Investigations of the Two-loop Massive Anomalous Dimension in QCD

- Master Thesis -

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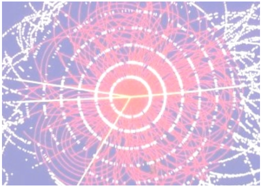

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- Matrix of Anomalous Dimensions
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- Analytical Investigation of the Master Integral
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4 Discussion and Conclusions

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- Apply method to specific known amplitude.
- Compare results in specific cases to provide evidence for validity of method.
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New Calculation Method

- Aim: computation of infrared contribution to QCD scattering amplitudes.
- Technical subset: solution of Integration by Parts identities to reduce number and complexity of integrals involved.
- Implementation: Laporta algorithm.
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- Structure of infrared poles in all orders of perturbation theory determined by matrix of anomalous dimensions.
- i-th order of matrix computed from coefficients of poles in dimensional regulator ε, in amplitudes of corresponding i-th-loop diagrams.

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Process Amplitude and Two-loop Anomalous Dimension

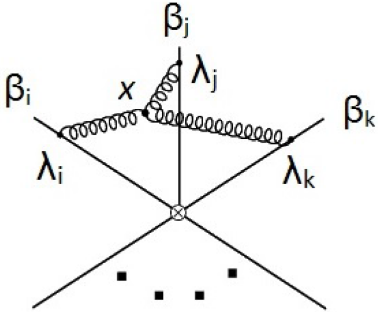

Figure: Two-loop three-einkonal diagram [1].

- Testing new method: Investigation of known two-loop three-eikonal diagram.
- Calculation in position space.
- Integration of distance $\lambda_{i,j,k}$ along each eikonal from 0 to ∞.
- Integration of position x of three-gluon vertex over all space.

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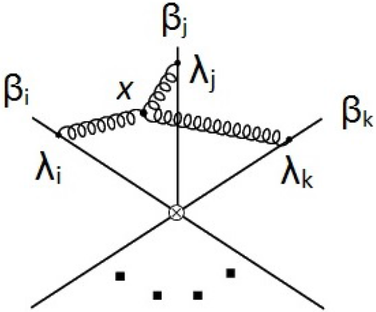

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Process Amplitude and Two-loop Anomalous Dimension

Amplitude of two-loop three-eikonal diagram obtained with new method [1]:

$$F_{3E}^{(2)}(\beta) \propto \int_0^\infty \prod_{i=1}^{3} \left(\frac{d\lambda_i}{\lambda_i}\right) \frac{A(\lambda_i,\beta)}{B(\lambda_i,\beta)} I(\lambda_i,\beta)$$

In total six integrations:

- Three integrations over λ_i.
- Three integrations over x_i replacing D-dim. integration over x: Master Integral $I(\lambda_i,\beta)$.

State of the Art

Previous Master thesis:

- Already first trivial integration over λ_1 gives $\propto 1/\varepsilon$.
- Assumption: $\varepsilon = 0$ in remaining integrals over $\lambda_2, \lambda_3, x_i$.

Goal of this project:

- Proof that ε can be set to 0 in remaining integrals.

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- Evaluate specific case of equal parton momenta.

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Solution of First Two Integrations of Master Integral

- Master Integral in Feynman parametrization consists of three integrations over x_i.
- Integrations over x_1 and x_2 give rise to integrand containing hypergeometric function

$$_2F_1(1-2\varepsilon, 1-2\varepsilon, 2-\varepsilon; z) \qquad \text{with} \qquad z = z(\beta, \lambda_i, x_3)$$

and prefactor

$$((1-x_3)x_3)^{-1+\varepsilon}.$$

- Third integration over x_3 highly non-trivial.
- Several integration approaches attempted.

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Expansion of Integrand in Powers of Dimensional Regulator

Expand ${}_2F_1$ in powers of ε with Mathematica package HypExp [2].

