

Soft Theorems in Gauge and Gravity Theories

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Based mainly on: [1410.1616](#) (super-soft theorem)
& [work in progress](#) (the scattering equations)

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Outline

A Brief Introduction to Scattering Amplitudes

- Tree amplitudes in gauge theory and gravity
- On-shell BCFW recursion relations
- The Scattering equations and CHY formulas

Soft theorems in maximally supersymmetric theories

- Soft theorem in $\mathcal{N}=4$ super Yang-Mills
- Soft theorem in $\mathcal{N}=8$ supergravity

Summary and outlook

Partial amplitudes in gauge theory

In non-Abelian gauge theory, a full tree scattering amplitude of gluons can be written as

$$\mathcal{A}_n^{\text{tree}}(\{k_i, h_i, a_i\}) = g^{n-2} \sum_{\sigma \in S_{n-1}} \text{Tr}(T^{a_1} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}) \mathcal{A}_n(1, \sigma(2), \dots, \sigma(n))$$

Inside the summation, the second factor is called (color-ordered) **partial amplitude**, which is purely kinematic

$$\mathcal{A}_n(1, 2, \dots, n) \equiv \delta^4(p) A_n(\{k_1, h_1\}, \{k_2, h_2\}, \dots, \{k_n, h_n\})$$

There exists many interesting relations between the partial amplitudes:

Cyclicity $A_n(1, 2, \dots, n) = A_n(2, \dots, n, 1) = \cdots = A_n(n, 1, \dots, n - 1)$

KK relations $A_n(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n_\beta} \sum_{\sigma \in \text{OP}(\{\alpha\}, \{\beta^T\})} A_n(1, \sigma, n)$
[Kleiss & Kuijf, NPB 312 (1989) 616]

BCJ relations

$$A_n(1, 2, \{\alpha\}, 3, \{\beta\}) = \sum_{\sigma \in \text{POP}(\{\alpha\}, \{\beta\})} A_n(1, 2, 3, \sigma) \prod_{k=4}^n \frac{\mathcal{F}_k(3, \sigma, 1)}{s_{2, 4, \dots, k}}$$

[Bern-Carrasco-Johansson, 0805.3993, see also a recent review by Carrasco, 1506.00974]

Graviton amplitudes

The full tree-level graviton amplitude including the gravitational coupling constant is

$$\mathcal{M}_n(1, \dots, n) = \kappa^{n-2} \mathcal{M}_n(1, \dots, n) = \delta^4(p) \kappa^{n-2} M_n(1, \dots, n)$$

where $\kappa = \sqrt{8\pi G_N}$. There exists the remarkable relations between gauge and gravity amplitudes at tree level, such as

$$M_3(1, 2, 3) = A_3(1, 2, 3)A_3(1, 2, 3),$$

$$M_4(1, 2, 3, 4) = -s_{12} A_4(1, 2, 3, 4)A_4(1, 2, 4, 3),$$

$$M_5(1, 2, 3, 4, 5) = s_{12}s_{34}A_5(1, 2, 3, 4, 5)A_5(2, 1, 4, 3, 5) + s_{13}s_{24}A_5(1, 3, 2, 4, 5)A_5(1, 4, 2, 5, 3).$$

More generally,

$$M_n(1, 2, \dots, n) = (-1)^{n+1} \sum_{\sigma, \rho \in S_{n-3}} A_n(n-1, n, \sigma, 1) \mathcal{S}[\sigma | \rho]_{k_1} A_n(1, \rho, n-1, n)$$

where the momentum kernel \mathcal{S} is the function of momenta, which also depends on the perturbations $\sigma, \rho \in S_{n-3}$.

[Bern, [gr-qc/0206071](#); Kawai, Lewellen, Tye, [NPB 269 \(1986\) 1](#); Bern, Dixon, Perelstein, Rozowsky, [hep-th/9811140](#);
Bjerrum-Bohr, Damgaard, Feng & Sondergaard, [1007.3111](#), [1005.4367](#), [1006.3214](#);
Bjerrum-Bohr, Damgaard, Sondergaard & Vanhove, [1010.3933](#)]

Britto-Cachazo-Feng-Witten Recursion Relations

Take the following shift of momenta for two chosen external legs i and j :

$$k_i \rightarrow k_i(z) = k_i + zq, \quad k_j \rightarrow k_j(z) = k_j - zq.$$

Obviously, $k_i(z) + k_j(z) = k_i + k_j$. The on-shell conditions for two shifted legs are needed

$$0 = k_i(z)^2 = 2z k_i \cdot q + z^2 q^2, \quad 0 = k_j(z)^2 = -2z k_j \cdot q + z^2 q^2.$$

Then the amplitude A becomes a complex function $A(z)$, and the original amplitude we need is just $A(0) \equiv A(z=0)$. The marrow of on-shell BCFW recursion is the understanding of the behavior of complex function $A(z)$. Consider the following contour integration

$$\frac{1}{2\pi i} \oint_{\infty} \frac{A(z)}{z} dz,$$

If $A(z) \rightarrow 0$ as $z \rightarrow \infty$, then the residue theorem gives

$$0 = \oint_{\infty} \frac{dz}{2\pi i} \frac{A(z)}{z} = A(0) + \sum_{\text{pole } z_a} \text{Res} \left(\frac{A(z)}{z} \right) \Big|_{z_a} \implies A(0) = - \sum_{\text{pole } z_a} \text{Res} \left(\frac{A(z)}{z} \right) \Big|_{z_a}$$

[BCF, [hep-th/0412308](#); BCFW, [hep-th/0501052](#); Cachazo & Svrcek, [hep-th/0502160](#); Bedford, Brandhuber, Spence & Travaglini, [hep-th/0502146](#); Arkani-Hamed, Cachazo & Kaplan, [0808.1446](#); Arkani-Hamed & Kaplan, [0801.2385](#); Cheung [0808.0504](#)]

