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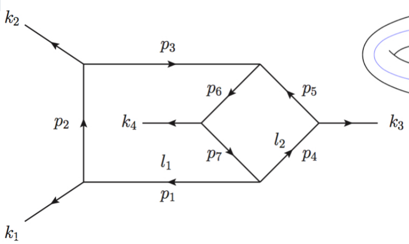

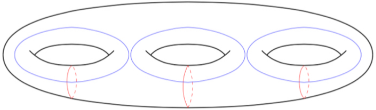

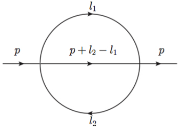

Two Loop Integral Reduction From Elliptic and Hyperelliptic Curves.

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Motivation

- **LHC Run2**

We need precision calculation of NLO and NNLO processes. Higher precision requires a better understanding of background processes. Reveal BSM physics.

- **Interesting hidden structure of QFT**

$\mathcal{N} = 4$ have given us a lot of insight on hidden symmetries and simplifications of scattering amplitudes, e.g. MHV amplitudes and integrability.

- **Beautiful Mathematics**

A lot of connection with interesting topics in mathematics, e.g. Algebraic geometry and Number theory.

Integral Basis

Every L-loop amplitude can be written in terms of **Master Integrals** (MIs) $\{I_k\}$:

$$A_n^{L\text{-loop}} = \sum_k c_k I_k + \text{rational terms}\,,$$

We have changed our problem :

BEFORE
Solve for every diagram one very complicated Integral.

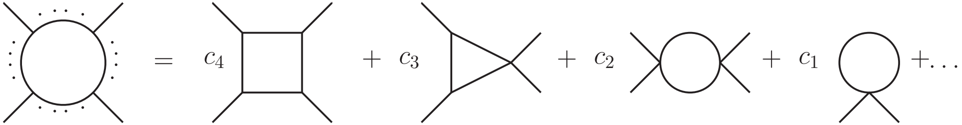
AFTER
Find the coefficent c_k and the I_k.

- Is more general, same I_k for different diagrams.
- Highly automatizable.

Unitarity

- Cutkosky Rules $\frac{1}{l^2 - m^2 + i\epsilon} \rightarrow 2\pi i \delta(l^2 - m^2)$

1-loop Amplitude (D-dimension):



- At 1-loop the basis is unique.
- We can extract the coefficents c_k by applying different cuts.

IBP

Another method for reducing the Integrals (in D-dimensio) is by **integral-by-parts** (IBP) identities.

$$\int \frac{d^D l_1}{(2\pi)^D} \cdots \frac{d^D l_L}{(2\pi)^D} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = 0.$$

- Mostly based on Laporta algorithm (FIRE,AIR,Reduze,. . .).
- New methods based on computational algebraic geometry.
- Two loop takes very long time (several scales).

Maximal Unitarity and IBP

We can now define the on-shell IBP relations,

$$\oint \frac{d^D l_1}{(2\pi)^D} \dots \frac{d^D l_L}{(2\pi)^D} N(l_1, \dots l_L)\delta(D_1)\dots\delta(D_k) = \sum_j w_j \oint_{\mathscr{C}_j} \omega$$

We can study the maximal cut as an ***algebraic variety***,

$$V : D_1 = \dots = D_k = 0.$$

If V is a curve, the number of propagators equal DL-1, the IBP relations futher simplify.

$$\oint \omega = 0, \quad \omega = dF.$$

Riemannian Surfaces

We are interested in a particular set of algebraic curves, $y^2 = h(x)$. If we see them as functions from $\mathbb{C} \to \mathbb{C}$,

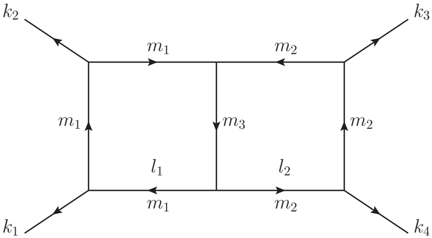
Theorem (Riemann-Hurwitz)

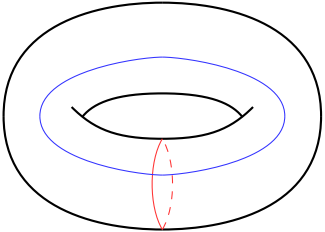
We can associate a Riemann surface of a specific genus to every curve based on h(x)

Remark

We need to find exact 1-forms on the associated Riemann surface. The topological properties are determined by the maximal cut.

