if the intermediate states lie close to a closed shell, their energies can be approximated by an average value

  $\hookrightarrow$ this also allows to once more use $\sum_n |n\rangle\langle n| = \mathbb{1}$

- $\Rightarrow$ hence:

$$\mathcal{M}_{0\nu\beta\beta} = \frac{-i G_F^2 m_{ee}}{2^4 \pi^2 \sqrt{E_1 E_2}} \int d^3x_1\, d^3x_2\, d^3q\, \frac{e^{+i\vec{q}(\vec{x}_1 - \vec{x}_2)}}{|\vec{q}|}\, e^{-i(\vec{p}_1\vec{x}_1 + \vec{p}_2\vec{x}_2)} \cdot$$

$$\cdot\, \delta(E_f - E_i - E_1 - E_2) \left[\frac{\langle A'| J_\mu(\vec{x}_1) J_\mu(\vec{x}_2)|A\rangle}{E_i - E_2 - \langle E_n\rangle - |\vec{q}|} + \frac{\langle A'| J_\mu(\vec{x}_2) J_\mu(\vec{x}_1)|A\rangle}{E_i - E_1 - \langle E_n\rangle - |\vec{q}|} \right] \cdot$$

$$\cdot\, \bar{u}^{s_1}(p_1)\gamma^\mu C P_L \gamma^{\mu T} \bar{u}^{s_2 T}(p_2) - (p_1 \leftrightarrow p_2,\ s_1 \leftrightarrow s_2)$$

- non-relativistic nuclear currents:

$$J_\alpha(\vec{x}) \approx \sum_m \tau_m^+ \left[g_V\, g_{\alpha 0} + g_A\, g_{\alpha j}\, \sigma_m^j \right] \delta^{(3)}(\vec{x} - \vec{r}_m)$$

sum over all nucleons

nuclear isospin raising operator: $\tau_m^+ |n\rangle_m = |p\rangle_m$, $\tau_m^+ |p\rangle_m = 0$

spin-flip of the m-th nucleon in the j-direction

position of the m-th nucleon

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$\Rightarrow$ <u>this implies $J_\mu(\vec{x}_1) J_\mu(\vec{x}_2) = J_\mu(\vec{x}_2) J_\mu(\vec{x}_1)$:</u>

× g_V^2-term:

$$\left[\sum_m \tau_m^+ g_V g_{\mu 0} \delta^{(3)}(\vec{x}_1 - \vec{r}_m), \sum_{m'} \tau_{m'}^+ g_V g_{\mu 0} \delta^{(3)}(\vec{x}_2 - \vec{r}_{m'})\right] =$$

$$= \sum_{m,m'} g_V^2 g_{\mu 0} g_{\mu 0} \delta^{(3)}(\vec{x}_1 - \vec{r}_m) \delta^{(3)}(\vec{x}_2 - \vec{r}_{m'}) \underbrace{[\tau_m^+, \tau_{m'}^+]}_{0} = \underline{\underline{0}}$$

$$\cancel{\delta_{mm'} [\tau_m^+, \tau_{m'}^+] = \delta_{mm'} \ldots = 0}$$

$\Rightarrow$ commutes

× g_A^2-term:

$$\left[\sum_m \tau_m^+ g_A g_{\mu j} \sigma_m^j \delta^{(3)}(\vec{x}_1 - \vec{r}_m), \sum_{m'} \tau_{m'}^+ g_A g_{\mu k} \sigma_{m'}^k \delta^{(3)}(\vec{x}_2 - \vec{r}_{m'})\right] =$$

$$= \sum_{m,m'} g_A^2 \tau_m^+ \tau_{m'}^+ g_{\mu j} g_{\mu k} \delta^{(3)}(\vec{x}_1 - \vec{r}_m) \delta^{(3)}(\vec{x}_2 - \vec{r}_{m'}) [\sigma_m^j, \sigma_{m'}^k] =$$

↑ raising operators always commute

$$[\sigma_m^j, \sigma_{m'}^k] = \delta_{mm'} [\sigma_m^j, \sigma_{m}^k] = \delta_{mm'} \cdot 2i\varepsilon^{jkl} \sigma_m^l$$

↑ only relevant if the same nucleon

$$= \sum_m g_A^2 (\tau_m^+)^2 g_{\mu j} g_{\mu k} 2i\varepsilon^{jkl} \sigma_m^l \delta^{(3)}(\vec{x}_1 - \vec{r}_m) \delta^{(3)}(\vec{x}_2 - \vec{r}_m) = \underline{\underline{0}}$$

the isospin of one and the same nucleon cannot be raised twice:

