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Lecture 5: Theory of lepton flavour

Leptonic mixing pattern:

- leptonic mixing looks different from quark mixing:

angle	leptons (nu-fit.org)	quarks (PDG)
θ_{12}	$33.5°$	$13.0°$
θ_{13}	$8.5°$	$0.2°$
θ_{23}	$42.3°/49.5°$	$2.4°$

⇒ leptonic mixings significantly larger ⇒ explanation ?!?

- before θ_{13} was measured in 2012, it could have been zero
  ⇒ the proposed mass pattern was TBM:

$$U_{TBM} = \begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \Rightarrow \begin{cases} \tan\theta_{12} = 1/\sqrt{2} \\ \theta_{13} = 0 \\ \theta_{23} = \frac{\pi}{4} \end{cases}$$

tri – bimaximal

↳ this pattern seems so simple that it suggests that some theoretical explanation might be behind

⇒ most popular idea: $\boxed{\text{discrete groups}}$

⇒ using discrete groups, one can impose certain structures on the mass matrices which translate into a mixing pattern

Basics of discrete groups:

- definition: group G

Set of elements $G = \{g_1, g_2, \ldots\}$ with a closed and associative operation "$\circ$",

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such that ~~every elem~~ a neutral element e exists and that every element g has an inverse g^{-1}.

↳ Abelian: "$\circ$" is commutative, $\forall g, g' \in G: g \circ g' = g' \circ g$

↳ discrete: elements can be numbered

↳ finite: $|G| < \infty$

- example: A_4 (= even permutations of 4 objects)

↳ this group is finite: $|A_4| = 4!/2 = 12$

- all elements of a group can be obtained from combinations of only a few elements ("generators")

↳ ~~this~~ the generators are a specific set of elements which obey relations called "presentations"

⇒ A_4-example:
- generators: S, T
- presentations: $S^2 = T^3 = (ST)^3 = \mathbb{1}$
- A_4-elements in terms of S & T:

$\mathbb{1}, S, T, T^2, TS, ST, STS, TST, ST^2, T^2S, TST^2, T^2ST$

- the abstract group elements can also be displayed as certain ~~a~~ $n \times n$-matrices ("representations")

↳ if these cannot be split any further (into independent objects of lower dimensions), they are called "irreducible" ("irreps")

⇒ A_4-example:

Representation	S	T
$\underline{1}$	1	1
$\underline{1}'$	1	$w^2 \equiv e^{4\pi i/3}$
$\underline{1}''$	1	$w \equiv e^{2\pi i/3}$
$\underline{3}$	$\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & w^2 & 0 \\ 0 & 0 & w \end{pmatrix}$

↳ sanity check: $1^2 + 1^2 + 1^2 + 3^2 = 3 + 9 = 12 = |A_4|$ ✓

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$\Rightarrow$ e.g.: $ST^2|_{\underline{1}} = 1$, $ST^2|_{\underline{1}'} = \omega$, $ST^2|_{\underline{1}''} = \omega^2$, $ST^2|_{\underline{3}} = \frac{1}{3}\begin{pmatrix} -1 & 2\omega^2 & -2\omega \\ 2 & -\omega^2 & -2\omega \\ 2 & 2\omega^2 & \omega \end{pmatrix}$

- these group elements tell us how certain objects ("tensors") transform under the action of the group
  - ↳ a triplet $a \equiv (a_1, a_2, a_3)$ under A_4 transforms under the action of T as:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ \omega^2 a_2 \\ \omega a_3 \end{pmatrix}$$

  - ↳ a $\underline{1}''$-singlet b transforms as $b \xrightarrow{T} \omega b$

$\Rightarrow$ this is where the structure comes in

- last ingredient: we need to know how different tensors combine

$\Rightarrow$ "tensor products"

↳ e.g. two A_4-triplets $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ combine as follows:

$$\boxed{\underline{3} \otimes \underline{3} = \underline{1} \oplus \underline{1}' \oplus \underline{1}'' \oplus \underline{3}_S \oplus 3_A}$$