- Explicit factor $(1 - x_3)x_3$ in first order of ${}_2F_1$ expansion cancels prefactor $((1 - x_3)x_3)^{-1+\varepsilon}$ for small ε.
- Hence no term $\propto 1/\varepsilon$ generated by integrand $\rightarrow$ can set $\varepsilon = 0$.
- *Master Integral is finite.*
- Highly non-trivial proof, numerically checked with Monte Carlo integration as well as Mathematica package SecDec [3].

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Final Integration over x_3

Approach chosen in previous Master thesis:
Split integrand into several terms and perform integrations.

- Obtain several logarithms and dilogarithms, identify denominator with polynomial $B(\beta, \lambda_i) \rightarrow$ Huge simplification.
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- However, for other parameter sets: result complex and only real part in accordance with numerical result.

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Final Integration over x_3

Mistake found in previous integration approach:

- Riemann/Lebesgue integration conditions not always fulfilled.
- Only Cauchy Principal Value exists in these cases, gives rise to terms $\propto i\pi(1 + 2n)$ with $n \in \mathbb{Z}$.
- Scattering amplitude and Master Integral have to be real [4].

Conclusion:

- Result obtained in previous Master thesis not valid.
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Case of Equal Momenta

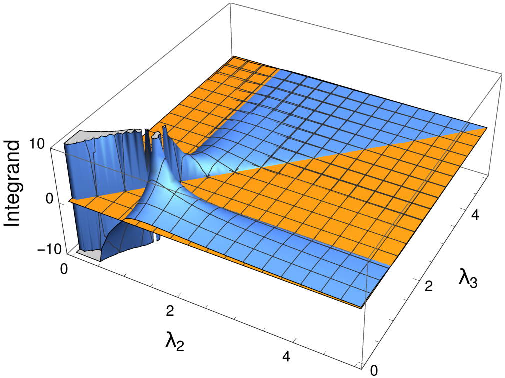
Case of equal parton momenta: $\beta_i \cdot \beta_j = \beta^2$.

- Integration without occurrence of integration conditions and imaginary parts.
- Resulting integrand simplified to expression antisymmetric in remaining integration variables $\lambda_{2,3}$.

Numerical Evaluation of Process Amplitude

Case of equal parton momenta:

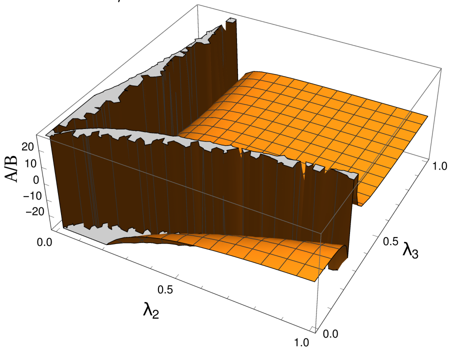
- Integrand of process amplitude numerically integrable to 0.



Numerical Evaluation of Process Amplitude

Case of arbitrary parton momenta:

- Numerical evaluation of integrals with Quasi Monte Carlo algorithm diverges due to numerical inaccuracy.
- Reason: Singularities in prefactor A/B.



Application of Symmetry Properties of Integrand

Final approach:

- Split integrand into symmetric and antisymmetric parts under exchange of momenta.
- Final result is known to be antisymmetric in momenta.

Result:

- Symmetric part vanishes as expected.
- Integration of antisymmetric part still diverges.

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For equal parton momenta:

- Antisymmetric part equals 0 for all λ_i.
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- Hence, process amplitude vanishes for equal momenta.
  $\rightarrow$ Coincides with analytical/numerical evaluation and theoretical expectation.

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  - Numerical integration accuracy not high enough.
  - Pole $\propto 1/\varepsilon$ extracted. $\rightarrow$ Computation possible in principle.
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Applicability of new method:

- New calculation method applicable to specific cases, e.g. equal momenta or massless eikonals [1].
- In both cases reproduction of known result.
  $\rightarrow$ Probably applicable to three-loop four-eikonal amplitude.
- In general: fast reduction of number & complexity of integrals.
  $\rightarrow$ Method very efficient and applicable to any process, hence promising for solving several undissolved problems in HEP.

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Thank you

Do you have any questions?

