Dualities for String Theory and SUGRA

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String Theory

String theory is an attempt to explain all of the particles and fundamental forces of nature in one theory by modeling them as vibrations of strings.

- Open and Closed Bosonic String (dim 26)

Incorporate fermions and supersymmetry (dim 10)

- Type I superstring
- Type IIA & IIB superstring
- Heterotic SO(32) & $E_8 \times E_8$ superstring

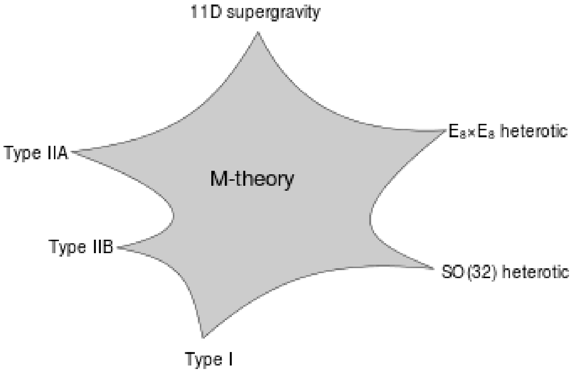
Unifying 5 types of superstring theories (dim 11)

- M-theory

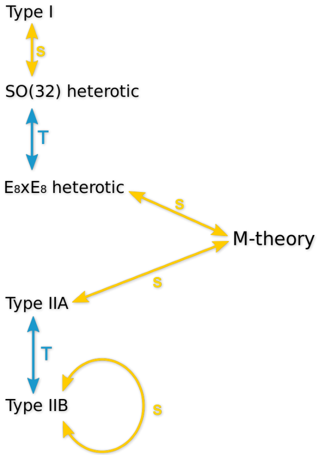
SL(2,Z) S-duality manifested type IIB superstring (dim 12)

- F-theory

String Theory



String Theory



T-duality

To make string theory contact with four-dim world, one needs to compactify on compact spaces. This feature a symmetry, T-duality.

- Bosonic String: Toroidal Compactification
- Type IIB $\Leftrightarrow$ Type IIA
- Heterotic $SO(32) \Leftrightarrow E8 \times E8$
- Mirror Symmetry

Toroidal Compactification of closed Bosonic Strnig

- Simplest case: compactification on a circle of radius R, e.g. $x^{25} \sim x^{25} + 2\pi RL$, $L \in \mathbb{Z}$.
  x^{25} parametrizes a 1-dim circle S^1 of radius R.
- The coordinate $x^{25}(\sigma, \tau), 0 \leqslant \sigma \leqslant 2\pi$, maps the closed string onto the spatial circle $0 \leqslant x^{25} \leqslant 2\pi R$. Thus the closed string is modified to $x^{25}(\sigma + 2\pi, \tau) = x^{25}(\sigma, \tau) + 2\pi RL\,(*)$
  The term $2\pi RL$ gives rise to strings which are closed on the circle S^1.
- After quantization this leads to new states called winding states which are characterized by the winding number L.
- T-duality: spectrum is invariant under $R \to \frac{\alpha'}{R}$.
  Simultaneously, the winding and momentum numbers interchange $L \Leftrightarrow M$, maps winding states to momentum states, vice versa.