BCFW Recursion Relations

At tree level, the only poles of the integrand are at the solutions of

$$P_I(z)^2 = (k_{a_1} + \cdots + k_i(z) + \cdots + k_{a_m})^2 = 0 \implies z_I = -\frac{P_I^2}{2q \cdot P_I},$$

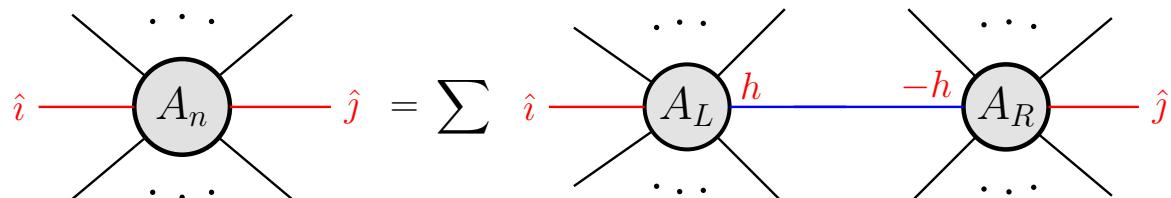
where $I = \{k_{a_1}, \dots, k_{a_m}\} \subset \{1, 2, \dots, n\}$ and $P_I \equiv P_I(0)$. Remarkably, we have

$$\text{Res} \left(\frac{A(z)}{z} \right) \Big|_{z_I} = \sum_h A_L^h(z_I) \frac{-1}{P_I^2} A_R^{-h}(z_I).$$

Finally, we obtain BCFW recursion relation for an on-shell n -point tree-level amplitude

$$\mathcal{A}_n = \sum_I \sum_h \mathcal{A}_L(\dots, k_i(z_I), \{-\hat{P}_I(z_I), h\}) \frac{1}{P_I^2} \mathcal{A}_R(\{P_I(z_I), -h\}, k_j(z_I), \dots).$$

Here two sub-amplitudes with fewer external legs all are **physical**, i.e. all particles are on-shell and momentum conservation is preserved, while the propagator P_I^2 is on-shell.



The scattering equations

The scattering equations connect the on-shell momenta k_a of massless particles with n marked points σ_a on \mathbb{CP}^1 .

$$f_a \equiv \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_{ab}} = 0, \quad a = 1, 2, \dots, n.$$

where $\sigma_{ab} = \sigma_a - \sigma_b$. This system of equations is invariant under Möbius transformations, and because of this symmetry, only $n - 3$ out of n equations are linearly independent. The scattering equations have $(n - 3)!$ independent solutions.

Cachazo-He-Yuan formulas:

$$A_n = \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod'_a \delta(f_a) \frac{1}{\sigma_{12} \cdots \sigma_{n1}} \text{Pf}' \Psi,$$
$$M_n = \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod'_a \delta(f_a) (\text{Pf}' \Psi)^2.$$

The similar formulas were constructed for the amplitudes in massless ϕ^3 theory, Einstein-Yang-Mills, $U(N)$ non-linear sigma model, Dirac-Born-Infeld theory, etc.

[Cachazo-He-Yuan, [1307.2199](#); [1309.0885](#); [1306.6575](#); [1409.8256](#); [1412.3479](#); Dolan-Goddard, [1311.5200](#)]

Cachazo-He-Yuan formulas

Matrix Ψ is defined as

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix},$$

where A , B and C are $n \times n$ antisymmetric matrices:

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}}, & a \neq b, \\ 0, & a = b, \end{cases} \quad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}}, & a \neq b, \\ 0, & a = b, \end{cases} \quad C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}}, & a \neq b, \\ -\sum_{c \neq a} C_{ac}, & a = b. \end{cases}$$

The determinant of matrix Ψ is vanishing, so is its pfaffian. If we remove rows i , j and columns i , j with $1 \leq i, j \leq n$, we will get a $(2n - 2) \times (2n - 2)$ Skew-symmetric matrix Ψ_{ij}^{ij} with full rank. And then we can define the reduced pfaffian:

$$\text{Pf}' \Psi \equiv \frac{2(-1)^{i+j}}{\sigma_{ij}} \text{Pf} (\Psi_{ij}^{ij}).$$

It is easy to show that the reduced pfaffian $\text{Pf}' \Psi$ is independent of the choice of labels $\{i, j\}$.

Cachazo-He-Yuan formulas

The δ -distribution is defined as

$$\prod_a' \delta(f_a) \equiv \sigma_{ij}\sigma_{jk}\sigma_{ki} \prod_{a \neq i,j,k} \delta(f_a)$$

Obviously, in CHY formulas the $(n-3)$ -fold integrals are fully localized by $(n-3)$ δ -functions

$$A_n = \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod_a' \delta(f_a) \frac{\text{Pf}' \Psi}{\sigma_{12} \cdots \sigma_{n1}} = \sum_{\text{all solutions}} \frac{1}{\sigma_{12} \cdots \sigma_{n1}} \frac{\text{Pf}' \Psi(k, \epsilon, \sigma)}{\det' \Phi(k, \sigma)},$$
$$M_n = \int \frac{d^n \sigma}{\text{vol SL}(2, \mathbb{C})} \prod_a' \delta(f_a) (\text{Pf}' \Psi)^2 = \sum_{\text{all solutions}} \frac{\det' \Psi(k, \epsilon, \sigma)}{\det' \Phi(k, \sigma)}$$

where

$$\Phi_{ab} \equiv \frac{\partial f_a}{\partial \sigma_b} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}^2}, & a \neq b, \\ - \sum_{c \neq a} \Phi_{ac}, & a = b, \end{cases} \quad \det' \Phi \equiv \frac{|\Phi|^{ijk}_{rst}}{(\sigma_{rs}\sigma_{st}\sigma_{tr})(\sigma_{ij}\sigma_{jk}\sigma_{ki})}$$

Soft theorems in maximally supersymmetric theories

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ABSTRACT: In this paper we study the supersymmetric generalization of the new soft theorem which was proposed by Cachazo and Strominger recently. At tree level, we prove the validity of the super soft theorems in both $\mathcal{N} = 4$ super-Yang-Mills theory and $\mathcal{N} = 8$ supergravity using super-BCFW recursion relations. We verify these theorems exactly by showing some examples.