$(\tau_m^+)^2 |p\rangle_m = \tau_m^+ |n\rangle_m = 0$

$(\tau_m^+)^2 |n\rangle_m = \tau_m^+ \cdot 0 = 0$

$\Downarrow$

0

σ_m^l $\Downarrow$ would flip only one spin, which is disfavoured, since the spin-conserving transitions $0^+ \to 0^+$ are strongly favoured in $0\nu\beta\beta$

$\Downarrow$

0

$\Rightarrow$ commutes

× $g_V g_A$-terms:

▫ 1st term:

$$\left[\sum_m \tau_m^+ g_V g_{\mu 0} \delta^{(3)}(\vec{x}_1 - \vec{r}_m), \sum_{m'} \tau_{m'}^+ g_A g_{\mu j} \sigma_{m'}^j \delta^{(3)}(\vec{x}_2 - \vec{r}_{m'})\right] =$$

$$= \sum_{m,m'} g_V g_A \tau_m^+ \tau_{m'}^+ g_{\mu 0} g_{\nu j} \sigma_{m'}^j \delta^{(3)}(\vec{x}_1 - \vec{r}_m) \delta^{(3)}(\vec{x}_2 - \vec{r}_{m'}) \underbrace{[1,1]}_{=0} = \underline{\underline{0}}$$

$\Rightarrow$ commutes

- 2nd term: commutes by the same argument

$\Rightarrow$ <u>this also simplifies the nuclear part:</u>

$$\langle A' | J_\mu(\vec{x}_1) J_\nu(\vec{x}_2) | A \rangle =$$

$$\cancel{= \sum_{m,m'} \tau_m^+ \tau_{m'}^+ [g_V^2 \langle A'|}$$

$$= \sum_{m,m'} [g_V^2 g_{\mu 0} g_{\nu 0} \langle A' | \tau_m^+ \tau_{m'}^+ | A \rangle +$$

$$+ g_A^2 g_{\mu j} g_{\nu k} \langle A' | \tau_m^+ \tau_{m'}^+ \sigma_m^j \sigma_{m'}^k | A \rangle +$$

$$+ g_V g_A g_{\mu 0} g_{\nu k} \underline{\langle A' | \tau_m^+ \tau_{m'}^+ \sigma_{m'}^k | A \rangle} +$$

$\rightarrow$ only one spin changed $\Rightarrow$ no contribution to $0^+ \rightarrow 0^+$ transition

$$+ g_V g_A g_{\mu j} g_{\nu 0} \overline{\langle A' | \tau_m^+ \tau_{m'}^+ \sigma_m^j | A \rangle}] .$$

$$\cdot \delta^{(3)}(\vec{x}_1 - \vec{r}_m) \delta^{(3)}(\vec{x}_2 - \vec{r}_{m'}) =$$

$$= \sum_{m,m'} (g_V^2 g_{\mu 0} g_{\nu 0} \langle A' | \tau_m^+ \tau_{m'}^+ | A \rangle +$$

$$+ g_A^2 g_{\mu j} g_{\nu k} \langle A' | \tau_m^+ \tau_{m'}^+ \sigma_m^j \sigma_{m'}^k | A \rangle] \delta^{(3)}(\vec{x}_1 - \vec{r}_m) \delta^{(3)}(\vec{x}_2 - \vec{r}_{m'})$$

- <u>long wavelength approximation:</u>

$$\left. \begin{array}{l} |\vec{x}_i| \sim |\vec{r}_m| \sim R \sim (100\ \mathrm{MeV})^{-1} \\ |\vec{p}_i| \underset{\uparrow}{\simeq} Q \ll 100\ \mathrm{MeV} \end{array} \right\} \Rightarrow |\vec{p}_i||\vec{x}_i| \sim 0 \Rightarrow e^{-i(\vec{p}_1 \vec{x}_1 + \vec{p}_2 \vec{x}_2)} \simeq 1$$

<u>Q-value:</u>

$$Q \equiv M_i - M_f \simeq E_i - E_f$$

(non-relativistic: $E_f \simeq M_f + \frac{\vec{p}^2}{2M_f} \simeq M_f$)

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~~• also the leptonic part can~~