Mode Expansion

- The mode expansion for $x^{25}(\sigma,\tau)$ respect to (*) reads $x^{25}(\sigma,\tau) = x^{25} + \alpha' p^{25}\tau + LR\sigma + osc.$
- x^{25} and p^{25} obey the usual commutation relation. $[x^{25}, p^{25}] = i$, p^{25} generates translation of x^{25}. Single valued of the wave function $e^{ip^{25}x^{25}}$ restricts the allowed internal momenta to discrete values: $p^{25} = \frac{M}{R}$, $M \in \mathbb{Z}$ The quantized momentum states are called Kaluza-Klein modes.
- Split x^{25} into left and right movers
  $x_R^{25}(\tau-\sigma) = \frac{1}{2}(x^{25} - c) + \frac{\alpha'}{2}(\frac{M}{R} - \frac{LR}{\alpha'})(\tau-\sigma) + osc.$
  $x_L^{25}(\tau+\sigma) = \frac{1}{2}(x^{25} + c) + \frac{\alpha'}{2}(\frac{M}{R} + \frac{LR}{\alpha'})(\tau+\sigma) + osc.$
- The mass operator receives contributions from winding states
  $\alpha' m_L^2 = \frac{\alpha'}{2}(\frac{M}{R} + \frac{LR}{\alpha'})^2 + 2(N_L - 1)$
  $\alpha' m_R^2 = \frac{\alpha'}{2}(\frac{M}{R} - \frac{LR}{\alpha'})^2 + 2(N_R - 1)$
  $\alpha' m^2 = \alpha'(m_L^2 + m_R^2) = \alpha' \frac{M^2}{R^2} - \frac{1}{\alpha'}L^2R^2 + 2(N_L + N_R - 2)$

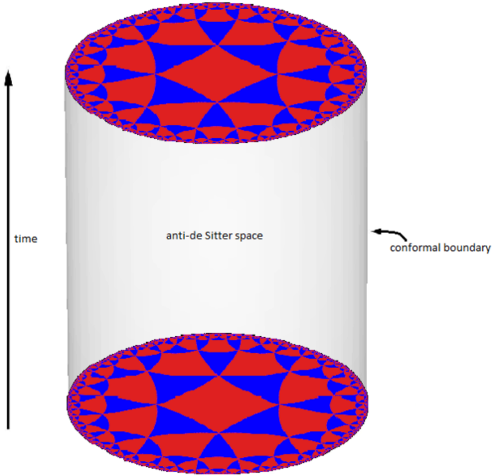
Dualities in String Theory

- Toroidal Compactification + asymmetric reflection for fermion: Type IIB $\Leftrightarrow$ Type IIA
- Heterotic SO(32) $\Leftrightarrow$ $E_8 \times E_8$:
  Break the gauge symmetry to $SO(16) \times SO(16)$. One can show that the two theories have identical spectra and symmetries under $R_1 R_2 = \alpha'/2$.
- Mirror Symmetry: Calabi-Yau compatification to preserve supersymmetry. $10 - CY_3\,(6) = 4$ dim. Spectrum invariant with Complex structure $U \Leftrightarrow$ Kähler modulus T
- S-duality: Strong and weak coupling. Type IIB self S-dual etc.

Gauge/gravity duality

- AdS/CFT: anti-de Sitter/conformal field theory correspondence, sometimes called gauge/gravity duality, holographic duality is a conjectured relationship between two kinds of physical theories.
- On one side are String Theory on anti-de Sitter spaces (AdS) which are used in theories of quantum gravity.
- On the other side of the correspondence are conformal field theories which are quantum field theories, including theories similar to the Yang–Mills theories that describe elementary particles.
- e.g. Type IIB superstring on $AdS_5 \times S^5$
  $\Leftrightarrow$ N=4 supersymmetric Yang-Mills Theory in 4-dim.

Gauge/gravity duality



AdS_3 space(stack of hyperbolic disks) dual to CFT on cylinder boundary.

Effective action of Heterotic String

- The low-energy effective action for heterotic string massless bosonic sector is described by

$$S = \int dx \sqrt{g}\, e^{-2\phi} \left(R + 4(\partial\phi)^2 - \frac{1}{12} H^{ijk} H_{ijk} - \frac{1}{4} G^{ij}{}_{\alpha} G_{ij}{}^{\alpha} \right)$$

- The field strength of the non-abelian gauge fields

$$G_{ij}{}^{\alpha} = \partial_i A_j{}^{\alpha} - \partial_j A_i{}^{\alpha} + g_0 \, [A_i, A_j]^{\alpha}$$

- The strength of the Kalb-Ramond field is modified by the Chern-Simons three-form,

$$H_{ijk} = 3\Big(\partial_{[i} B_{jk]} - \kappa_{\alpha\beta} A_{[i}{}^{\alpha} \partial_j A_{k]}{}^{\beta} - \frac{1}{3} g_0 \, \kappa_{\alpha\beta} \, A_{[i}{}^{\alpha} [A_j, A_{k]}]^{\beta} \Big)$$

