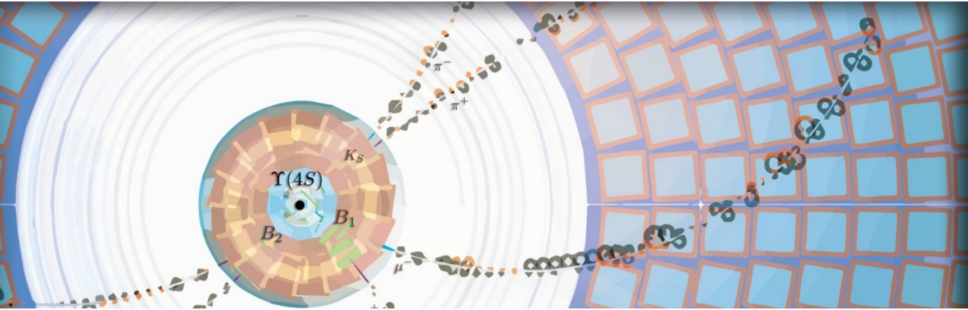
Helix gymnastics.

F2F Meeting - Karlsruhe 2015



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DESY HELMHOLTZ ASSOCIATION Belle II

Overview

> Addressing of points in perigee parameter representation

> Issues converting perigee uncertainties to Cartesian uncertainties.

> Questions

Addressing of points in perigee parameter representation

Perigee parameterisation - currently implemented

Definition

The perigee is the point on the helix closest to the beam line.

Parameters related to *xy* only

- d_0 - signed distance to the origin
  - Sign is the sign of z coordinate of $\vec{x} \times \vec{p}$
- ϕ_0 - direction of the momentum at the perigee
- ω - curvature in the *xy* plane signed by the charge.
  - Unfortunately this is the opposite of the mathematical curvature

Parameters related to *z*

- z_0 - z coordinate of the perigee
- $\tan\lambda$ - angle out of the xy plane $= \frac{p_z}{p_t} = \frac{\partial z}{\partial s}$

Addressing points on the helix

Naive addressing of the helix points

Using the opening angle χ from the perigee viewed from the center of the helix

$$h(s) = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -\frac{\sin\chi}{\omega} \\ \frac{\cos\chi - 1}{\omega} - d_0 \\ -\tan\lambda \cdot \frac{\chi}{\omega} + z_0 \end{pmatrix}$$

Weaknesses of this parameterisation

- Isolated singularity at $\omega \sim 0$ in x and y
- Unstabilities for low curvatures
- Unstabilities when close to the helix and make small extrapolations, because the helix is perceived as almost straight.
- Cannot represent a straight line $\omega = 0$.

Demostration of the weakness of χ

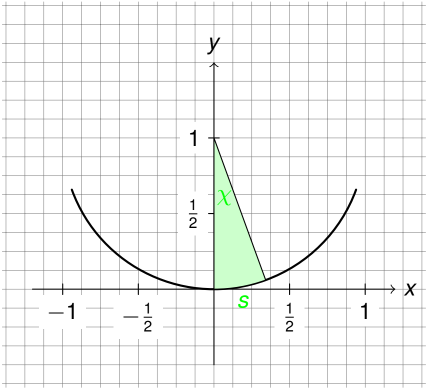

Figure 1: χ parameter okay for rather large ω and large extrapolation steps.

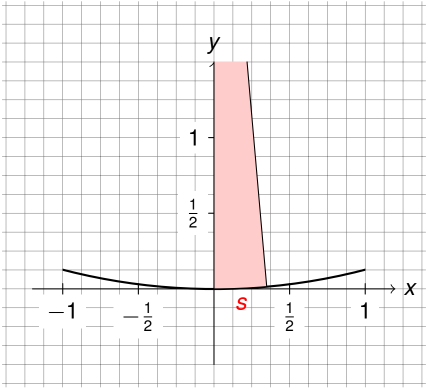

Figure 2: As $w \to 0$ positions become more sensitive to χ

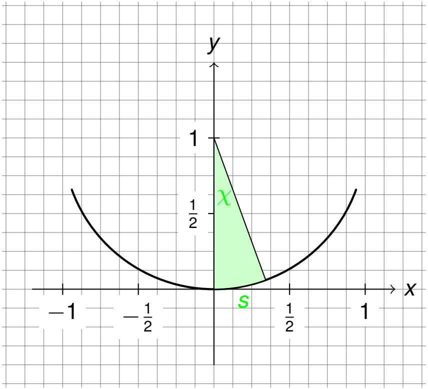

Figure 3: χ parameter okay for rather large ω and large extrapolation steps.

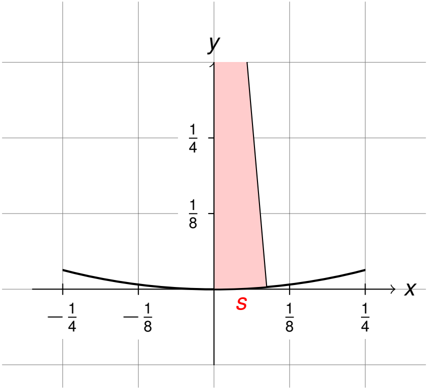

Figure 4: Same weakness occures on small scales, e.g. extrapolating near the origin.