$\Rightarrow$ thus:

$$\mathcal{M}_{0\nu\beta\beta} = \frac{-i G_F^2 m_{ee}}{2^4 \pi^2 \sqrt{E_1 E_2}} \sum_{m,m'} \int d^3 q \, \frac{e^{+i\vec{q}(\vec{r}_m - \vec{r}_{m'})}}{|\vec{q}|} \cdot \delta(Q - E_1 - E_2) \cdot$$

$$\cdot \left[g_V^2 g_{\mu 0} g_{\mu 0} \langle A' | \tau_m^+ \tau_{m'}^+ | A \rangle + g_A^2 g_{\mu j} g_{\mu k} \langle A' | \tau_m^+ \tau_{m'}^+ \sigma_m^j \sigma_{m'}^k | A \rangle \right] \cdot$$

$$\cdot \left[\frac{1}{E_i - E_1 - \langle E_n \rangle - |\vec{q}|} + \frac{1}{E_i - E_2 - \langle E_n \rangle - |\vec{q}|} \right] \cdot \bar{u}^{s_1}(p_1) \gamma^\mu C P_L \gamma^{\nu T} \bar{u}^{s_2 T}(p_2) -$$

$$- (p_1 \leftrightarrow p_2, s_1 \leftrightarrow s_2)$$

• the leptonic part can be simplified further using $C^2 = -1$ and $C \gamma^{\mu T} C = -\gamma^\mu$:

$$\bar{u}^{s_1}(p_1) \gamma^\mu \underbrace{C P_L}_{= P_L C} \gamma^{\nu T} \bar{u}^{s_2 T}(p_2) = - \bar{u}^{s_1}(p_1) \gamma^\mu P_L \underbrace{(C \gamma^{\nu T} C)}_{= -\gamma^\nu} \underbrace{C \bar{u}^{s_2 T}(p_2)}_{= (u^{s_2}(p_2))^c = v^{s_2}(p_2)} =$$

$$= \bar{u}^{s_1}(p_1) \gamma^\mu P_L \gamma^\nu v^{s_2}(p_2) = \bar{u}^{s_1}(p_1) \gamma^\mu \gamma^\nu P_R v^{s_2}(p_2) =$$

$$= \bar{u}^{s_1}(p_1) \Big[\underbrace{g^{\mu\nu}}_{\text{symmetric}} + \tfrac{1}{2} \underbrace{[\gamma^\mu, \gamma^\nu]}_{\text{antisymmetric}} \Big] P_R v^{s_2}(p_2)$$

symmetric antisymmetric in $\mu \leftrightarrow \nu$

• to show that the nuclear part is also symmetric, it is useful to go back to the form $\langle A' | J_\mu(\vec{x}_1) J_\mu(\vec{x}_2) | A \rangle$, but to make use of the fact that the nuclear currents commute and of the long wavelengths and closure approximations:

$$\int d^3 x_1 \, d^3 x_2 \, d^3 q \, \frac{e^{i\vec{q}(\vec{x}_1 - \vec{x}_2)}}{|\vec{q}|} \langle A' | J_\mu(\vec{x}_1) J_\mu(\vec{x}_2) | A \rangle \overset{\text{commutation}}{=}$$

$$= \int d^3 x_1 \, d^3 x_2 \, d^3 q \, \frac{e^{i\vec{q}(\vec{x}_1 - \vec{x}_2)}}{|\vec{q}|} \langle A' | J_\mu(\vec{x}_2) J_\mu(\vec{x}_1) | A \rangle \overset{\text{rename: } \vec{x}_1 \leftrightarrow \vec{x}_2}{=}$$

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$$= \underbrace{\int d^3x_2\, d^3x_1}_{= d^3x_1\, d^3x_2} d^3q \;\frac{e^{i\vec{q}(\vec{x}_2-\vec{x}_1)}}{|\vec{q}|} \langle A'| J_\mu(\vec{x}_1) J_\nu(\vec{x}_2)|A\rangle =$$

$$= \int d^3x_1\, d^3x_2\, d^3q \;\frac{e^{i\vec{q}(\vec{x}_1-\vec{x}_2)}}{|\vec{q}|} \langle A'| J_\mu(\vec{x}_1) J_\nu(\vec{x}_2)|A\rangle$$

$\uparrow$

substitute: $\vec{q} \to -\vec{q} \Rightarrow d^3q \to d^3q,\ |\vec{q}| \to |-\vec{q}| = |\vec{q}|$ ($\Rightarrow$ also in energy-denominators)