The New Method [1]

- The two-loop three-eikonal amplitude in Feynman gauge can be written as

$$F_{3E}^{(2)}(\beta) = \int d^D x \int_0^\infty \prod_{i=1}^{3} (d\lambda_i) V(x, \beta).$$

- The integrand is given by a sum over six terms,

$$V(x, \beta) = \sum_{i,j,k} \varepsilon_{ijk} v_{ijk}(x, \beta).$$

The New Method [1]

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$$V(x,\boldsymbol{\beta}) = \sum_{i,j,k} \varepsilon_{ijk} v_{ijk}(x,\boldsymbol{\beta}).$$

- Here it was defined

$$v_{ijk} = -i(g\varepsilon)^4 \beta_i \cdot \beta_j \Delta(x-\lambda_j\beta_j)\Delta(x-\lambda_k\beta_k)\beta_k \cdot \partial_x \Delta(x-\lambda_i\beta_i).$$

- The propagators are given by

$$\Delta(x-\lambda_i\beta_i) = -\frac{\Gamma(1-\varepsilon)}{4\pi^{2-\varepsilon}} \frac{1}{(x-\lambda_i\beta_i)^{2(1-\varepsilon)}}.$$

The New Method [1]

- v_{ijk} can be expressed as a linear comb. of Feynman integrals,

$$v_{ijk} \propto I(0, 2-\varepsilon, 1-\varepsilon, 1-\varepsilon) - I(-1, 2-\varepsilon, 1-\varepsilon, 1-\varepsilon) + I(0, 2-\varepsilon, 1-\varepsilon, 0-\varepsilon)$$

- Feynman integrals are defined as

$$I(a_1, \dots, a_n) := \int \dots \int \prod_i (d^D k_i) \frac{1}{E_1^{a_1}(\boldsymbol{k}, \boldsymbol{p}) \dots E_n^{a_n}(\boldsymbol{k}, \boldsymbol{p})}$$

where $\boldsymbol{k}$ and $\boldsymbol{p}$ are called internal and external momenta in analogy with momentum-space integrals.

- The propagators are $E_0 = x^2$ and $E_i = (x - \lambda_i \beta_i)^2$.

The New Method [1]

- Reduce linear combination of Feynman integrals (FI) by using Integration By Parts identities:

$$0 = \int \dots \int \prod_i (d^D k_i) \frac{\partial}{\partial k^\mu} \frac{\eta^\mu}{E_1^{a_1} \dots E_n^{a_n}}$$

  where η can take values $\{k_1, \dots\}$.
- Using these relations, express given FI as linear combination of simpler FIs.
- Method of reduction: Laporta algorithm, solves a system of IBPs by carrying out Gaussian elimination.

The New Method [1]

- Program AIR (Automatic Integral Reduction) implements Laporta algorithm. Propagators need to have integer powers.
- However, only relative difference between powers of propagators relevant for reduction.
- Hence one can shift the powers $a_i \to a_i - \varepsilon$.
  $\to$ For reduction one shifts powers, i.e. temporarily sets $\varepsilon = 0$, and re-introduces ε at later stage.

Result:

- Reduction of initially 12 integrals to single Master Integral $I(0, 1, 1, 1)$.

The New Method [1]

- Remember Feynman integral notation

$$I(a_1, ..., a_n) := \int ... \int \prod_i (d^D k_i) \frac{1}{E_1^{a_1}(\boldsymbol{k}, \boldsymbol{p}) ... E_n^{a_n}(\boldsymbol{k}, \boldsymbol{p})},$$

where $k := x$ and $p = \beta$.