Lifting the isolated singularity by replacing χ

Two dimensional arc length s

$$s = -\frac{\chi}{\omega}$$

$$\chi = -s \cdot \omega$$

In contrast to the opening angle χ the two dimensional arc length s is also defined for the straight line.

The sinus cardinalis

The divergent quotient in the x coordinate

$$x(s) = -\frac{\sin\chi}{\omega} = s \cdot \frac{\sin(s \cdot \omega)}{s \cdot \omega} = s \cdot \mathrm{sinc}(s \cdot \omega)$$

contains the same sort of divergence as the sinus cardinalis function. Similarly

$$y(s) = -s \cdot \sin\left(\frac{s \cdot \omega}{2}\right) \cdot \mathrm{sinc}\left(\frac{s \cdot \omega}{2}\right) - d_0$$

Addressing points on the helix

Improved addressing of the helix points

Using two dimensional arc length s

$$h(s) = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s \cdot \operatorname{sinc}(s \cdot \omega) \\ -s \cdot \sin\left(\frac{s \cdot \omega}{2}\right) \cdot \operatorname{sinc}\left(\frac{s \cdot \omega}{2}\right) - d_0 \\ \tan\lambda \cdot s + z_0 \end{pmatrix}$$

Issues converting perigee uncertainties to Cartesian uncertainties.

Continuted conversation with Anze.

Summary

- Anze: Using the Belle II code I do not get the same 6x6 covariance matrix for position and momentum as Belle I when unpacking from the 5x5 covariance matrices of the perigee parameters.
- Me: The 6x6 Cartesian covariances are not unique and are partly abitrary in the sense, that a convention has to be established to get them. Probably the convention of Belle I is different to the current implementation of Belle II, which is why there cannot be a match in the 6x6 covariance matrix.
- ...
- We also found some sign errors in the UncertainHelix, but still the above statement holds.

Converting to cartesian covariances

Task

Given a 5x5 covariance matrix of the perigee parameters find the 6x6 covariance of the cartesian parameters.

Constraints

- The 6x6 covariance matrix can be of rank 5 at best.
- Corollary there is one linear combination of Cartisian parameters that has a variance of 0, meaning that it is know to **infinite** precision.
- Which combination is it?
- Equivalently how is the left over rank filled?

Suspicion

There are many equivalently valid answers!

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Converting to cartesian parameters

Perigee - first three parameters

$$\vec{x} = \begin{pmatrix} d_0 \cdot \sin\phi_0 \\ -d_0 \cdot \cos\phi_0 \\ z_0 \end{pmatrix}$$

Momentum - last three parameters

$$\vec{p} = p_t \cdot \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \\ \tan\lambda \end{pmatrix}$$

where

$$p_t = \frac{B_z \cdot c \cdot 10^{11}}{|\omega|}$$

Demostration

Themes

- Covariance propagation C to C' using similarity transformations with Jacobi matrix of derivations J

$$C' = J \cdot C \cdot J^T$$

- Always derive first before inserting the known values.
- The origin is a abitrary construct.
- Different choice of the origin does not invalidate the results.
- Moving to a new origin in Cartesian parameters is an identity for the Cartesian covariance.
- Moving to a new origin in perigee parameters is none trivial.

In the following slides

- Concentrating on the xy coordinates only
- Assuming $\phi_0 = 0$ without loss of generality (rotations are a trivial matter)

Investigated situation

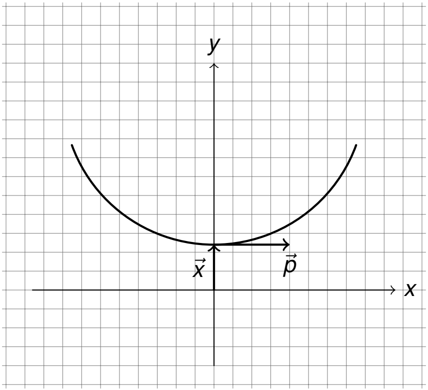

Figure 5: Projection of a helix after rotating to $\phi_0 = 0$

Translation from the current reference point \ *xy* part only

Using a similarity transformation, where the Jacobian matrix of the transformation from perigee parameters to Cartesian parameters is

$$J_{inflate} = \begin{pmatrix} \frac{\partial x}{\partial d_0} & \frac{\partial x}{\partial \phi_0} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial d_0} & \frac{\partial y}{\partial \phi_0} & \frac{\partial y}{\partial \omega} \\ \frac{\partial p_x}{\partial d_0} & \frac{\partial p_x}{\partial \phi_0} & \frac{\partial p_x}{\partial \omega} \\ \frac{\partial p_y}{\partial d_0} & \frac{\partial p_y}{\partial \phi_0} & \frac{\partial p_y}{\partial \omega} \end{pmatrix} = \begin{pmatrix} 0 & d_0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{p_t}{\omega} \\ 0 & p_t & 0 \end{pmatrix}$$

Constrained direction with covariance 0

$$\frac{1}{d_0} \cdot x - \frac{1}{p_t} \cdot p_y$$

depends on the parameters themselves.