Heterotic Double Field Theory

- As a generalization of T-duality, the global symmetry group of heterotic Double Field Theory is enhanced to $O(D, D+n)$
- Heterotic DFT lives on $2D+n$ dimensional space, coordinates $X^M = (\tilde{x}_i, x^i, y^\alpha)$, $O(D, D+n)$ vector X^M: $X'^M = h^M{}_N X^N$, $h \in O(D, D+n)$.
- The heterotic DFT action is expressed in terms of generalized metric H_{MN} and an $O(D, D+n)$ invariant dilation d, defined by $e^{-2d} = \sqrt{g} e^{-2\phi}$.
- The abelian bosonic subsector of heterotic action under the so-called strong constraint $\tilde{\partial}^i = \partial_\alpha = 0$, by $S = \int dx\, e^{-2d} \Big(\frac{1}{8} H^{ij} \partial_i H^{KL} \partial_j H_{KL} - \frac{1}{2} H^{Mi} \partial_i H_{Kj} \partial_j H_{MK} - 2\partial_i d \partial_j H^{ij} + 4 H^{ij} \partial_i d \partial_j d \Big)$

Lie algebroid for heterotic SUGRA

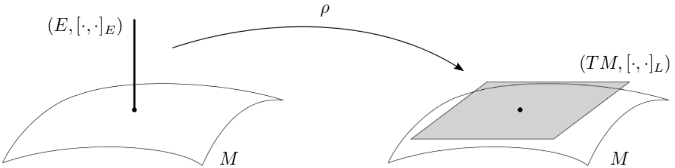


Figure 1: Illustration of a Lie algebroid. On the left, one can see a manifold M together with a bundle E and a bracket $[\cdot,\cdot]_E$. This structure is mapped via the anchor ρ to the tangent bundle TM with Lie bracket $[\cdot,\cdot]_L$, which is shown on the right.

- A Lie algebroid is specified by three pieces of information:
  a vector bundle E over a manifold M,
  a bracket $[\cdot,\cdot]_E : E \times E \to E$,
  a homomorphism $\rho : E \to TM$ called the anchor.

- We considers a D-dimensional manifold M with usual coordinates x^i, equipped with a generalized bundle $E = TM \oplus T^*M \oplus V$.
- On this bundle E one defines a generalized metric $\mathcal{H}_{MN}$ in terms of the fundamental fields g_{ij}, B_{ij} and $A_i{}^\alpha$ etc.
- An $O(D, D+n)$ transformation $\mathcal{M}$ acts on the generalized metric via conjugation, i.e. $\hat{H}(\hat{g}, \hat{B}, \hat{A}) = \mathcal{M}^t \, H(g, B, A) \, \mathcal{M}$, $\mathcal{M}^t \, \mathcal{M} = 1$, and therefore defines a field redefinition $(g, B, A) \longrightarrow (\hat{g}, \hat{B}, \hat{A})$.

$$
\begin{aligned}
\hat{g} &= \rho^t \, g \, \rho = \tilde{g} \\
\hat{C} &= \rho^t \, \mathfrak{C} \, \rho = C^t \, g^{-1} \, \tilde{g} \\
\hat{A} &= \rho^t \, \mathfrak{A} = -(1 + C^t \, g^{-1}) A
\end{aligned}
\qquad
\begin{aligned}
\rho^* &= (\rho^t)^{-1} = -(g + C) \, \tilde{g}^{-1} \\
\delta &= -\tilde{g} \\
\mathfrak{A} &= A
\end{aligned}
$$

The redefined heterotic action

- Gravitational quantities transform as
  $\hat{R}^q{}_{mnp} = (\rho^{-1})^q{}_l\, \rho^i{}_m\, \rho^j{}_n\, \rho^k{}_p\, R^l{}_{ijk}\,, \quad \hat{R}_{mn} = \rho^i{}_m\, \rho^j{}_n\, R_{ij}\,,$
  $\hat{R} = R\,, \quad \sqrt{|\hat{g}|} = \sqrt{|g|}|\rho^t|\,, \quad \hat{\phi} = \phi, \qquad D_i = (\rho^t)_i{}^j\, \partial_j$
- For the gauge field strength $G = dA$, we have
  $(\Lambda^2\rho^*)d_E\hat{A} = d(\rho^*\hat{A}) = dA$