ARXIV EPRINT: [1410.1616](https://arxiv.org/abs/1410.1616)

Soft graviton theorems

In the limit $k_s \rightarrow 0$, a generic $(n+1)$ -graviton on-shell scattering amplitude behaves as

$$\mathcal{M}_{n+1}(k_1, \dots, k_n, k_s) \xrightarrow{k_s \rightarrow 0} \left(S^{(0)} + S^{(1)} + S^{(2)} \right) \mathcal{M}_n(k_1, \dots, k_n) + \mathcal{O}(k_s^2),$$

where the soft factor are given by

$$S^{(0)} \equiv \sum_{a=1}^n \frac{E_{\mu\nu} k_a^\mu k_a^\nu}{k_s \cdot k_a}, \quad S^{(1)} \equiv \sum_{a=1}^n \frac{E_{\mu\nu} k_a^\mu (k_{s\sigma} J_a^{\sigma\nu})}{k_s \cdot k_a}, \quad S^{(2)} \equiv \sum_{a=1}^n \frac{E_{\mu\nu} (k_{s\rho} J_a^{\rho\mu}) (k_{s\sigma} J_a^{\sigma\nu})}{k_s \cdot k_a}$$

In 4d, it is natural to use the spinor-helicity formulism, $k^2 = 0 \iff k_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$. In the holomorphic soft limit, i.e. $\lambda_s \rightarrow \epsilon \lambda_s$, $\tilde{\lambda}_s \rightarrow \tilde{\lambda}_s$

$$\mathcal{M}_{n+1}(\dots, \{\epsilon \lambda_s, \tilde{\lambda}_s\}) = \left(\frac{1}{\epsilon^3} S^{(0)} + \frac{1}{\epsilon^2} S^{(1)} + \frac{1}{\epsilon} S^{(2)} \right) \mathcal{M}_n + \mathcal{O}(\epsilon^0),$$

where the soft operators are given by

$$S^{(0)} = \sum_{a=1}^n \frac{[s a] \langle x a \rangle \langle y a \rangle}{\langle s a \rangle \langle x s \rangle \langle y s \rangle}, \quad S^{(1)} = \frac{1}{2} \sum_{a=1}^n \frac{[s a]}{\langle s a \rangle} \left(\frac{\langle x a \rangle}{\langle x s \rangle} + \frac{\langle y a \rangle}{\langle y s \rangle} \right) \tilde{\lambda}_{s\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{a\dot{\alpha}}},$$

$$S^{(2)} = \frac{1}{2} \sum_{a=1}^n \frac{[s a]}{\langle s a \rangle} \tilde{\lambda}_{s\dot{\alpha}} \tilde{\lambda}_{s\dot{\beta}} \frac{\partial^2}{\partial \tilde{\lambda}_{a\dot{\alpha}} \partial \tilde{\lambda}_{a\dot{\beta}}}, \quad \langle i j \rangle = \lambda_{i\alpha} \lambda_j^\alpha, \quad [i j] = \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_j^{\dot{\alpha}}$$

[Weinberg 1965; Low 1954, 1958; Gross-Jackiw 1968; White, 1103.2981, Laenen-Stavenga-White, 0811.2067; Cachazo-Strominger, 1404.4091; Strominger, 1312.2229, 1401.7026, 1406.3312]

Soft gluon theorems

In the holomorphic soft limit, $\lambda_s \rightarrow \epsilon \lambda_s$, an on-shell gluon amplitude \mathcal{A}_{n+1} behaves as

$$\mathcal{A}_{n+1}(\dots, \{\epsilon \lambda_s, \tilde{\lambda}_s\}) = \left(\frac{1}{\epsilon^2} S^{(0)} + \frac{1}{\epsilon} S^{(1)} \right) \mathcal{A}_n + \mathcal{O}(\epsilon^0).$$

where the soft operators is given by

$$S_{\text{YM}}^{(0)} = \frac{\langle x n \rangle}{\langle s n \rangle \langle x s \rangle} + \frac{\langle x 1 \rangle}{\langle s 1 \rangle \langle x s \rangle}, \quad S_{\text{YM}}^{(1)} = \frac{1}{\langle n s \rangle} \tilde{\lambda}_{s\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{n\dot{\alpha}}} + \frac{1}{\langle s 1 \rangle} \tilde{\lambda}_{s\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{1\dot{\alpha}}}$$

[Low 1958; Burnett-Kroll 1968; Berends-Giele 1989; Casali, 1404.5551]

Cachazo-Strominger's new soft graviton theorem has inspired a lot of works:

- symmetry Strominger et al. 1407.3814, 1407.3789, 1406.3312, 1411.5745, 1412.2763, 1502.06120, 1502.07644, ...
- Soft theorem in arbitrary dimensions from the scattering equations [CHY 1307.2199]
Schwab-Volovich 1404.7749; Afkhami-Jeddi 1405.3533; Zlotnikov 1407.5936; Kalousios-Rojas 1407.5982
- Loop corrections Bern-Davies-Nohle 1405.1015; He-Huang-Wen 1405.1410; Cachazo-Yuan 1405.3413, ...
- Supersymmetric theories He-Huang-Wen 1405.1410; Liu 1410.1616; Rao 1410.5047
- Ambitwistor strings
Geyer-Lipstein-Mason 1404.6219, 1406.1462; Lipstein 1504.01364; Adamo-Casali-Skinner 1405.5122, 1504.02304, ...
- Double soft CHY 1503.04816; Plefka et al. 1504.05558; Volovich-Wen-Zlotnikov 1504.05559; Du-Luo 1505.04411
- ...