1. First move the origin into the perigee.
2. Then translate to Cartesian coordinates

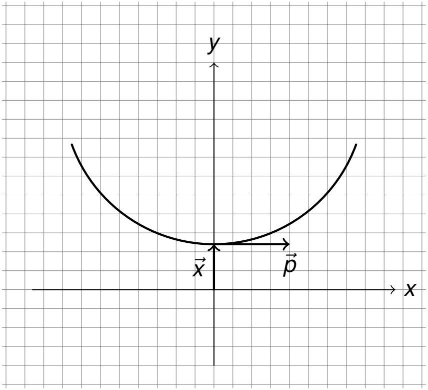

Figure 6: Original situation

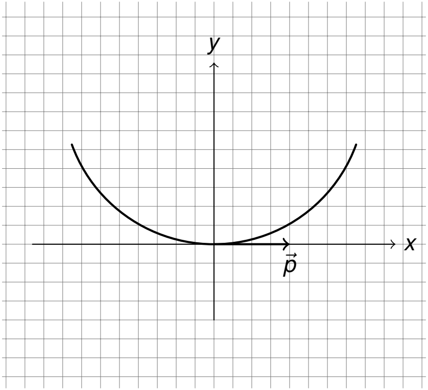

Figure 7: After the moving the origin

Alternative approach - first move covariance into perigee \ *xy* part only

Moving the perigee covariance matrix from the reference point to the perigee first in a passive move.

$$J_{\text{move, orthogonal to } \phi_0} = \begin{pmatrix} \frac{\partial d_0'}{\partial d_0} & \frac{\partial d_0'}{\partial \phi_0} & \frac{\partial d_0'}{\partial \omega} \\ \frac{\partial \phi_0'}{\partial d_0} & \frac{\partial \phi_0'}{\partial \phi_0} & \frac{\partial \phi_0'}{\partial \omega} \\ \frac{\partial \omega'}{\partial d_0} & \frac{\partial \omega'}{\partial \phi_0} & \frac{\partial \omega'}{\partial \omega} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1 + \omega \cdot d_0}{1 + \omega \cdot d_0'} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(Karimaki 1990)

Notes

- Moving parallel to the position vector does not change the perigee.
- Moving preserves all information in the covariance matrix. A reverse move of the origin cancels out exactly.
- Could also move to a different position inbetween or further away.

Alternative approach - second inflate to cartesian covariance \ xy part only

Same inflation jacobian matrix as before, but we can insert $d_0 = 0$, because we moved to the perigee first

$$J_{inflate} = \begin{pmatrix} \frac{\partial x}{\partial d_0} & \frac{\partial x}{\partial \phi_0} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial d_0} & \frac{\partial y}{\partial \phi_0} & \frac{\partial y}{\partial \omega} \\ \frac{\partial p_x}{\partial d_0} & \frac{\partial p_x}{\partial \phi_0} & \frac{\partial p_x}{\partial \omega} \\ \frac{\partial p_y}{\partial d_0} & \frac{\partial p_y}{\partial \phi_0} & \frac{\partial p_y}{\partial \omega} \end{pmatrix} = \begin{pmatrix} 0 & d_0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{p_t}{\omega} \\ 0 & p_t & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{p_t}{\omega} \\ 0 & p_t & 0 \end{pmatrix}$$

Total jacobian matrix of the conversion is

$$J = J_{inflate} \cdot J_{move}$$

Constrained direction with covariance 0

x

- Does not depend on the concrete parameters!
- Resulting covariance matrix is clearly different from the previous approach.
- The constrained direction is easier to spot.

Summary of the demonstration

Many valid conversion procedures from 5x5 to 6x6

- Converting from the origin (probably Belle I)
- Move to the perigee than inflate (current Belle II)
- Move halfway between origin and perigee, than inflate.
- Move one radius away from the helix.
- . . .

Conclusion

- Obtaining a 6x6 Cartesian covariance from the 5x5 perigee covariance is ambiguous and depends on a particular convention.
- There is always one linear combination of the Cartesian coordinates, that is known to abitrary precision.
- To keep the coordinate that is know in check I propose to always move to the perigee first, before the inflation.
- The constained coordinate is x (rotated by ϕ_0) the spatial component along the helix in this case.

Questions

First my questions

- What does it even mean for physics analysis to have covariances of rank 5?
  - Ultimately the results should not depend on the particular choice of translation convention.
- Should we rather stick to inverse Cartesian covariances C^{-1} to signify that one direction is **infinitely unconstrained** in contrast to **infinitely constrained**?
- Are we throwing to much of the covariance out of the window?
  - The missing 6th coordinate is s
  - Although s is 0 at the perigee and can be neglected for storage, its covariance with the other perigee parameters might unambiguosly fill the missing rank.