Field Strength

$$\hat{G} := d_E\hat{A} = (\Lambda^2\rho^t)\, G$$

Three-form Flux

$$\hat{H} := d_E\hat{B} - \tfrac{1}{2}\hat{A}\wedge d_E\hat{A} = (\Lambda^3\rho^t)\, H$$

- so that the action in the redefined fields can be expressed as
  $\mathcal{S} = \int dx\sqrt{\hat{g}}\ |\rho^*|\ e^{-2\phi}\Big(\hat{R} + 4(D\phi)^2 - \tfrac{1}{12}\hat{H}^{ijk}\hat{H}_{ijk} - \tfrac{1}{4}\hat{G}^{ij\alpha}\hat{G}_{ij\alpha}\Big)$
- Upshot: the first order α' correction of Buscher rules is naturally included in the form of the gauge field terms.

A The Buscher rules derived from heterotic DFT

Using the implementation of T-duality in heterotic DFT, one can now quite generally (re-)derive the Buscher from the conjugation of the generalized metric with the corresponding T-duality matrix. Carrying out this procedure for a T-duality in the x^θ direction, we get precisely the α' corrected Buscher rules presented in [42]

$$\begin{aligned}
G'_{\theta\theta} &= \frac{G_{\theta\theta}}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)^2} \\
G'_{\theta i} &= -\frac{G_{\theta\theta}B_{\theta i} + \frac{\alpha'}{2}G_{\theta i}A_\theta^2 - \frac{\alpha'}{2}G_{\theta\theta}\,A_\theta A_i}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)^2} \\
G'_{ij} &= G_{ij} - \frac{G_{\theta i}G_{\theta j} - B_{\theta i}B_{\theta j}}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)} \\
&\quad - \frac{1}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)^2}\Bigg(G_{\theta\theta}\left[\frac{\alpha'}{2}B_{\theta j}A_\theta A_i + \frac{\alpha'}{2}B_{\theta i}A_\theta A_j - \frac{\alpha'^2}{4}A_\theta A_i\,A_\theta A_j\right] \\
&\qquad + \frac{\alpha'}{2}A_\theta^2\left[(G_{\theta i} - B_{\theta i})(G_{\theta j} - B_{\theta j}) + \frac{\alpha'}{2}(G_{\theta i}A_\theta A_j + G_{\theta j}A_\theta A_i)\right]\Bigg) \qquad \text{(A.1)} \\
B'_{\theta i} &= -\frac{G_{\theta i} + \frac{\alpha'}{2}A_\theta A_i}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)} \\
B'_{ij} &= B_{ij} - \frac{(G_{\theta i} + \frac{\alpha'}{2}A_\theta A_i)B_{\theta j} - (G_{\theta j} + \frac{\alpha'}{2}A_\theta A_j)B_{\theta i}}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)} \\
{A'_\theta}^\alpha &= -\frac{{A_\theta}^\alpha}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)} \\
{A'_i}^\alpha &= {A_i}^\alpha - {A_\theta}^\alpha\frac{G_{\theta i} - B_{\theta i} + \frac{\alpha'}{2}A_\theta A_i}{\left(G_{\theta\theta} + \frac{\alpha'}{2}A_\theta^2\right)}
\end{aligned}$$

where e.g. $A_\theta A_i = A_\theta^\alpha A_{i\alpha}$. Here the metric and the Kalb-Ramond field have dimension $[l]^0$ and the gauge field $[l]^{-1}$.

Conclusion

Dualities for String Theory and SUGRA

- T-duality, Mirror Symmetry
- S-duality: Strong and weak coupling
- Gauge/gravity duality(AdS/CFT)
- Lie algebroid mapping, dual gravity

Thank you very much!