R-evolution

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Objective

- Convert between mass schemes without large logs — use a new RGE
- a threshold "MSR mass" that smoothly matches on to $\overline{\text{MS}}$ mass without threshold corrections
- a new way to analyze renormalons in OPE

Renormalon Formals

Borel Summation

$$f(\alpha) = f(0) + \sum_{n=0}^{\infty} f_n \alpha^{n+1}$$

asymptotic series

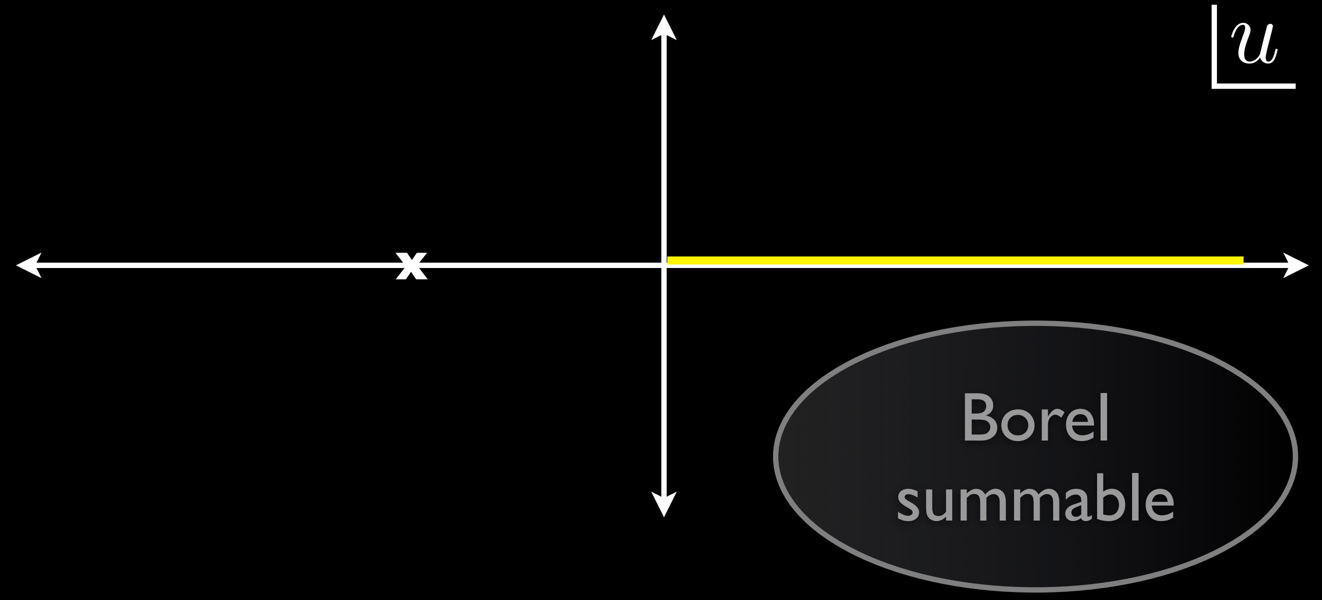
$$B[f](u) = f(0)\delta(u) + \sum_{n=0}^{\infty} \frac{f_n u^n}{n!}$$