- Reduction with AIR to single Master Integral

$$I(0,1,1,1) = \int d^D x \frac{1}{[(x-\gamma_1)^2]^{(1-\varepsilon)}[(x-\gamma_2)^2]^{(1-\varepsilon)}[(x-\gamma_3)^2]^{(1-\varepsilon)}}$$

Master Integral in Feynman Parametrization

Starting point: single Master Integral

$$I(0,1,1,1) = \int d^D x \frac{1}{[(x-\gamma_1)^2]^{(1-\varepsilon)}[(x-\gamma_2)^2]^{(1-\varepsilon)}[(x-\gamma_3)^2]^{(1-\varepsilon)}}.$$

Integral written in Feynman parametrization

$$\propto \int_0^1 \int_0^1 \int_0^1 dx_1 dx_2 dx_3 \frac{\delta(1-x_1-x_2-x_3)(x_1 x_2 x_3)^{(-\varepsilon)}}{(x_1 x_2 \sigma_{12} + x_2 x_3 \sigma_{23} + x_3 x_1 \sigma_{31})^{(1-2\varepsilon)}}$$

with

$$\sigma_{ij} := (\gamma_i - \gamma_j)^2$$
$$\gamma_i := \lambda_i \beta_i.$$

Solution of First Two Integrations

Master Integral after x_1 integration

$$\propto \int_0^1 \int_0^1 dx_2 dx_3 (1-x_2)(x_2 x_3 (1-x_2)(1-x_3))^{(-\varepsilon)} \cdot [x_2 x_3 (1-x_2)\sigma_{12} + x_3 (1-x_2)^2 (1-x_3)\sigma_{23} + (1-x_2)(1-x_3) x_2 \sigma_{31}]^{(-1+2\varepsilon)}.$$

Master Integral after x_2 integration

$$\propto \frac{1}{\sigma_{23}} \int_0^1 dx_3 \, ((1-x_3)x_3)^{-1+\varepsilon} {}_2\tilde{F}_1(1-2\varepsilon, 1-2\varepsilon, 2-\varepsilon; z),$$

with argument z defined as

$$z := 1 - \frac{1}{\sigma_{23}} \left(\frac{\sigma_{12}}{1-x_3} + \frac{\sigma_{13}}{x_3} \right).$$

Expansion of Integrand in Powers of ε

First order of expansion of hypergeometric function in powers of ε is

$$
{}_2F_1(1-2\varepsilon, 1-2\varepsilon, 2-\varepsilon; z) \approx -\frac{\ln(1-z)}{z}
$$

$$
\approx \frac{(1-x_3)x_3}{-x_3(1-x_3)+\frac{1}{\sigma_{23}}(x_3\sigma_{12}+(1-x_3)\sigma_{13})} \ln\left[\frac{\sigma_{12}x_3+\sigma_{13}(1-x_3)}{\sigma_{23}(1-x_3)x_3}\right].
$$

Expansion of prefactor $((1-x_3)x_3)^{-1+\varepsilon}$ in powers of ε done with

$$
\xi^{-1+\varepsilon} = \frac{1}{\varepsilon}\delta(\xi) + \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\left[\frac{\ln^n(\xi)}{\xi}\right]_+ ,
$$

where plus prescription $[f(x)]_+$ defined via

$$
\int_0^1 dx[f(x)]_+ g(x) = \int_0^1 dx f(x)(g(x)-g(0)).
$$

Third Integration

Final expression for Master Integral

$$I(0,1,1,1) = \frac{\pi^2}{\sigma_{23}} \int_0^1 dx_3 \, \frac{\ln\left[\frac{\sigma_{12}x_3+\sigma_{13}(1-x_3)}{\sigma_{23}(1-x_3)x_3}\right]}{-x_3(1-x_3) + \frac{1}{\sigma_{23}}(x_3\sigma_{12} + (1-x_3)\sigma_{13})}.$$

Approach in previous Master thesis: Split logarithm into

$$\ln(\sigma_{12}x_3 + \sigma_{13}(1-x_3)) - \ln(1-x_3) - \ln(x_3) - \ln(\sigma_{23}),$$

integrate over x_3 and obtain (wrong complex) result consisting of logarithms and dilogarithms.