$\Rightarrow$ singlets:

$$\underline{1} = a_1 b_1 + a_2 b_3 + a_3 b_2$$
$$\underline{1}' = a_1 b_2 + a_2 b_1 + a_3 b_3$$
$$\underline{1}'' = a_1 b_3 + a_2 b_2 + a_3 b_1$$

$\Rightarrow$ triplets:

$$\underline{3}_S = \frac{1}{3}(2a_1b_1 - a_2b_3 - a_3b_2, -a_1b_2 - a_2b_1 + 2a_3b_3, -a_1b_3 + 2a_2b_2 - a_3b_1)$$
$$\underline{3}_A = \frac{1}{2}(a_2b_3 - a_3b_2, a_1b_2 - a_2b_1, a_1b_3 - a_3b_1)$$

What does all the math tell us?

- this we will see using a toy model
- very simple setting: neutrino mass by Weinberg operator

$$\mathcal{L}_{Weinberg} = -\frac{y_{ij}}{\Lambda}(\overline{L_i^c}\,\tilde{H}^*)(\tilde{H}^\dagger L_j)$$

↳ charged lepton masses: $\mathcal{L}_{cl} = y_{e,ij}\, \overline{L}_i H e_{R_j} + h.c.$

$\Rightarrow$ a priori: $\boxed{y_{ij} \; \& \; y_{e,ij} \text{ are arbitrary complex matrices (BUT: } y_{ij} = y_{ji})}$

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- idea: suppose that different flavours have a certain transformation behaviour under a discrete group

↳ try the following "assignment" with A_4:

$$L = \begin{pmatrix} L_e \\ L_\mu \\ L_\tau \end{pmatrix} \sim \underline{3}\ ,\ e_R \sim \underline{1}\ ,\ \mu_R \sim \underline{1}''\ ,\ \tau_R \sim \underline{1}'\ ,\ H \sim \underline{1}$$

- since $\overline{L^c} \sim L^T$ and $H \sim \underline{1}$, one can only use the trivial singlet combination from $\underline{3} \otimes \underline{3}$ for the Weinberg operator:

$$(\overline{L^c}\tilde{H}^*)(\tilde{H}^\dagger L) \xrightarrow{VEV} v^2 \overline{(\nu_L)^c}\nu_L = v^2 \underbrace{\left[\overline{(\nu_{eL})^c}\nu_{eL} + \overline{(\nu_{\mu L})^c}\nu_{\tau L} + \overline{(\nu_{\tau L})^c}\nu_{\mu L}\right]}_{\sim \underline{1}\text{ - combination} \Rightarrow \text{ allowed in } \mathcal{L}}$$

$$\Rightarrow \mathcal{L} \supset -\frac{y v^2}{\Lambda}\left[\ldots\right] = -\frac{yv^2}{\Lambda}\left(\overline{(\nu_{eL})^c}, \overline{(\nu_{\mu L})^c}, \overline{(\nu_{\tau L})^c}\right)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \nu_{eL} \\ \nu_{\mu L} \\ \nu_{\tau L} \end{pmatrix}$$

the matrix (y_{ij}) is NOT arbitrary anymore!

↳ BUT: • does not yet yield acceptable mixing

• charged leptons massless ↯ (because: $\underline{3} \otimes \underline{1}_i \not\supset \underline{1}_0$)

- the issue with the charged leptons can be cured by promoting the Yukawa couplings to effective operators which involve "Higgs" fields that are SM-singlets but charged under the flavour symmetry:

"flavons"

$$\phi_S \sim \underline{3} \Rightarrow \langle \phi_S \rangle = 3\alpha_S \Lambda_F \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \leftarrow \text{"VEV alignment"}$$

$$u \sim \underline{1} \Rightarrow \langle u \rangle = \alpha_0 \Lambda_F$$

"flavour breaking scale"

⇒ this makes a charged lepton mass possible:

$$\mathcal{L} \supset \underbrace{\overline{L}}_{\sim \underline{3}} \frac{\overbrace{\langle \phi_S \rangle}^{\sim \underline{3}}}{\Lambda_F} \underbrace{H}_{\sim \underline{1}} \underbrace{e_R}_{\sim \left(\substack{\underline{1} \\ \underline{1}'' \\ \underline{1}'}\right)} + \text{h.c.} \longrightarrow$$