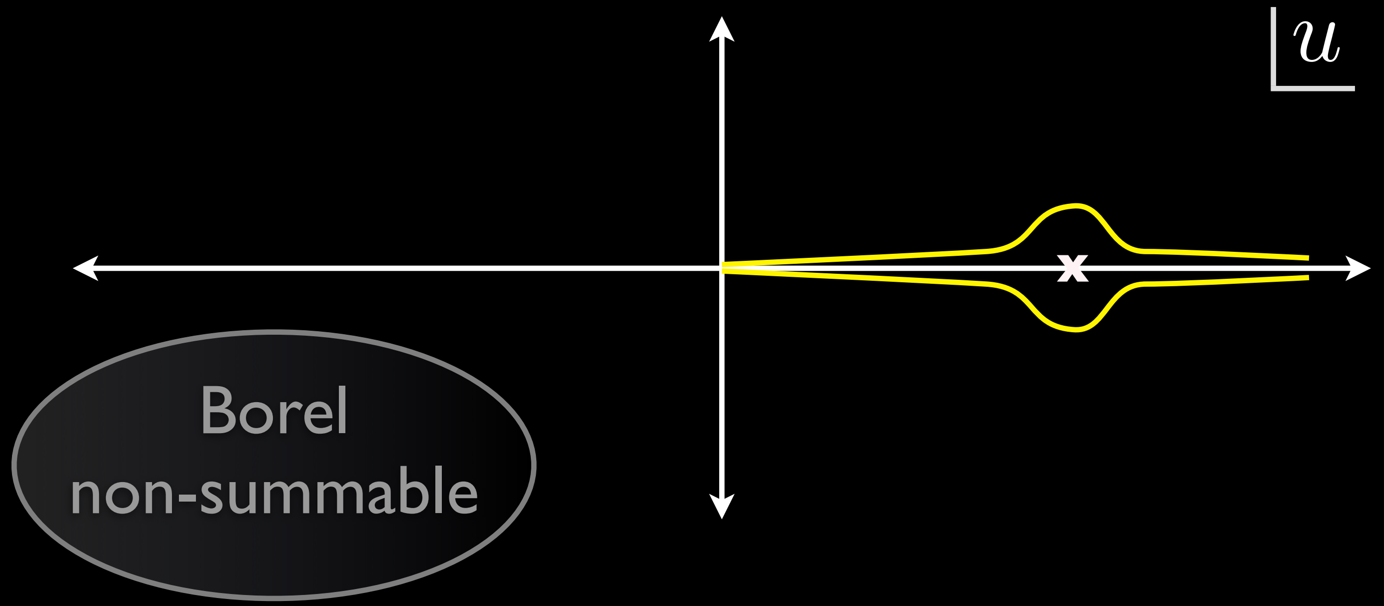
Borel transform

$$f(\alpha) = \int_0^{\infty} du\, e^{-u/\alpha} B[f](u)$$

Borel inverse

- factorial growth implies pole in complex Borel plane
- inverse Borel is ill-defined due to pole on real axis
- gives rise to ambiguity in Borel summed f(α)





pole mass with bubble chain

$$iS(p,m) = \frac{i}{\not p \ - m - \Sigma(p,m)}$$

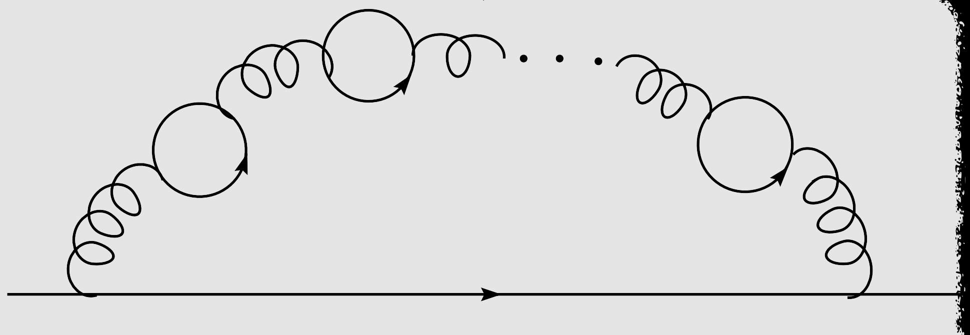
full quark propagator

‘t Hooft (1976)
Bigi et. al. (1994)
Beneke & Braun (1994)

$$\not p \ - m - \Sigma(p,m)|_{p^2=m^2_{\rm pole}} = 0$$

pole mass

$$\Sigma(\text{p,m}) = \sum_{\text{n}}$$



$$\left(\frac{\beta_0\alpha_s}{4\pi}\right)^{n+1} \longrightarrow \frac{u^n}{n!}$$