Super soft theorem in $\mathcal{N}=4$ SYM

- IIB superstring on $AdS_5 \times S^5 \longleftrightarrow \mathcal{N}=4$ SYM on the 4d boundary
- $\mathcal{N}=4$ SYM: 21st century's harmonic oscillator (Hydrogen atom)

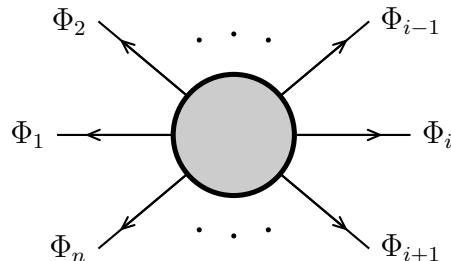
In $\mathcal{N}=4$ SYM, all on-shell fields can be organised into a single Nair's on-shell superfield:

$$\Phi(p, \eta) = g^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2!} \eta^A \eta^B S_{AB}(p) + \frac{1}{3!} \eta^A \eta^B \eta^C \epsilon_{ABCD} \bar{\Gamma}^D(p) + \eta^1 \eta^2 \eta^3 \eta^4 g^-(p)$$

Here η^A , $A = 1, 2, 3, 4$, are Grassmann odd variables.

In Nair's on-shell superspace, we can nicely define the superamplitude

$$\mathcal{A}_n \equiv \langle \Phi_1 \Phi_1 \cdots \Phi_n \rangle = \mathcal{A}_n(\{\lambda_1, \tilde{\lambda}_1, \eta_1\}, \dots, \{\lambda_n, \tilde{\lambda}_n, \eta_n\}).$$



The component field amplitudes are then obtained by projecting upon the relevant terms in the η -expansion of the superamplitude.

Super soft theorem in $\mathcal{N}=4$ SYM

Let us choose the soft particle and its adjacent particle “ n ” to shift:

$$\lambda_s(z) = \lambda_s + z\lambda_n, \quad \tilde{\lambda}_n(z) = \tilde{\lambda}_n - z\tilde{\lambda}_s, \quad \eta_n(z) = \eta_n - z\eta_s.$$

where we work in spinor notations. Super-BCFW recursion gives:

$$\begin{aligned} \mathbb{A}_{n+1} &= \sum_{a=1}^{n-2} \int d^4\eta_I \mathbb{A}_{a+2}(\{\lambda_s(z^*), \tilde{\lambda}_s, \eta_s\}, 1, \dots, a, \{I(z^*), \eta_I\}) \\ &\quad \times \frac{1}{P_I^2} \mathbb{A}_{n+1-a}(\{-I(z^*), \eta_I\}, a+1, \dots, n-1, \{\lambda_n, \tilde{\lambda}_n(z^*), \eta_n(z^*)\}). \end{aligned}$$

The singular terms only come from the term with $a = 1$, so we can drop the terms with $a > 1$, and then write

$$\begin{aligned} \mathbb{A}_{n+1} &= \int d^4\eta_I \mathbb{A}_3^{\overline{\text{MHV}}}(\{\hat{s}(z^*), \eta_s\}, \{1, \eta_1\}, \{I(z^*), \eta_I(z^*)\}) \\ &\quad \times \frac{1}{P_I^2} \mathbb{A}_n(\{-I(z^*), \eta_I(z^*)\}, \{2, \eta_2\}, \dots, \{\hat{n}(z^*), \eta_n(z^*)\}) \end{aligned}$$

Super soft theorem in $\mathcal{N}=4$ SYM

After a long computation, we obtain

$$\mathcal{A}_{n+1} = \frac{\langle n 1 \rangle}{\langle n s \rangle \langle s 1 \rangle} \mathcal{A}_n \left(\{ \lambda_1, \tilde{\lambda}_1 + \frac{\langle n s \rangle}{\langle n 1 \rangle} \tilde{\lambda}_s, \eta_1 + \frac{\langle n s \rangle}{\langle n 1 \rangle} \eta_s \}, \dots, \{ \lambda_n, \tilde{\lambda}_n + \frac{\langle s 1 \rangle}{\langle n 1 \rangle} \tilde{\lambda}_s, \eta_n + \frac{\langle s 1 \rangle}{\langle n 1 \rangle} \eta_s \} \right)$$

In the holomorphic soft limit $\lambda_s \rightarrow \epsilon \lambda_s$,

$$\begin{aligned} \mathcal{A}_{n+1}(\epsilon) &= \frac{1}{\epsilon^2} \frac{\langle n 1 \rangle}{\langle n s \rangle \langle s 1 \rangle} \mathcal{A}_n \left(\{ \lambda_1, \tilde{\lambda}_1 + \epsilon \frac{\langle n s \rangle}{\langle n 1 \rangle} \tilde{\lambda}_s, \eta_1 + \epsilon \frac{\langle n s \rangle}{\langle n 1 \rangle} \eta_s \}, \{ \lambda_2, \tilde{\lambda}_2, \eta_2 \}, \right. \\ &\quad \dots, \left. \{ \lambda_n, \tilde{\lambda}_n + \epsilon \frac{\langle s 1 \rangle}{\langle n 1 \rangle} \tilde{\lambda}_s, \eta_n + \epsilon \frac{\langle s 1 \rangle}{\langle n 1 \rangle} \eta_s \} \right). \end{aligned}$$