$\Rightarrow$ nuclear part symmetric in $\mu \Leftrightarrow \nu$

$\Rightarrow [\gamma^\mu, \gamma^\nu]$-part in the leptonic piece does NOT contribute

$\Rightarrow \bar{u}^{s_1}(p_1)\gamma^\mu C P_L \gamma^{\nu T} \bar{u}^{s_2 T}(p_2) \to g^{\mu\nu} \cdot \bar{u}^{s_1}(p_1) P_R v^{s_2}(p_2)$

- with $g^{\mu\nu} g_{\mu 0} g_{\nu 0} = g^{00} = +1$, $g^{\mu\nu} g_{\mu j} g_{\nu k} \sigma^j_m \sigma^k_{m'} =$

$$= g^{jk} \sigma^j_m \sigma^k_{m'} = -\delta^{jk} \sigma^j_m \sigma^k_{m'} = -\sigma^j_m \sigma^j_{m'} = -\vec{\sigma}_m \vec{\sigma}_{m'}:$$

$$\mathcal{M}_{0\nu\beta\beta} = \frac{-i G_F^2 m_{ee}}{2^4 \pi^2 \sqrt{E_1 E_2}} \delta(Q - E_1 - E_2) \sum_{m,m'} \int d^3q \;\frac{e^{+i\vec{q}(\vec{r}_m - \vec{r}_{m'})}}{|\vec{q}|} \cdot$$

$$\cdot \left[g_V^2 \langle A'| \tau^+_m \tau^+_{m'} |A\rangle - g_A^2 \langle A'| \tau^+_m \tau^+_{m'} \vec{\sigma}_m \vec{\sigma}_{m'} |A\rangle \right] \cdot [\ldots] \cdot$$

$$\cdot \bar{u}^{s_1}(p_1) P_R v^{s_2}(p_2) - (p_1 \Leftrightarrow p_2,\ s_1 \Leftrightarrow s_2)$$

$\uparrow$ energy denominators

- exchange-term: go back to previous form (rest symmetric anyway)

$$\bar{u}^{s_1}(p_1)\gamma^\mu C P_L \gamma^{\nu T} \bar{u}^{s_2 T}(p_2)\Big|_{p_1 \Leftrightarrow p_2,\ s_1 \Leftrightarrow s_2} = \bar{u}^{s_2}(p_2)\gamma^\mu C P_L \gamma^{\nu T} \bar{u}^{s_1 T}(p_1) =$$

$$= \left[\bar{u}^{s_2}(p_2)\gamma^\mu C P_L \gamma^{\nu T} \bar{u}^{s_1 T}(p_1)\right]^T = \bar{u}^{s_1}(p_1)\gamma^\nu \underbrace{P_L^T}_{= P_L} \underbrace{C^T}_{= -C} \gamma^{\mu T} \bar{u}^{s_2 T}(p_2) =$$

$$= -\bar{u}^{s_1}(p_1)\gamma^\nu \underbrace{P_L C}_{= C P_L} \gamma^{\mu T} \bar{u}^{s_2 T}(p_2) = -\bar{u}^{s_1}(p_1)\gamma^\nu C P_L \gamma^{\mu T} \bar{u}^{s_2 T}(p_2) =$$

$$= -\bar{u}^{s_1}(p_1)\Big[\underbrace{g^{\nu\mu}}_{= g^{\mu\nu}} + \underbrace{\tfrac{1}{2}[\gamma^\nu, \gamma^\mu]}_{\to \text{ does not contribute}}\Big] P_R\, v^{s_2}(p_2) =$$

$$= -g^{\mu\nu} \cdot \bar{u}^{s_1}(p_1) P_R v^{s_2}(p_2) \Rightarrow \text{exchange term gives an equal contribution}$$

$\Rightarrow$ only factor 2:

$$\mathcal{M}_{0\nu\beta\beta} = \frac{+iG_F^2 m_{ee}}{2^3\pi^2\sqrt{E_1E_2}}\,\delta(Q-E_1-E_2)\cdot\sum_{m,m'}\int d^3q\,\frac{e^{+i\vec{q}(\vec{r}_m-\vec{r}_{m'})}}{|\vec{q}|}\cdot$$

$$\cdot\left[\frac{1}{|\vec{q}|+(E_1+\langle E_n\rangle-E_i)}+\frac{1}{|\vec{q}|+(E_2+\langle E_n\rangle-E_i)}\right]\cdot\bar{u}^{s_2}(p_2)P_R v^{s_1}(p_1)\cdot$$

$$\cdot\left[g_V^2\langle A'|\tau_m^+\tau_{m'}^+|A\rangle - g_A^2\langle A'|\tau_m^+\tau_{m'}^+\vec{\sigma}_m\vec{\sigma}_{m'}|A\rangle\right].$$