Borel transform

$$B[m_{\rm pole}-\bar{m}](u) \underset{u\sim\frac{1}{2}}{\sim} \boxed{\frac{4\mu}{3\pi\beta_0}\left(\frac{e^{5/6}}{u-1/2}\right)} + \dots$$

$$\boxed{\text{Amb}[m_{\rm pole}] \ = \ i\,\Lambda_{\rm QCD}\frac{8e^{5/6}}{3\beta_0}}$$

IR Scale & Short Distance Masses

pole mass relation to other schemes

$$\begin{aligned} m(R) &= m_{\text{pole}} - \delta m(R, \mu = R) \\ \delta m(R) &= R\left[a_1\left(\frac{\alpha_s(R)}{4\pi}\right) + a_2\left(\frac{\alpha_s(R)}{4\pi}\right)^2 + \ldots\right] \end{aligned}$$

Introduces an IR scale "R"

converting between schemes

$$\begin{aligned} m^A(R_0) - m^B(R_1) &= R_1\left[a_1'\left(\frac{\alpha_s(R_1)}{4\pi}\right) + a_2'\left(\frac{\alpha_s(R_1)}{4\pi}\right)^2 + \ldots\right] \\ &- R_0\left[a_1\left(\frac{\alpha_s(R_1)}{4\pi}\right) + (a_2 + \beta_0 \log(R_1/R_0))\left(\frac{\alpha_s(R_1)}{4\pi}\right)^2 + \ldots\right] \end{aligned}$$

must have α_s at the same scale to cancel the renormalon !!

Introduces log(R_1/R_0) which directly effects accuracy of conversion

IR Scale & Matrix Elements

Observable

$$\mathcal{O} = C_1(M,\mu)Q_1(\mu) + C_2(M,\mu)\frac{Q_2(\mu)}{M} + \ldots$$

OPE

$$\mathrm{Amb}[C_1] \sim \frac{\Lambda_{\mathrm{QCD}}}{M}$$

&

$$\left[Q_1\right] = d$$

$\Rightarrow$

$$\mathrm{Amb}[Q_2] \sim \Lambda^{d+1}$$

Luke, Manohar, Savage (1994)

Define:

$$C_1^{(R)}(M,\mu) = C_1(M,\mu) - \frac{R}{M}\delta C_1(M,\mu,R)$$

$$\mathcal{O} = C_1^{(R)}(M,\mu)Q_1(\mu) + C_2(M,\mu)\frac{Q_2^{(R)}(\mu)}{M} + \ldots$$

$$\delta Q_2(\mu,R) = -R\frac{\delta C_1(M,\mu,R)}{C_2(M,\mu)}Q_1(\mu)$$

$R \gtrsim \Lambda_{\mathrm{QCD}}$

How to define δC_1?
Will see later!

Review by ElKhadra, Luke (2002)

Scheme Mania

$$m_{\text{pole}} = m(R) + R \sum_{n=0}^{\infty} a_n \, \alpha_s^n(R)$$

scheme	R	R_{bottom}	R_{top}	process	Reference
$\overline{\text{MS}}$	$\overline{\text{m}}$	4.2	163		text book
1S	$m_{1S} C_F \alpha_s$	1.5	25	ϒ decay, $e^+ e^- \to Q\overline{Q}$(threshold)	Hoang, Ligeti, Manohar (1999)
PS	μ_f	2	20	$e^+ e^- \to Q\overline{Q}$(threshold)	Beneke (1998)
kinetic	$\mu_{f(\text{kin})}$	1	20	$b \to c$ decays	Bigi ,Shifman, Uraltsev (1997)
SF	$\mu_{f(\text{SF})}$	1	–	$b \to X_s \gamma$	Bosch, Lange, Neubert, Paz (2004)
RGI	m_{RGI}	5	170	Lattice QCD	Floratos et. al. (1979)
jet	R_{jet}	–	2	$e^+ e^- \to t\, t \to j\, j$	AJ, Scimemi, Stewart (2008)
MSR	R	!!	!!		Hoang, AJ, Scimemi, Stewart (2008)

Review by ElKhadra, Luke
(2002)