Performing Taylor expansion at $\epsilon = 0$, we have

$$\mathcal{A}_{n+1}(\epsilon) = \left(\frac{1}{\epsilon^2} \mathcal{S}_{\text{SYM}}^{(0)} + \frac{1}{\epsilon} \mathcal{S}_{\text{SYM}}^{(1)} \right) \mathcal{A}_n + \mathcal{O}(\epsilon^0)$$

with

$$\mathcal{S}_{\text{SYM}}^{(0)} = \frac{\langle n 1 \rangle}{\langle n s \rangle \langle s 1 \rangle},$$

$$\mathcal{S}_{\text{SYM}}^{(1)} = \tilde{\lambda}_s^{\dot{\alpha}} \left(\frac{1}{\langle n s \rangle} \frac{\partial}{\partial \tilde{\lambda}_n^{\dot{\alpha}}} + \frac{1}{\langle s 1 \rangle} \frac{\partial}{\partial \tilde{\lambda}_1^{\dot{\alpha}}} \right) + \eta_s^A \mathcal{F}_A^{(0)}, \quad \mathcal{F}_A^{(0)} \equiv \frac{1}{\langle s 1 \rangle} \frac{\partial}{\partial \eta_1^A} + \frac{1}{\langle n s \rangle} \frac{\partial}{\partial \eta_n^A}$$

[Zheng-Wen Liu, [1410.1616](#); Junjie Rao, [1410.5047](#)]

Super soft theorem in $\mathcal{N}=4$ SYM

Let us expand the superamplitude \mathcal{A}_{n+1} in odd Grassmann odd variables η_s

$$\mathcal{A}_{n+1}(\dots, \Phi_s) = \mathcal{A}_{n+1}(\dots, g_s^+) + \eta_s^A \mathcal{A}_{n+1}(\dots, \Gamma_{sA}) + \frac{1}{2!} \eta_s^A \eta_s^B \mathcal{A}_{n+1}(\dots, S_{sAB}) + \dots$$

According to the degrees of the Grassmann odd η_s , we can express super soft theorem as:

$$\begin{aligned}\mathcal{A}_{n+1}(\dots, g_s^+)(\epsilon) &= \left(\frac{1}{\epsilon^2} S_{\text{YM}}^{(0)} + \frac{1}{\epsilon} S_{\text{YM}}^{(1)} \right) \mathcal{A}_n + \mathcal{O}(\epsilon^0), \\ \mathcal{A}_{n+1}(\dots, \Gamma_{sA})(\epsilon) &= \frac{1}{\epsilon} \mathcal{F}_A^{(0)} \mathcal{A}_n + \mathcal{O}(\epsilon^0), \\ \mathcal{A}_{n+1}(\dots, S_{sAB})(\epsilon) &= \frac{0}{\epsilon} + \mathcal{O}(\epsilon^0).\end{aligned}$$

I also obtained the super soft theorem by studying Drummond-Henn's formulas [0808.2475]

$$\begin{aligned}\mathcal{A}_n &= \mathcal{A}_n^{\text{MHV}} \mathcal{P}_n, & \mathcal{P}_n &= \mathcal{P}_n^{\text{MHV}} + \mathcal{P}_n^{\text{NMHV}} + \dots + \mathcal{P}_n^{\overline{\text{MHV}}}, & \mathcal{P}_n^{\text{MHV}} &= 1 \\ \mathcal{P}_n^{\text{NMHV}} &= \sum_{2 \leq a < b \leq n-1} R_{n;ab} & \text{dual superconformal invariants} \\ \mathcal{P}_n^{\text{NNMHV}} &= \sum_{2 \leq a_1, b_1 \leq n-1} R_{n,a_1 b_1}^{0;0} \left(\sum_{a_1+1 \leq a_2, b_2 \leq b_1} R_{n;b_1 a_1; a_2 b_2}^{0;a_1 b_1} + \sum_{b_1 \leq a_2, b_2 \leq n-1} R_{n;a_2 b_2}^{a_1 b_1;0} \right) \\ &\dots \dots \dots & [\text{Drummond \& Henn, 0808.2475}]\end{aligned}$$

