Towards a Quantum Theory of Solitons

Tehseen Rug

Max-Planck-Institut für Physik, Ludwig-Maximilians Universität

November, 13th 2015

Contents

An Example

Classical electric field:

$$\partial_i E_i = \rho \tag{1}$$

- given charge distribution $\rho \Rightarrow$ classical electric field ρ
- QED $\Rightarrow$ particles are fundamental, classical field as emergent phenomenon.
- classical field E_i is coherent state of N (longitudinal) photons
- semi-classical limit: $N \to \infty$

Semiclassical Approach

Quantum effects encoded in fluctuations around background

- expand $E_i \to E_i + \delta E_i$
- compute loops of "δE_i-particles" in classical background E_i
- classical limit: $\hbar \to 0$

.

Semiclassical Approach

Quantum effects encoded in fluctuations around background

- expand $E_i \to E_i + \delta E_i$
- compute loops of "δE_i-particles" in classical background E_i
- classical limit: $\hbar \to 0$

.

Levels of Quantumness

$$\begin{aligned} \text{Fully Quantum} &: N, \hbar \text{ finite} \\ \text{Semi-classical} &: N \to \infty,\ \hbar \text{ finite} \\ \text{Classical Limit} &: N \to \infty, \hbar \to 0 \end{aligned} \tag{2}$$

Solitons: Definition and Examples

Definition: Soliton is a static, localized, finite energy solution

Examples:

- non-topological soliton:
  $\mathscr{L} = (\partial_\mu \phi)^2 - m^2\phi^2 + g^2\phi^4 \rightarrow \phi_c(x) = \frac{m}{g}\text{sech}(mx)$
  characteristic size: $L = m^{-1}$
  Energy: $E_{non-top} = \int dx(\partial_x \phi)^2 = \frac{4m^3}{3g^2}$

Solitons: Definition and Examples

Definition: Soliton is a static, localized, finite energy solution

Examples:

- non-topological soliton:
  $\mathscr{L} = (\partial_\mu \phi)^2 - m^2\phi^2 + g^2\phi^4 \rightarrow \phi_c(x) = \frac{m}{g}\mathrm{sech}(mx)$
  characteristic size: $L = m^{-1}$
  Energy: $E_{non-top} = \int dx (\partial_x \phi)^2 = \frac{4m^3}{3g^2}$

- topological soliton:
  $\mathscr{L} = (\partial_\mu \phi)^2 - g^2(\phi^2 - m^2/g^2)^2 \rightarrow \phi_c(x) = \frac{m}{g}\tanh(xm)$
  characteristic size: $L = m^{-1}$, Energy: $E_{top} = \frac{8m^3}{3g^2}$
  new characteristic: topological charge Q

Basic Idea (Dvali, Gomez, Gruending, Rug; Nucl. Phys. B)

Idea: Understand Solitons and their properties quantum mechanically!

- expand $\phi_c(x) = \sqrt{R}\int \frac{dk}{\sqrt{4\pi|k|}}\left(e^{ikx}\alpha_k + e^{-ikx}\alpha_k^*\right)$

Basic Idea (Dvali, Gomez, Gruending, Rug; Nucl. Phys. B)

Idea: Understand Solitons and their properties quantum mechanically!

- expand $\phi_c(x) = \sqrt{R} \int \frac{dk}{\sqrt{4\pi|k|}} (e^{ikx} \alpha_k + e^{-ikx} \alpha_k^*)$
- promote coefficients to creation and annihilation operators $\hat{a}_k^\dagger$, $\hat{a}_k$ with $[\hat{a}_k, \hat{a}_p^\dagger] = \delta_{kp}$

Basic Idea (Dvali, Gomez, Gruending, Rug; Nucl. Phys. B)

Idea: Understand Solitons and their properties quantum mechanically!

- expand $\phi_c(x) = \sqrt{R}\int \frac{dk}{\sqrt{4\pi|k|}}(e^{ikx}\alpha_k + e^{-ikx}\alpha_k^*)$
- promote coefficients to creation and annihilation operators $\hat{a}_k^\dagger$, $\hat{a}_k$ with $[\hat{a}_k, \hat{a}_p^\dagger] = \delta_{kp}$
- construct quantum soliton state self-consistently as coherent state: $|\mathrm{sol}\rangle = e^{-\frac{1}{2}|\alpha_k|^2}\sum_{n_k=0}^{\infty}\frac{\alpha_k^{n_k}}{\sqrt{n_k!}}|n_k\rangle \rightarrow \hat{a}_k|\mathrm{sol}\rangle = \sqrt{N_k}|\mathrm{sol}\rangle$

Basic Idea (Dvali, Gomez, Gruending, Rug; Nucl. Phys. B)

Idea: Understand Solitons and their properties quantum mechanically!

- expand $\phi_c(x) = \sqrt{R}\int \frac{dk}{\sqrt{4\pi|k|}}(e^{ikx}\alpha_k + e^{-ikx}\alpha_k^*)$
- promote coefficients to creation and annihilation operators $\hat{a}_k^\dagger$, $\hat{a}_k$ with $[\hat{a}_k, \hat{a}_p^\dagger] = \delta_{kp}$
- construct quantum soliton state self-consistently as coherent state: $|\mathrm{sol}\rangle = e^{-\frac{1}{2}|\alpha_k|^2}\sum_{n_k=0}^{\infty}\frac{\alpha_k^{n_k}}{\sqrt{n_k!}}|n_k\rangle \rightarrow \hat{a}_k|\mathrm{sol}\rangle = \sqrt{N_k}|\mathrm{sol}\rangle$
- classical limit: $\langle\mathrm{sol}|\hat{\phi}_c|\mathrm{sol}\rangle = \phi_c$, $|\alpha_k|^2 \equiv N_k$: occupation of corpuscles in mode k.