$\underline{3} \otimes \underline{3} \supset \underline{1}_0 \oplus \underline{1}_1 \oplus \underline{1}_2 \oplus \ldots$

$\underline{1}_a \otimes \underline{1}_b = \underline{1}_{(a+b) \bmod 3}$

$(\underline{1}_0 = \underline{1},\ \underline{1}_1 = \underline{1}',\ \underline{1}_2 = \underline{1}'')$

$$\longrightarrow 3\alpha_S \left[k_e \underbrace{(\overline{L}_e + \overline{L}_\mu + \overline{L}_\tau)}_{\text{from } \underline{1}} H e_R + k_\mu \underbrace{(\overline{L}_e + \overline{L}_\mu + \overline{L}_\tau)}_{\text{from } \underline{1}'} H \mu_R + k_\tau \underbrace{(\overline{L}_e + \overline{L}_\mu + \overline{L}_\tau)}_{\text{from } \underline{1}''} H \tau_R \right]$$

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$\Rightarrow$ charged lepton mass matrix: $\begin{pmatrix} k_e & k_\mu & k_\tau \\ k_e & k_\mu & k_\tau \\ k_e & k_\mu & k_\tau \end{pmatrix}$

look similar due to (1,1,1)-alignment

$\Rightarrow$ eigenvalues: $(m_e, m_\mu, m_\tau) = (0, 0, 3\alpha_s v(k_e + k_\mu + k_\tau))$

$\Rightarrow$ better than before, but still not accurate

- also for the neutrinos, further terms become possible:

$$\frac{1}{\Lambda}(\overline{L^c}\tilde{H}^*)(\tilde{H}^+ L)\frac{\langle\phi_s\rangle}{\Lambda_F} \Rightarrow \boxed{M_M = \frac{v^2}{\Lambda}\begin{pmatrix} \alpha_0 + 2\alpha_s & -\alpha_s & -\alpha_s \\ -\alpha_s & 2\alpha_s & \alpha_0 - \alpha_s \\ -\alpha_s & \alpha_0 - \alpha_s & 2\alpha_s \end{pmatrix}}$$

structure in the ν-mass matrix!

$\Rightarrow$ ν-masses: $\frac{v^2}{\Lambda}(\alpha_0, 3\alpha_s - \alpha_0, 3\alpha_s + \alpha_0) \equiv \underbrace{(\tilde{m}_2, \tilde{m}_3, \tilde{m}_1)}_{\text{complex masses (incl. phases)}}$

$\Rightarrow$ ν-mixing: $U_{TBM}^T M_M U_{TBM} = \text{diagonal}$

$$\Rightarrow \tan\theta_{12} = \frac{1}{\sqrt{2}}, \quad \theta_{13} = 0, \quad \theta_{23} = \frac{\pi}{4}$$

okay $\qquad$ ↯ $\qquad$ disfavoured

BUT: charged lepton mass matrix not diagonal

$\Rightarrow$ can correct the mixing...

.... you see its gets more complicated, the closer we come to reality

Mass sum rules:

- benefit of (some) flavour models: they yield testable correlations between ~~obs~~ certain observables
- the above toy model yields a "neutrino mass sum rule":

$$\tilde{m}_1 - \tilde{m}_3 = \frac{v^2}{\Lambda}(3\alpha_s + \alpha_0 - 3\alpha_s + \alpha_0) = 2\cdot\frac{v^2}{\Lambda}\alpha_0 = 2\tilde{m}_2$$

$$\Rightarrow \text{thus:} \quad \boxed{\tilde{m}_1 - 2\tilde{m}_2 - \tilde{m}_3 = 0}$$