Soft gluino

$$\begin{aligned}
A_6(g_1^-, g_2^-, \bar{\Gamma}_3^A, \Gamma_{4B}, g_5^+, g_6^+) &\xrightarrow{\lambda_4 \rightarrow \epsilon \lambda_4} \delta_B^A \frac{1}{\epsilon} \left(\frac{1}{\langle 34 \rangle} A_5(g_1^-, g_2^-, g_3^-, g_5^+, g_6^+) + \frac{1}{\langle 45 \rangle} A_5(g_1^-, g_2^-, \bar{\Gamma}_3^C, \Gamma_{5C}, g_6^+) \right) + \mathcal{O}(\epsilon^0), \\
A_6(g_1^-, \bar{\Gamma}_2^A, g_3^-, \Gamma_{4B}, g_5^+, g_6^+) &\xrightarrow{\lambda_4 \rightarrow \epsilon \lambda_4} \delta_B^A \frac{1}{\epsilon} \frac{1}{\langle 45 \rangle} A_5(g_1^-, \bar{\Gamma}_2^A, g_3^-, \Gamma_{5A}, g_6^+) + \mathcal{O}(\epsilon^0), \\
A_6(\bar{\Gamma}_1^A, g_2^-, g_3^-, \Gamma_{4B}, g_5^+, g_6^+) &\xrightarrow{\lambda_4 \rightarrow \epsilon \lambda_4} \frac{1}{\epsilon} \frac{1}{\langle 45 \rangle} A_5(\bar{\Gamma}_1^A, g_2^-, g_3^-, \Gamma_{5B}, g_6^+) + \mathcal{O}(\epsilon^0), \\
A_6(g_1^-, \bar{\Gamma}_2^A, g_3^-, g_4^+, \Gamma_{5B}, g_6^+) &\xrightarrow{\lambda_5 \rightarrow \epsilon \lambda_5} \frac{1}{\epsilon} \frac{1}{\langle 45 \rangle} A_5(g_1^-, \bar{\Gamma}_2^A, g_3^-, \Gamma_{4B}, g_6^+) + \frac{1}{\epsilon} \frac{1}{\langle 56 \rangle} A_5(g_1^-, \bar{\Gamma}_2^A, g_3^-, g_4^+, \Gamma_{6B}) + \mathcal{O}(\epsilon^0), \\
A_6(\bar{\Gamma}_1^A, g_2^-, g_3^-, g_4^+, \Gamma_{5B}, g_6^+) &\xrightarrow{\lambda_5 \rightarrow \epsilon \lambda_5} \frac{1}{\epsilon} \frac{1}{\langle 45 \rangle} A_5(\bar{\Gamma}_1^A, g_2^-, g_3^-, \Gamma_{4B}, g_6^+) + \frac{1}{\epsilon} \frac{1}{\langle 56 \rangle} A_5(\bar{\Gamma}_1^A, g_2^-, g_3^-, g_4^+, \Gamma_{6B}) + \mathcal{O}(\epsilon^0), \\
A_6(\bar{\Gamma}_1^A, g_2^-, \Gamma_{3B}, g_4^-, g_5^+, g_6^+) &\xrightarrow{\lambda_3 \rightarrow \epsilon \lambda_3} \frac{0}{\epsilon} + \mathcal{O}(\epsilon^0), \quad A_6(g_1^-, \Gamma_{2B}, g_3^-, g_4^+, \bar{\Gamma}_5^A, g_6^+) \xrightarrow{\lambda_2 \rightarrow \epsilon \lambda_2} \frac{0}{\epsilon} + \mathcal{O}(\epsilon^0), \\
A_6(g_1^-, \bar{\Gamma}_2^A, \Gamma_{3B}, g_4^-, g_5^+, g_6^+) &\xrightarrow{\lambda_3 \rightarrow \epsilon \lambda_3} \frac{1}{\epsilon} \frac{1}{\langle 23 \rangle} \delta_B^A A_5(g_1^-, g_2^-, g_4^-, g_5^+, g_6^+) + \mathcal{O}(\epsilon^0), \\
A_6(g_1^-, g_2^-, \Gamma_{3B}, \bar{\Gamma}_4^A, g_5^+, g_6^+) &\xrightarrow{\lambda_3 \rightarrow \epsilon \lambda_3} \frac{1}{\epsilon} \frac{1}{\langle 34 \rangle} \delta_B^A A_5(g_1^-, g_2^-, g_4^-, g_5^+, g_6^+) + \mathcal{O}(\epsilon^0), \\
A_6(\bar{\Gamma}_1^A, \Gamma_{2B}, g_3^-, g_4^+, g_5^-, g_6^+) &\xrightarrow{\lambda_2 \rightarrow \epsilon \lambda_2} \frac{1}{\epsilon} \frac{1}{\langle 12 \rangle} \delta_B^A A_5(g_1^-, g_3^-, g_4^+, g_5^-, g_6^+) + \mathcal{O}(\epsilon^0), \\
A_6(g_1^-, \Gamma_{2B}, \bar{\Gamma}_3^A, g_4^+, g_5^-, g_6^+) &\xrightarrow{\lambda_2 \rightarrow \epsilon \lambda_2} \frac{1}{\epsilon} \frac{1}{\langle 23 \rangle} \delta_B^A A_5(g_1^-, g_3^-, g_4^+, g_5^-, g_6^+) + \mathcal{O}(\epsilon^0).
\end{aligned}$$

[Kunszt 1986; Luo-Wen, [hep-th/0501121](#); Bidder-Dunbar-Perkins, [hep-th/0505249](#); Luo-Wen, [hep-th/0502009](#)]

Super soft theorem in $\mathcal{N}=8$ supergravity

- largest amount of supersymmetry and spins smaller than 2
- $\mathcal{N}=8$ SUGRA: the simplest quantum field theory [Arkani-Hamed, Cachazo, Kaplan, 0808.1446]
- $(\mathcal{N}=8 \text{ SUGRA}) \sim (\mathcal{N}=4 \text{ SYM}) \otimes (\mathcal{N}=4 \text{ SYM})$

$\mathcal{N}=8$ SUGRA consists of 256 massless on-shell fields which form a single on-shell superfield

$$\begin{aligned}\Phi(p, \eta) = & h^+(p) + \eta^A \psi_A(p) + \frac{1}{2!} \eta^A \eta^B v_{AB}(p) + \frac{1}{3!} \eta^A \eta^B \eta^C \chi_{ABC}(p) \\ & + \frac{1}{4!} \eta^A \eta^B \eta^C \eta^D S_{ABCD}(p) + \dots + \eta^1 \eta^2 \eta^3 \eta^4 \eta^5 \eta^6 \eta^7 \eta^8 h^-(p)\end{aligned}$$

Using super-BCFW, we obtain

$$\begin{aligned}\mathcal{M}_{n+1}(\epsilon) = & \frac{1}{\epsilon^3} \sum_{a=1}^{n-1} \frac{[s a] \langle n a \rangle^2}{\langle s a \rangle \langle n s \rangle^2} \mathcal{M}_n \left(\dots, \{\lambda_a, \tilde{\lambda}_a + \epsilon \frac{\langle n s \rangle}{\langle n a \rangle} \tilde{\lambda}_s, \eta_a + \epsilon \frac{\langle n s \rangle}{\langle n a \rangle} \eta_s\}, \right. \\ & \left. \dots, \{\lambda_n, \tilde{\lambda}_n + \epsilon \frac{\langle s a \rangle}{\langle n a \rangle} \tilde{\lambda}_s, \eta_n + \epsilon \frac{\langle s a \rangle}{\langle n a \rangle} \eta_s\} \right)\end{aligned}$$

in the holomorphic soft limit $\lambda_s \rightarrow \epsilon \lambda_s$.