Implications: Non-Topological Soliton

- total constituent number: $N = \int_k N_k = \frac{m^2}{g^2}\left(\frac{\log(2)8}{\pi}\right)$
- energy: $E_{non-top} = \int_k |k| N_k = \frac{4m^3}{3g^2}$

  dominant contribution to total number and energy: $k \sim m$!

Implications: Non-Topological Soliton

- total constituent number: $N = \int_k N_k = \frac{m^2}{g^2}\left(\frac{\log(2)8}{\pi}\right)$
- energy: $E_{non-top} = \int_k |k| N_k = \frac{4m^3}{3g^2}$

  dominant contribution to total number and energy: $k \sim m$!
- vacuum instability: $\langle 0|sol\rangle = e^{-\frac{N}{2}} \neq 0$

  vacuum can decay into soliton quantum mechanically!

Implications: Topological Soliton

- total constituent number: $N = \int_k N_k \sim \log(k_0)|_{k_o \to 0} \to \infty$

  dominant contribution to N: $k = 0$!
- $E_{top} = \int_k |k| N_k = \frac{8m^3}{3g^2}$

  dominant contribution to energy: $k \sim m$!

Implications: Topological Soliton

- total constituent number: $N = \int_k N_k \sim \log(k_0)|_{k_o \to 0} \to \infty$

  dominant contribution to N: $k = 0$!
- $E_{top} = \int_k |k| N_k = \frac{8m^3}{3g^2}$

  dominant contribution to energy: $k \sim m$!
- quantum stability: $\langle 0 | sol \rangle = e^{-\frac{N}{2}} = 0$

  quantum origin of topological charge: infinite occupation of $k = 0$ corpuscles!

Topological Soliton as Convolution

Idea: use convolution to disentangle energy and topology

- $\phi_c(x) = \frac{m}{g}\left(\mathrm{sign} \star \mathrm{sech}^2\right)(mx)$

Topological Soliton as Convolution

Idea: use convolution to disentangle energy and topology

- $\phi_c(x) = \frac{m}{g}\left(\mathrm{sign} \star \mathrm{sech}^2\right)(mx)$
- convolution theorem: $\alpha_k = t_k\, c_k$

  $c_k \equiv \sqrt{\pi m}\frac{k}{g}\mathrm{csch}\left(\frac{\pi k}{2m}\right), \; t_k \equiv \frac{i}{\sqrt{k}}$

  energy encoded in c_k quanta: $H = \int c_k^\dagger c_k$

  topological charge encoded in pole of t_k at $k = 0$!

Topological Soliton as Convolution

Idea: use convolution to disentangle energy and topology

- $\phi_c(x) = \frac{m}{g}\left(\mathrm{sign} \star \mathrm{sech}^2\right)(mx)$
- convolution theorem: $\alpha_k = t_k\, c_k$

  $c_k \equiv \sqrt{\pi m}\frac{k}{g}\mathrm{csch}\left(\frac{\pi k}{2m}\right),\; t_k \equiv \frac{i}{\sqrt{k}}$

  energy encoded in c_k quanta: $H = \int c_k^\dagger c_k$

  topological charge encoded in pole of t_k at $k = 0$!
- non-topological soliton: t_k trivial

Soliton-Anti-Soliton Interaction

- topological charge of soliton-anti-soliton pair:
  $Q = 1 + (-1) = 0$

Soliton-Anti-Soliton Interaction

- topological charge of soliton-anti-soliton pair: $Q = 1 + (-1) = 0$
- semi-classical result: interaction among pair of solitons seperated by distance a: $\sim -e^{-am}$

Soliton-Anti-Soliton Interaction

- topological charge of soliton-anti-soliton pair: $Q = 1 + (-1) = 0$
- semi-classical result: interaction among pair of solitons seperated by distance a: $\sim -e^{-am}$
- quantum picture: $H = \int d(k/m)\left(\hat{c}_k^\dagger \hat{c}_k + \hat{\tilde{c}}_k^\dagger \hat{\tilde{c}}_k + \hat{c}_k^\dagger \hat{\tilde{c}}_k + \hat{\tilde{c}}_k^\dagger \hat{c}_k\right)$
- first and second term: energy of individual soliton
- third and fourth term: interaction between solitons
- total number finite → vacuum decay due to attractive interactions of corpuscles in the different solitons

Summary and Outlook

Summary:

- represent solitons quantum mechanically as coherent states
- properties such as energy and topology encoded in distribution of corpuscles
- attractive potential due to interaction of corpuscles.

Summary and Outlook

Summary:

- represent solitons quantum mechanically as coherent states
- properties such as energy and topology encoded in distribution of corpuscles
- attractive potential due to interaction of corpuscles.

Outlook:

- include $1/N$ corrections
- corpuscular theory of instantons
- coherent state picture of gravitational backgrounds

Thank You for Your Attention