Quantum entanglement, it's entropy, and why we calculate it

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The classical system

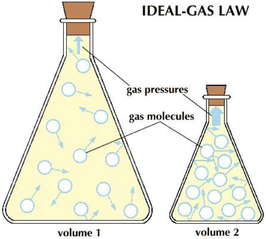


Figure : Credit: [Reich-chemistry]

$$P(V_l, V_r) = P_l(V_l)P_r(V_r)$$

The quantum system

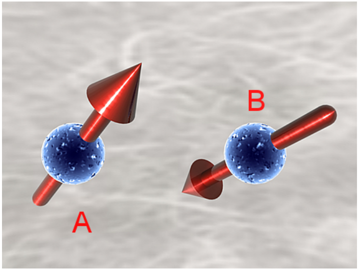


Figure : Credit: university of Delft

Hilbert space: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$
Base states: $|0\rangle_A |0\rangle_B$, $|1\rangle_A |1\rangle_B$, $|0\rangle_A |1\rangle_B$, $|1\rangle_A |0\rangle_B$

Not every vector of $\mathcal{H}$ (quantum state) can be decomposed as product of vectors from $\mathcal{H}_A$ & $\mathcal{H}_B$!

$$\frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right)$$

We call such non-decomposable states *entangled*.

If system is in an entangled state measurements on its sub-systems are not independent – the probabilities do not factorise $P(S_l, S_r) \neq P_l(S_l)P_r(S_r)$

Density matrix

Instead of $A \in \mathcal{H}$ use $\rho \in L(\mathcal{H})$ – a matrix (operator), such that $\mathrm{Tr}\left[\rho\right] = 1$ and ρ is Hermitian and positive definite. Also $\mathrm{Tr}\left[\rho^2\right] \leq 1$, equality for pure states (isolated system $\rho = \left|\phi\right\rangle \left\langle\phi\right|, \ \phi \in \mathcal{H}$)

Reduced density matrix

if Hilbert space decomposes $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ we can trace out states form one subsystem (say A) to obtain ρ_B – reduced density matrix.

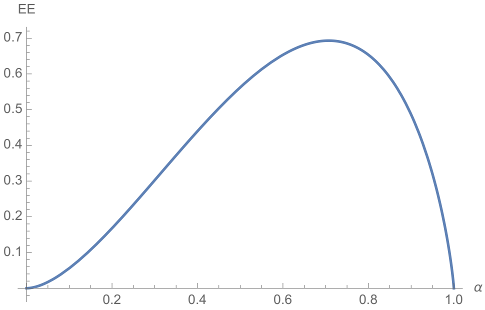
Entanglement entropy

$S_B = -\mathrm{Tr}\left[\rho_B \log \rho_B\right]$

Example

$$|\phi\rangle = \alpha\,|0\rangle_A\,|0\rangle_B + \sqrt{1-\alpha^2}\,|1\rangle_A\,|1\rangle_B$$

$$\rho = |\phi\rangle\,\langle\phi|\,,\ \ \rho_B = \alpha^2\,|0\rangle_B\,\langle 0|_B + (1-\alpha^2)\,|1\rangle_B\,\langle 1|_B$$



Many body systems

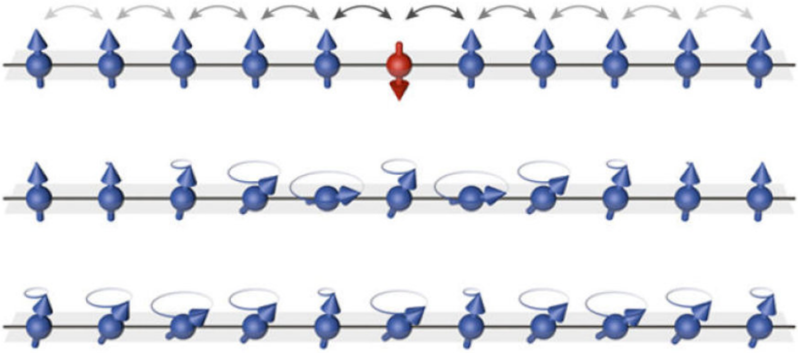


Figure : Credit: [Science Daily]

The very many bodies limit

- The continuous limit of many bodies – a quantum field theory!
- Still complicated :(
- Some cases – eg. near critical point – QFT becomes *conformal* – much simpler!

Cardy-Calabrese formula

1+1 dim. CFT, thermal state, Entanglement between interval of length l and the rest of the system:

$$S(l) = c/3 \log \left(\frac{\beta}{\pi \epsilon} \sinh \left(\frac{\pi l}{\beta} \right) \right)$$

$\beta = 1/kT$, c – central charge (density of degrees of freedom), ϵ– cut-off (lattice spacing)

How to treat more complicated states?

How to treat more complicated states?