- nuclear matrix element:
  - neutrino potential functions:

$$H(|\vec{x}|,a)\equiv\frac{1}{2\pi^2}\int\frac{e^{+i\vec{q}\vec{x}}\,d^3q}{|\vec{q}|(|\vec{q}|+a)}$$

$\Rightarrow$ here:

$$\mathcal{M}_{0\nu\beta\beta} = \frac{iG_F^2 m_{ee}}{2^2\sqrt{E_1E_2}}\,\delta(Q-E_1-E_2)\,\cancel{\sum_{m,m'}}\cdot\bar{u}^{s_2}(p_2)P_R v^{s_1}(p_1)\cdot$$

$$\cdot\sum_{m,m'}\Big(\underbrace{H(|\vec{r}_m-\vec{r}_{m'}|,E_1+\langle E_n\rangle-E_i)+H(|\vec{r}_m-\vec{r}_{m'}|,E_2+\langle E_n\rangle-E_i}_{\equiv\frac{2}{R}h(|\vec{r}_m-\vec{r}_{m'}|)}\Big]\cdot$$

$$\cdot\left[g_V^2\langle A'|\tau_m^+\tau_{m'}^+|A\rangle - g_A^2\langle A'|\tau_m^+\tau_{m'}^+\vec{\sigma}_m\vec{\sigma}_{m'}|A\rangle\right]$$

- nuclear matrix elements:
  - × Fermi-matrix element:

$$M_F\equiv\langle A'|\sum_{m,m'}\tau_m^+\tau_{m'}^+\,h(|\vec{r}_m-\vec{r}_{m'}|)|A\rangle$$

  - × Gamov-Teller matrix element:

$$M_{GT}\equiv\langle A'|\sum_{m,m'}\tau_m^+\tau_{m'}^+\vec{\sigma}_m\vec{\sigma}_{m'}\,h(|\vec{r}_m-\vec{r}_{m'}|)|A\rangle$$

  - × total nuclear matrix element (NME):

$$M^{0\nu}\equiv M_{GT}-\frac{g_V^2}{g_A^2}M_F$$

$\Rightarrow$ hence:

$$\boxed{\mathcal{M}_{0\nu\beta\beta} = \underbrace{\frac{-i\, g_A^2 G_F^2 m_{ee} M^{0\nu}}{2R\sqrt{E_1E_2}}}_{\equiv C_R/\sqrt{E_1E_2}}\, \delta(Q-E_1-E_2)\cdot \bar u^{s_2}(p_2)\, P_R\, v^{s_1}(p_1)}$$

- squared and summed matrix element:

$$\overline{|\mathcal{M}_{0\nu\beta\beta}|^2} = \sum_{s_1 s_2} \frac{-i^2 |C_R|^2}{E_1E_2}\underbrace{[\delta(Q-E_1-E_2)]^2}_{=\frac{T}{2\pi}\delta(Q-E_1-E_2)}\, \bar u^{s_2}(p_2)P_R v^{s_1}(p_1)\bar v^{s_1}(p_1)P_L u^{s_2}(p_2) =$$

pseudoscalar is $i\gamma_5$ and not γ_5

$$= \frac{T}{2\pi}|C_R|^2\, \frac{\delta(Q-E_1-E_2)}{E_1E_2}\cdot \underbrace{\mathrm{Tr}[(\not p_2+m_e)P_R(\not p_1 - m_e)P_L]}_{\equiv \tilde T} =$$

$$\tilde T = \mathrm{Tr}[(\not p_2+m_e)(\not p_1 \underbrace{P_L^2}_{=P_L} - m_e \underbrace{P_R P_L}_{=0})] =$$

$$= \mathrm{Tr}[(\not p_2 + m_e)\not p_1 P_L] = \mathrm{Tr}[\not p_2 \not p_1 P_L] \overset{P_L = \frac12(1-\gamma_5),\ \mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma_5)=0}{=}$$

$$= \frac12\cdot \mathrm{Tr}(\not p_2\not p_1) = \frac12 \mathrm{Tr}(\not p_1\not p_2) = \frac12\cdot 4(p_1p_2) = 2(p_1p_2) =$$

$$= 2(E_1E_2 - \vec p_1\vec p_2)$$

$$= \frac{T}{\pi}|C_R|^2\,\delta(Q-E_1-E_2)\left[1-\frac{\vec p_1\vec p_2}{E_1E_2}\right]$$