Super soft theorem in $\mathcal{N}=8$ SUGRA

Performing Taylor expansion of $\mathcal{M}(\epsilon)$ around $\epsilon = 0$, we can obtain the super soft theorem:

$$\mathcal{M}_{n+1}(\epsilon) = \left(\frac{1}{\epsilon^3} \mathcal{S}^{(0)} + \frac{1}{\epsilon^2} \mathcal{S}^{(1)} + \frac{1}{\epsilon} \mathcal{S}^{(2)} \right) \mathcal{M}_n + \mathcal{O}(\epsilon^0).$$

Super-soft operators are given by

$$\mathcal{S}^{(0)} = \sum_{a=1}^{n-1} \frac{[s a] \langle n a \rangle^2}{\langle s a \rangle \langle n s \rangle^2} = S^{(0)},$$

$$\mathcal{S}^{(1)} = S^{(1)} + \eta_s^A \mathcal{S}_A^{(1)}, \quad S^{(1)} = \sum_{a=1}^{n-1} \frac{[s a] \langle n a \rangle}{\langle s a \rangle \langle n s \rangle} \tilde{\lambda}_{s\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{a\dot{\alpha}}}, \quad \mathcal{S}_A^{(1)} = \sum_{a=1}^{n-1} \frac{[s a] \langle n a \rangle}{\langle s a \rangle \langle n s \rangle} \frac{\partial}{\partial \eta_a^A},$$

$$\mathcal{S}^{(2)} = S^{(2)} + \eta_s^A \mathcal{S}_A^{(2)} + \frac{1}{2} \eta_s^A \eta_s^B \mathcal{S}_{AB}^{(2)}, \quad S^{(2)} = \frac{1}{2} \sum_{a=1}^n \frac{[s a]}{\langle s a \rangle} \tilde{\lambda}_{s\dot{\alpha}} \tilde{\lambda}_{s\dot{\beta}} \frac{\partial^2}{\partial \tilde{\lambda}_{a\dot{\alpha}} \partial \tilde{\lambda}_{a\dot{\beta}}},$$

$$\mathcal{S}_A^{(2)} = \sum_{a=1}^n \frac{[s a]}{\langle s a \rangle} \tilde{\lambda}_{s\dot{\alpha}} \frac{\partial^2}{\partial \tilde{\lambda}_{a\dot{\alpha}} \partial \eta_a^A}, \quad \mathcal{S}_{AB}^{(2)} = \sum_{a=1}^n \frac{[s a]}{\langle s a \rangle} \frac{\partial^2}{\partial \eta_a^B \partial \eta_a^A}.$$

Super soft theorem in $\mathcal{N}=8$ SUGRA

Expanding superamplitude \mathcal{M}_{n+1} in Grassmannian odd variables η_s , we have

$$\begin{aligned}\mathcal{M}_{n+1}(\Phi_1, \dots, \Phi_n, \Phi_s) &= \mathcal{M}_{n+1}(\Phi_1, \dots, \Phi_n, h_s^+) + \eta_s^A \mathcal{M}_{n+1}(\Phi_1, \dots, \Phi_n, \psi_{sA}) \\ &\quad + \frac{1}{2} \eta_s^A \eta_s^B \mathcal{M}_{n+1}(\Phi_1, \dots, \Phi_n, v_{sAB}) + \dots\end{aligned}$$

Thus we can express the super soft theorem in $\mathcal{N}=8$ SUGRA as

$$\text{soft graviton: } \mathcal{M}_{n+1}(\dots, h_s^+)(\epsilon) = \left(\frac{1}{\epsilon^3} S^{(0)} + \frac{1}{\epsilon^2} S^{(1)} + \frac{1}{\epsilon} S^{(2)} \right) \mathcal{M}_n + \mathcal{O}(\epsilon^0),$$

$$\text{soft gravitino: } \mathcal{M}_{n+1}(\dots, \psi_{sA})(\epsilon) = \left(\frac{1}{\epsilon^2} \mathcal{S}_A^{(1)} + \frac{1}{\epsilon} \mathcal{S}_A^{(2)} \right) \mathcal{M}_n + \mathcal{O}(\epsilon^0),$$

$$\text{soft gravi-photon: } \mathcal{M}_{n+1}(\dots, v_{sAB})(\epsilon) = \frac{1}{\epsilon} \mathcal{S}_{AB}^{(2)} \mathcal{M}_n + \mathcal{O}(\epsilon^0),$$

$$\text{soft gravi-photino: } \mathcal{M}_{n+1}(\dots, \chi_{sABC})(\epsilon) = \frac{0}{\epsilon} + \mathcal{O}(\epsilon^0),$$

$$\text{soft scalar: } \mathcal{M}_{n+1}(\dots, S_{sABCD})(\epsilon) = \frac{0}{\epsilon} + \mathcal{O}(\epsilon^0).$$

[Liu, 1410.1616]

Soft gravi-photon

4-gravi-photon amplitude [0811.3417, 0805.0757]:

$$\mathcal{M}_4(v^{AB}, v^{CD}, v_{EF}, v_{GH}) = \langle 12 \rangle^2 [34]^2 \left(\frac{1}{t} \delta_{EF}^{AB} \delta_{GH}^{CD} + \frac{1}{u} \delta_{GH}^{AB} \delta_{EF}^{CD} + \frac{1}{s} \delta_{EFGH}^{ABCD} \right)$$

where $s = s_{12}$, $t = s_{13}$, $u = s_{14}$ are Mandelstam variables.

In the holomorphic soft limit $\lambda_4 \rightarrow \epsilon \lambda_4$, the amplitude becomes

$$\begin{aligned} \mathcal{M}_4(v^{AB}, v^{CD}, v_{EF}, \epsilon v_{GH}) &= \frac{1}{\epsilon} \left(\frac{[24]}{\langle 24 \rangle} \mathcal{M}_3(v_1^{AB}, h_2^-, v_{3EF}) \delta_{GH}^{CD} \right. \\ &\quad \left. + \frac{[14]}{\langle 14 \rangle} \mathcal{M}_3(h_1^-, v_2^{CD}, v_{3EF}) \delta_{GH}^{AB} + \frac{[34]}{\langle 34 \rangle} \mathcal{M}_3(v_1^{AB}, v_2^{CD}, S_{3EFGH}) \right), \end{aligned}$$

where we have used the following 3-point amplitudes:

$$\begin{aligned} \mathcal{M}_3(h_1^-, v_2^{CD}, v_{3EF}) &= \frac{\langle 12 \rangle^4}{\langle 23 \rangle^2} \delta_{EF}^{CD}, & \mathcal{M}_3(v_1^{AB}, h_2^-, v_{3EF}) &= \frac{\langle 12 \rangle^4}{\langle 31 \rangle^2} \delta_{EF}^{AB}, \\ \mathcal{M}_3(v_1^{AB}, v_2^{CD}, S_{3EFGH}) &= \langle 12 \rangle^2 \delta_{EFGH}^{ABCD}. \end{aligned}$$