Massive Double Box





$$D_1 = l_1^2 - m_1^2\,, \qquad D_2 = (l_1 - k_1)^2 - m_1^2\,, \qquad D_3 = (l_1 - k_1 - k_2)^2 - m_1^2\,,$$
$$D_4 = l_2^2 - m_2^2\,, \qquad D_5 = (l_2 - k_4)^2 - m_2^2\,, \qquad D_6 = (l_2 - k_3 - k_4)^2 - m_2^2\,,$$
$$D_7 = (l_1 + l_2)^2 - m_3^2\,.$$

Loop momenta parametrization:

$$l_1^\mu = \alpha_1 k_1^\mu + \alpha_2 k_2^\mu + \alpha_3 \frac{s}{2} \frac{\langle 1|\,\gamma^\mu |2]}{\langle 14\rangle\,[42]} + \alpha_4 \frac{s}{2} \frac{\langle 2|\,\gamma^\mu |1]}{\langle 24\rangle\,[41]},$$
$$l_2^\mu = \beta_1 k_3^\mu + \beta_2 k_4^\mu + \beta_3 \frac{s}{2} \frac{\langle 3|\,\gamma^\mu |4]}{\langle 31\rangle\,[14]} + \beta_4 \frac{s}{2} \frac{\langle 4|\,\gamma^\mu |3]}{\langle 41\rangle\,[13]},$$

On-Shell solutions:

$$\alpha_1 = 1\,, \qquad \alpha_2 = 0\,, \qquad \alpha_3 = \frac{m_1^2 t(s+t)}{\alpha_4 s^3},$$
$$\beta_1 = 0\,, \qquad \beta_2 = 1\,, \qquad \beta_3 = \frac{m_2^2 t(s+t)}{\beta_4 s^3}\,,$$

Then the remaining one equation relates α_4 and β_4,

$$K(\alpha_4, \beta_4) = A(\alpha_4)\beta_4^2 + B(\alpha_4)\beta_4 + C(\alpha_4) = 0,$$

The scalar maximal cut is:

$$I|_{7-cut} = \frac{s^2 t}{16} \oint \frac{d\alpha_4}{\sqrt{\Delta}}$$

The Numerator insertions are then:

$$\oint N(\alpha_3, \alpha_4, \beta_3, \beta_4) \frac{d\alpha_4}{\sqrt{\Delta}}$$

Our IBP should have the form:

$$dF(\alpha_3, \alpha_4, \beta_3, \beta_4) = \frac{\partial F}{\partial \alpha_3} d\alpha_3 + \frac{\partial F}{\partial \alpha_4} d\alpha_4 + \frac{\partial F}{\partial \beta_3} d\beta_3 + \frac{\partial F}{\partial \beta_4} d\beta_4 \equiv f \frac{d\alpha_4}{\sqrt{\Delta}}$$

We only need to evaluate the seeds: $\{d\alpha_3, d\alpha_4, d\beta_3, d\beta_4\}$ and then use the chain rule to generate integral reduction relations.