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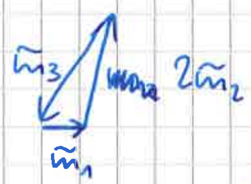
- crucial: this rule is a complex equation
  $\Rightarrow$ it constrains two things:
  - neutrino mass scale & pattern:
    - IO not possible:

$$|\tilde{m}_1 - \tilde{m}_3| \leq m_1 + m_3 = \sqrt{m^2 + \Delta m_A^2} + m < 2\sqrt{m^2 + \Delta m_A^2} <$$

$$< 2\sqrt{m^2 + \Delta m_A^2 + \Delta m_\odot^2} = 2 m_2 = |2\tilde{m}_2| \text{ ↯}$$

    - lowest mass for NO:



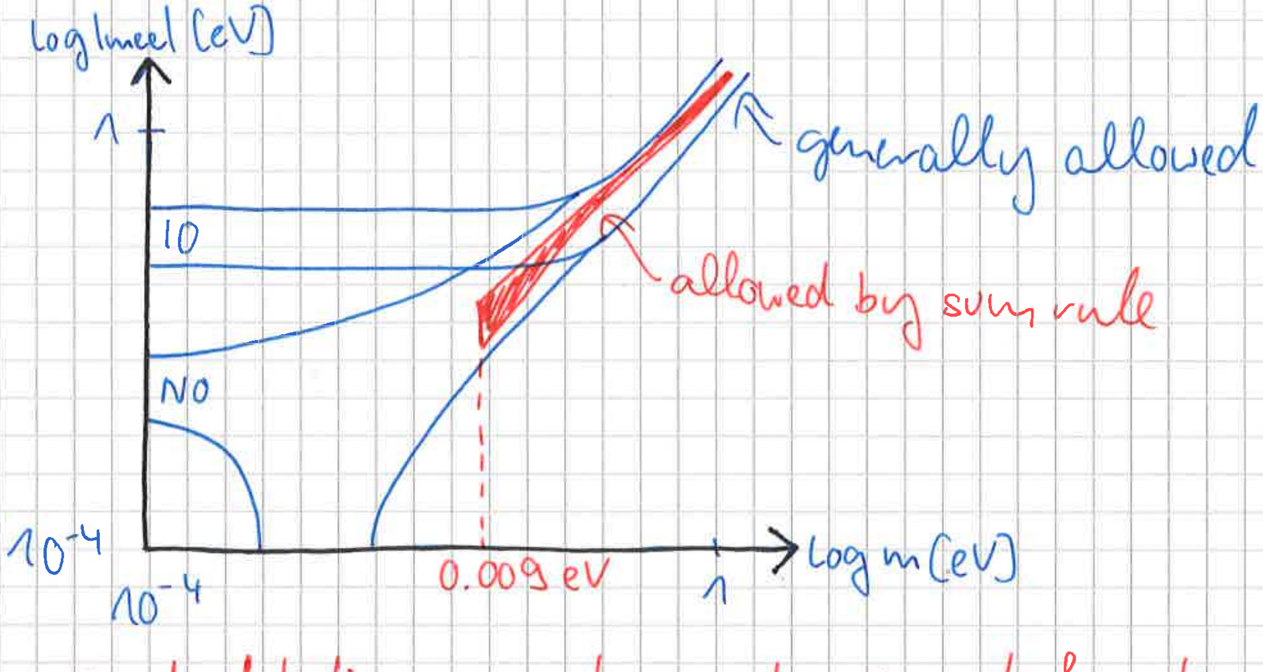
~~$\Rightarrow$ triangle until: $m_1 + m_3 = m + \sqrt{m^2 + \Delta m_A^2} \overset{!}{=} 2\sqrt{m^2 + \Delta m_A^2 + \Delta m_\odot^2}$~~

~~$\Rightarrow m \simeq$~~

$\Rightarrow$ triangle until: $m + \sqrt{m^2 + \Delta m_A^2} \overset{!}{=} 2\sqrt{m^2 + \Delta m_\odot^2}$

$$\Rightarrow \quad m_{lowest} \simeq \sqrt{\frac{\Delta m_A^2}{8} - 2\Delta m_\odot^2} \simeq 0.009\,\text{eV}$$

  - a certain combination of Majorana phases $\Rightarrow$ this is reflected particularly strongly in the effective mass:



$\Rightarrow$ good detection prospects + weaker impact of systematics