[Kallosh-Lee-Rube, 0811.3417; Bianchi-Elvang-Freedman, 0805.0757]

Soft gravi-photon

2-scalar-2-gravi-photon amplitude [0811.3417, 0805.0757]:

$$\begin{aligned} \mathcal{M}_4(v^{AB}, v_{CD}, S_{EFGH}, S_{IJKL}) \\ = \langle 13 \rangle^2 [23]^2 \left(\frac{1}{s} \delta_{CD}^{AB} \epsilon_{EFGHIJKL} + \frac{3!}{t} \delta_{[EF}^{AB} \epsilon_{GH]IJKLCD} + \frac{3!}{u} \delta_{[IJ}^{AB} \epsilon_{KL]EFGHCD} \right). \end{aligned}$$

In the holomorphic soft limit $\lambda_2 \rightarrow \epsilon \lambda_2$, this amplitude becomes

$$\begin{aligned} \mathcal{M}_4(v^{AB}, \epsilon v_{CD}, S_{EFGH}, S_{IJKL}) &= \frac{1}{\epsilon} \left(\frac{[21]}{\langle 21 \rangle} \delta_{CD}^{AB} \mathcal{M}_3(h_1^-, S_{3EFGH}, S_{4IJKL}) \right. \\ &\quad + \frac{1}{2} \frac{[23]}{\langle 23 \rangle} \epsilon_{MNEFGHCD} \mathcal{M}_3(v_1^{AB}, v_3^{MN}, S_{4IJKL}) \\ &\quad \left. + \frac{1}{2} \frac{[24]}{\langle 24 \rangle} \epsilon_{MNIJKLCD} \mathcal{M}_3(v_1^{AB}, S_{3EFGH}, v_4^{MN}) \right) \end{aligned}$$

Three 3-point amplitudes involved are given by

$$\mathcal{M}_3(h_1^-, S_{3EFGH}, S_{4IJKL}) = \frac{\langle 13 \rangle^2 \langle 14 \rangle^2}{\langle 34 \rangle^2} \epsilon_{EFGHIJKL},$$

$$\mathcal{M}_3(v_1^{AB}, v_3^{MN}, S_{4IJKL}) = \langle 13 \rangle^2 \delta_{IJKL}^{ABMN}, \quad \mathcal{M}_3(v_1^{AB}, S_{3EFGH}, v_4^{MN}) = \langle 14 \rangle^2 \delta_{EFGH}^{ABMN}.$$

Gravity = Yang-Mills²

Let us define

$$\mathfrak{S}^0(x, s, a) \equiv \frac{\langle x a \rangle}{\langle x s \rangle \langle s a \rangle}, \quad \mathfrak{S}^1(s, a) \equiv \frac{1}{\langle s a \rangle} \tilde{\lambda}_{s\dot{\alpha}} \frac{\partial}{\partial \tilde{\lambda}_{a\dot{\alpha}}}, \quad \mathfrak{S}_\eta^1(s, a) \equiv \frac{1}{\langle s a \rangle} \eta_s^A \frac{\partial}{\partial \eta_a^A}.$$

Then all the super soft operators can be written as

$$\mathcal{S}_{\text{SYM}}^{(0)} = \mathfrak{S}^0(x, s, n) + \mathfrak{S}^0(x, s, 1)$$

$$\mathcal{S}_{\text{SYM}}^{(1)} = (\mathfrak{S}^1(s, 1) - \mathfrak{S}^1(s, n)) + (\mathfrak{S}_\eta^1(s, 1) - \mathfrak{S}_\eta^1(s, n))$$

$$\mathcal{S}^{(0)} = \sum_{a=1}^n s_{sa} \mathfrak{S}^0(x, s, a) \mathfrak{S}^0(y, s, a)$$

$$\mathcal{S}^{(1)} = \frac{1}{2} \sum_{a=1}^n s_{sa} (\mathfrak{S}^0(x, s, a) + \mathfrak{S}^0(y, s, a)) (\mathfrak{S}^1(s, a) + \mathfrak{S}_\eta^1(s, a))$$

$$\mathcal{S}^{(2)} = \frac{1}{2} \sum_{a=1}^n s_{sa} (\mathfrak{S}^1(s, a) + \mathfrak{S}_\eta^1(s, a)) (\mathfrak{S}^1(s, a) + \mathfrak{S}_\eta^1(s, a))$$

Summary

- BCFW recursion relations and CHY formulas
- Soft theorems in $\mathcal{N}=4$ SYM and $\mathcal{N}=8$ SUGRA
 - soft-gluino, soft gravitino, soft gravi-photon divergences
- By the way, I also studied the M5-brane solutions in 11d M-theory in $AdS_4 \times Q^{1,1,1}$ spacetime with B. Chen, D.-S. Li and J.-B. Wu [[PRD 90, 066005; 1406.1892](#)].

Outlook

- How to understand the soft-gravitino (spin-3/2) behavior from a symmetry principle
- The scattering equations and CHY formulas
 - in 4d, MHV amplitudes, Fermions, SUSY, ...
- Scattering amplitudes in marginal deformations of $\mathcal{N}=4$ SYM from twistor strings
- Scattering amplitudes of Yang-Mills, gravity, strings, ambitwistor strings, etc.

Thank you very much and Welcome to China!

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Sanya, China

December 31, 2015 – January 5, 2016

Strings 2016

Beijing, China

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