The seeds are:

$$
\begin{aligned}
d\alpha_4 &= \eta \frac{d\alpha_4}{\eta} = (2A(\alpha_4)\beta_4 + B(\alpha_4)) \frac{d\alpha_4}{\eta}\,, \\
d\alpha_3 &= d\left(\frac{\lambda_1}{\alpha_4}\right) = -\lambda_1 \frac{1}{\alpha_4^2} d\alpha_4 = -\frac{\alpha_3^2}{\lambda_1} d\alpha_4 \quad \lambda_1 \equiv \frac{m_1^2 t(s+t)}{s^3}\,, \\
d\beta_4 &= -\left(A'(\alpha_4)\beta_4^2 + B'(\alpha_4)\beta_4 + C'(\alpha_4)\right) \frac{d\alpha_4}{\eta}\,, \\
d\beta_3 &= d\left(\frac{\lambda_2}{\beta_4}\right) = -\lambda_2 \frac{1}{\beta_4^2} d\beta_4 = -\frac{\beta_3^2}{\lambda_2} d\beta_4 \quad \lambda_2 \equiv \frac{m_2^2 t(s+t)}{s^3}\,.
\end{aligned}
$$

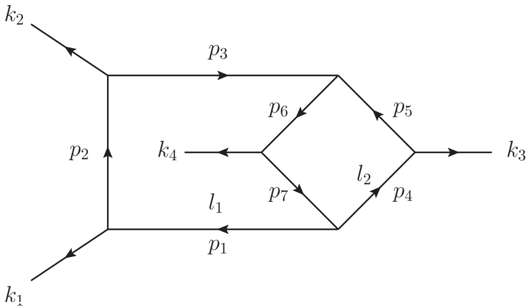
We can now find the MIs:

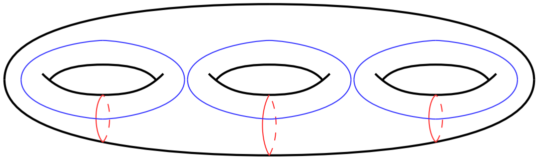
$$
\mathrm{MI}_{\mathrm{dbox}} = \left\{ I_{\mathrm{dbox}}[\alpha_4\beta_3], I_{\mathrm{dbox}}[\alpha_4^2], I_{\mathrm{dbox}}[\alpha_4], I_{\mathrm{dbox}}[\beta_3], I_{\mathrm{dbox}}[1] \right\}\,.
$$

An example of Integral reduction:

$$
\begin{aligned}
I_{\text{dbox}}[\alpha_4^3] \;=\;& \frac{1}{2s^4(4m_2^2-s)}\Bigg(3s^3\left(m_1^2 s - m_2^2 s - m_3^2 s - 4m_2^2 t + st\right) I_{\text{dbox}}[\alpha_4^2] \\
&+ s(4m_1^2 s^2 t - 2m_2^2 s^2 t - 2m_3^2 s^2 t + m_1^4 s^2 - 2m_2^2 m_1^2 s^2 - 2m_3^2 m_1^2 s^2 + m_2^4 s^2 + m_3^4 s^2 \\
&-2m_2^2 m_3^2 s^2 + 2m_1^2 s t^2 - 4m_2^2 s t^2 - 8m_2^2 m_1^2 st - 8m_2^2 m_1^2 t^2 + s^2 t^2) I_{\text{dbox}}[\alpha_4] \\
&+ m_1^2 t(s+t)\left(m_1^2 s - m_2^2 s - m_3^2 s - 4m_2^2 t + st\right) I_{\text{dbox}}[1]\Bigg) + \ldots
\end{aligned}
$$

Massive Cross-Box





$$I|_{7-cut} = \frac{s^3(s+t)}{16} \oint \frac{\alpha_4 d\alpha_4}{\sqrt{\Delta(\alpha_4)}} .$$

- 7 MIs
- Perform the **analityc** reduction.

Computation Outlook

Perform the maximal cut and determine the algebraic structure.

Generate the possible numerators insertion and reduce them by a Gröbner basis division.

Generate the complete set of IBP, exact one-forms, to reduce the integral to MIs with the correct coefficients.

Conclusion

Summary:

- No need to know the complete structure of the elliptic and hyperelliptic curves.
- Different from usual Unitarity methods.
- Very fast and easily generalisable to same genus curves.
- It can be solved for several internal masses.

Future Work:

- This method could be generalised to higher dimensional algebraic varieties.
- I am currently working with YZ on scattering equations to 1-Loop.