- then, the decay rate can be computed using Fermi's Golden Rule:

$$\Gamma_{0\nu\beta\beta} = \frac12 \cdot \frac{1/T}{(2\pi)^6}\int d^3p_1\, d^3p_2\, \overline{|\mathcal{M}_{0\nu\beta\beta}|^2} =$$

statistical factor (two indistinguishable/identical electrons in the final state)

$$* = \frac{1/T}{2^7\pi^6} \int dp_1\, p_1^2 \cdot 4\pi\, dp_2\, p_2^2 \cdot 2\pi\, d(\cos\theta) \cdot \frac{T}{\pi} |C_R|^2 \cdot$$

$\vec{p}_1 \vec{p}_2 = p_1 p_2 \cos\theta$
($p_i \equiv |\vec{p}_i|$)

$$\cdot\, \delta(Q-E_1-E_2)\left[1 - \frac{p_1 p_2}{E_1 E_2}\cos\theta\right] =$$

$\int_{-1}^{+1} d(\cos\theta) \cos\theta = 0$

factor 2 from cos-integral

$$= \frac{|C_R|^2}{2^3\pi^5} \cdot \int dp_1\, p_1^2\, dE_2\, E_2 \sqrt{E_2^2 - m_e^2}\, \delta(E_2 - (Q-E_1)) =$$

$E_2 dE_2 = p_2 dp_2$

$$= \frac{|C_R|^2}{2^3\pi^5} \int dp_1\, p_1^2\, (Q-E_1)\sqrt{(Q-E_1)^2 - m_e^2} =$$

$\cancel{|C_R|^2} = g_A^4 G_F^4$

$$= \frac{g_A^4 G_F^4}{(2\pi)^5 R^2} |M^{0\nu}|^2 |m_{ee}|^2 \overbrace{\int_{E_1=m_e}^{Q-m_e} dE_1\, E_1 (Q-E_1)\sqrt{E_1^2-m_e^2}\sqrt{(Q-E_1)^2-m_e^2}}^{\equiv I}$$

↳ we have neglected the influence of the Coulomb-potential from the nucleus (would have been done using so-called "Fermi-functions")

- the integral is complicated ... BUT: we can simplify it considerably in the approximation $m_e \simeq 0$ (okay for large Q-value)

⇒ then:

$$I \approx \int_{E_1=0}^{Q} dE_1 \cdot \overbrace{E_1 (Q-E_1) E_1 (Q-E_1)}^{= E_1^2 (Q-E_1)^2} =$$

$$= Q^2 \int_{E_1=0}^{Q} dE_1 \cdot E_1^2 - 2Q^3 \int_{E_1=0}^{Q} dE_1 \cdot E_1^3 + \int_{E_1=0}^{Q} dE_1 \cdot E_1^4 =$$

$$= Q^2 \cdot \frac{1}{3} Q^3 - 2Q^3 \cdot \frac{1}{4} Q^2 + \frac{1}{5} \cdot Q^5 =$$

$$= Q^5 \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right] = Q^5 \cdot \frac{10 - 15 + 6}{30} = \frac{Q^5}{30}$$

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⇒ final expression for the decay rate:

$$\boxed{\Gamma_{0\nu\beta\beta} \approx \frac{g_A^4 G_F^4}{(2\pi)^5 R^2} |M^{0\nu}|^2 |m_{ee}|^2 \cdot \frac{Q^5}{30}}$$

⇒ important: • $|M^{0\nu}|$: NME ⇒ ~~unknown to ab~~ only known to ~ 10%
• $|m_{ee}|$: particle physics part ⇒ can be tiny
• Q^5-dependence ⇒ strong preference for large Q

↳ nuclear radius: $R \simeq 1.2\,\text{fm}\,\sqrt[3]{A}$

- furthermore, one can determine from this:

• lifetime τ:
$$\tau \equiv \frac{1}{\Gamma_{0\nu\beta\beta}}$$

• half-life $T_{1/2}$:
$$T_{1/2} \equiv \frac{\ln 2}{\Gamma_{0\nu\beta\beta}}$$