Case of Equal Momenta

Master Integral for equal momenta

$$I(0,1,1,1) = \pi^2 \int_0^1 dx_3 \, \frac{\ln\left[\frac{\sigma_{12}x_3+\sigma_{13}(1-x_3)}{\sigma_{23}(1-x_3)x_3}\right]}{\sigma_{23}(x_3 - x_0)^2},$$

with double zero of denominator

$$x_0 := \frac{\lambda_1 - \lambda_3}{\lambda_2 - \lambda_3}.$$

Integral after x_3 integration

$$-\frac{\ln[(1-\lambda_2)^2]}{(\lambda_2-\lambda_3)(-1+\lambda_3)}+\frac{\ln[(1-\lambda_3)^2]}{(\lambda_2-\lambda_3)(-1+\lambda_2)}+\frac{\ln[(\lambda_2-\lambda_3)^2]}{(-1+\lambda_2)(-1+\lambda_3)}.$$

Case of Equal Momenta

Integrand of process amplitude with $\lambda_1 = 1$ and $m^2 = 1$ and for equal momenta

$$\frac{1}{\lambda_2\lambda_3}\frac{A(\gamma)}{B(\gamma)}I(0,1,1,1) = \frac{3\pi^2}{2(\lambda_3^2 - \lambda_2\lambda_3(1+\lambda_3) + \lambda_2^2(1-\lambda_3+\lambda_3^2))} \cdot \Big\{ + (1-\lambda_2)\cdot \ln[(1-\lambda_2)^2] - (1-\lambda_3)\cdot \ln[(1-\lambda_3)^2] + (\lambda_2-\lambda_3)\cdot \ln[(\lambda_2-\lambda_3)^2] \Big\}.$$

Application of Symmetry Properties of Integrand

Master Integral after x_1 integration

$$\propto \int_0^1 \int_0^1 dx_2 dx_3 \; [x_2 x_3 \sigma_{12} + x_3(1-x_2)(1-x_3)\sigma_{23} + (1-x_3)x_2\sigma_{31}]^{-1}$$

with $\sigma_{ij} := (\gamma_i - \gamma_j)^2$ and $\gamma_i := \lambda_i \beta_i$.

Simulate for example exchange of $\beta_1 \leftrightarrow \beta_2$:
Relabel $\lambda_1 \leftrightarrow \lambda_2$ and $(1-x_3) \leftrightarrow x_2 x_3$.

Further Approaches to Solve Master Integral

Master Integral

$$I(0,1,1,1) \propto \int_0^1 dx_3 \, (x_3(1-x_3))^{-1+\varepsilon} {}_2\tilde{F}_1(1-2\varepsilon, 1-2\varepsilon, 2-\varepsilon; z)$$

with z defined as

$$z := 1 - \frac{1}{\sigma_{23}} \left(\frac{\sigma_{12}}{1-x_3} + \frac{\sigma_{13}}{x_3} \right). \tag{1}$$

Identity of hypergeometric functions

$${}_3F_2(a_1, a_2, a_3, b_1, b_2; z) \propto \int_0^1 dy (y)^{a_3-1} (1-y)^{-a_3+b_2-1} {}_2F_1(a_1, a_2, b_1; yz).$$

Definition of $A(\gamma)$ and $B(\gamma)$

$A(\gamma)$ and $B(\gamma)$ are

$$\begin{aligned}
A(\gamma) &= \gamma_{12}[(v_{12}^2+1)\gamma_{12}-\gamma_3^2](\gamma_{23}-\gamma_{13}) \\
&\quad + \gamma_{13}[(v_{13}^2+1)\gamma_{13}-\gamma_2^2](\gamma_{12}-\gamma_{23}) \\
&\quad + \gamma_{23}[(v_{23}^2+1)\gamma_{23}-\gamma_1^2](\gamma_{13}-\gamma_{12}) \\
B(\gamma) &= -v_{12}^2\gamma_{12}^2 - v_{13}^2\gamma_{13}^2 - v_{23}^2\gamma_{23}^2 \\
&\quad + 2\gamma_{12}(\gamma_{13}-\gamma_3^2) + 2\gamma_{13}(\gamma_{23}-\gamma_2^2) + 2\gamma_{23}(\gamma_{12}-\gamma_1^2),
\end{aligned}$$

where $\gamma_i := \lambda_i\beta_i$, $\gamma_{ij} = \gamma_i \cdot \gamma_j$ and $\gamma_{ij}^2 v_{ij}^2 = \gamma_{ij}^2 - \gamma_i^2\gamma_j^2$.