Figure : AdS/CFT correspondence which "geometrises" CFT questions comes to the rescue!

Ryu-Takayanagi proposal

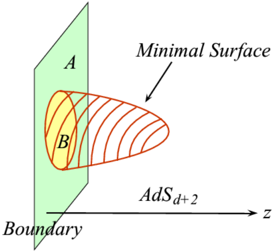


Figure : Credit: J.Phys. A42 (2009) 504008

The holographic entanglement entropy

$$S_B = \frac{\text{Area of minimal surface}}{4 G_N^{d+2}}$$

Example: "Local quench", or putting hot & cold together

- At $t = 0$, discontinuous temperature profile: $T = T_R,\ x > 0$, $T = T_L,\ x < 0$
- CFT stress-energy tensor at $t = 0$ diagonal
- From AdS/CFT we see the evolution of CFT with such initial condition

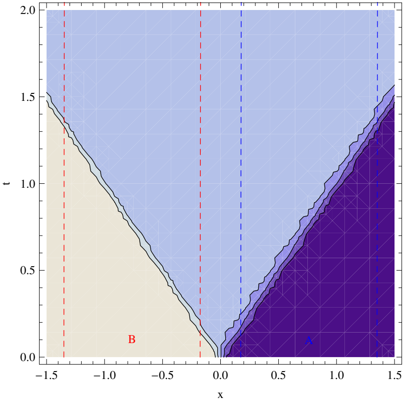


FIGURE : The dynamics of 1+1 CFT after local quench. Middle – steady state region, has non-zero current proportional to $T_L - T_R$ and temperature $\sqrt{T_L T_R}$. The "shockwave" travels with the speed of light

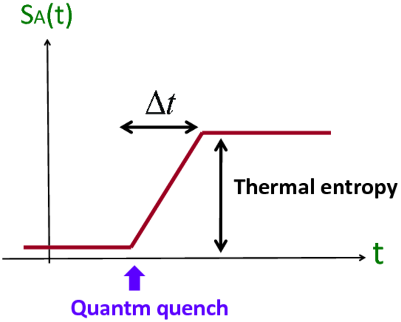


FIGURE : Naive expectation for entanglement entropy as a function of time. Credit: Class.Quant.Grav. 29 (2012) 153001

Evolution of EE

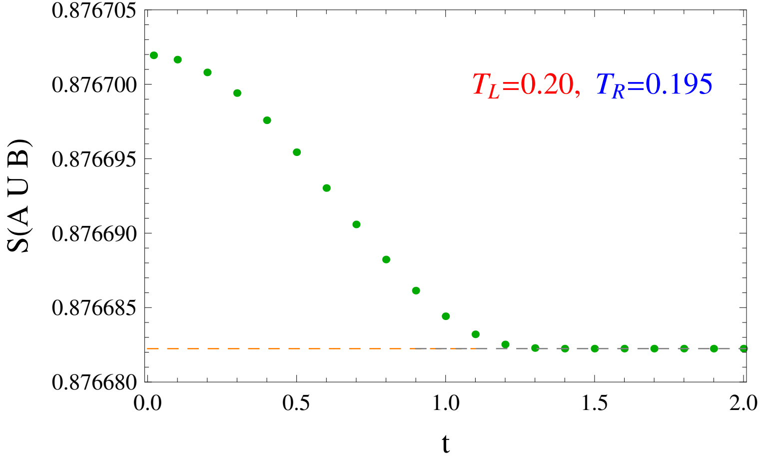


Figure : The entanglement entropy changes in much smoother way – numerics indicate $at^2 + bt^3$ instead of linear dependence! (*A preliminary result!*)

Summary

- Entanglement is a feature of quantum many-body systems – the measurements on independent parts of the system may not be statistically independent
- It's quantified by *entropy of entanglement* that is 0 for separable states and non-zero for entangled ones.
- For large many body systems near phase transitions we can use CFT methods and AdS/CFT
- In AdS/CFT higher dimensional space-time describes state of CFT, and area of minimal surface measures entanglement entropy
- Using AdS/CFT we can probe fancy, non-equilibrium problems for many-body systems (of course in some limits!)

Thank you for your attention

Bibliography

[1] D. Bernard, B. Doyon, "Conformal Field Theory out of equilibrium: a review", Arxiv: cond-mat 1603.07765

[2] P. Calabrese, J. Cardy, "Entanglement Entropy and Quantum Field Theory", Arxiv: hep-th 0405152

[3] M. Bahaseen, *et al.*, "Energy flow in quantum critical systems out of equilibrium", Nature Physics 11, 509–514 (2015)

[Reich-chemistry] Reich chemistry, `https://reich-chemistry.wikispaces.com/The+Ideal+Gas+Law.Bertino`

[Science Daily] Science Daily, `https://www.sciencedaily.com/releases/2013/03/130313095421.htm`