Scheme Mania

$$m_{\rm pole} = m(R) + R \sum_{n=0}^{\infty} a_n\, \alpha_s^n(R)$$

scheme	R	$R_{\rm bottom}$	$R_{\rm top}$	process	[?]
$\overline{\rm MS}$	$\bar{m}$	4.2	163		text book
1S	$m_{1S}\, C_F\, \alpha_s$	1.5	25	Υ decay, $e^+ e^- \to Q\bar{Q}$(threshold)	Hoang, Ligeti, Manohar (1999)
PS	μ_f	2	20	$e^+ e^- \to Q\bar{Q}$(threshold)	Beneke (1998)
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MSR	R	1 – 4.2	1 – 163		Hoang, AJ, Scimemi, Stewart (2008)

Schemes Mania contd. : λ_1

$$\boxed{\lambda_1 = \langle B|\bar{b}_v(iD_\perp)^2 b_v|B\rangle}$$

kinetic energy operator

(1) $$\lambda_1^{\rm kin}(R) = \lim_{\vec{v}\to 0}\lim_{m_Q\to 0}\frac{3}{\vec{v}^2}\frac{\int_0^R \omega^2 w(\omega,\vec{v})d\omega}{\int_0^R w(\omega,\vec{v})d\omega}$$

kinetic scheme

Czarnecki, Melnikov, Uraltsev
(1998)

(2) $$\lambda_1^{\rm SF}(\mu_f,\mu) = \frac{3\int_{-\mu_f}^{\infty} d\omega\,\omega^2 S(\omega,\mu)}{\int_{-\mu_f}^{\infty} d\omega\,S(\omega,\mu)}$$

$$-\mu_\pi^2(\mu_f,\mu_f) = \lambda_1^{\rm SF}(\mu_f,\mu_f) = \lambda_1 - \delta\lambda_1^{\rm SF}(\mu_f)$$

shape function scheme

Bosch, Lange,
Neubert, Paz (2004)

(3) $$\lambda_1^{\rm i} = \lambda_1 - \delta\lambda_1^{\rm i}$$

$$\delta\lambda_1^{\rm i} = \langle b_v|\bar{b}_v(iD_\perp)^2 b_v|b_v\rangle\Big|_R \sim R^2(\alpha_s^2+\ldots)$$

invisible scheme

Ligeti, Stewart, Tackmann (2008)

MSR mass

$$m_{\mathrm{pole}} \;-\; \overline{m}(\overline{m}) \;=\; \delta\bar{m}(\mu = \bar{m})$$

$$\mathrm{Amb}\big[\delta\overline{m}(\overline{m})\big] \sim \Lambda_{\mathrm{QCD}}$$

$$\delta\bar{m}(\bar{m}) \;=\; \overline{m}\,\big[\,\bar{a}_1\,\alpha_s(\overline{m}) + \bar{a}_2\,\alpha_s^2(\overline{m}) + \bar{a}_3\,\alpha_s^3(\overline{m}) + \ldots\big]$$

$$\sim \;\; \overline{m}\,\left[\frac{\Lambda_{\mathrm{QCD}}}{\overline{m}} \;+\; \ldots\right]$$

$$\overline{m} \longrightarrow R$$

$$R\,\big[\,\bar{a}_1\,\alpha_s(R) + \bar{a}_2\,\alpha_s^2(R) + \bar{a}_3\,\alpha_s^3(R) + \ldots\big] \equiv \delta m^{\mathrm{MSR}}(R)$$

$$m_{\mathrm{pole}} \;-\; \delta m^{\mathrm{MSR}}(R) \;=\; m^{\mathrm{MSR}}(R)$$

Definition of MSR mass

MSR and $\overline{\text{MS}}$ mass

$$\boxed{m_{\text{pole}} \; - \; m^{\text{MSR}}(R) \; = \; \delta m^{\text{MSR}}(R)}$$

$$\delta m^{\text{MSR}}(R) \equiv R \left[\bar{a}_1 \, \alpha_s(R) + \bar{a}_2 \, \alpha_s^2(R) + \bar{a}_3 \, \alpha_s^3(R) + \ldots \right]$$

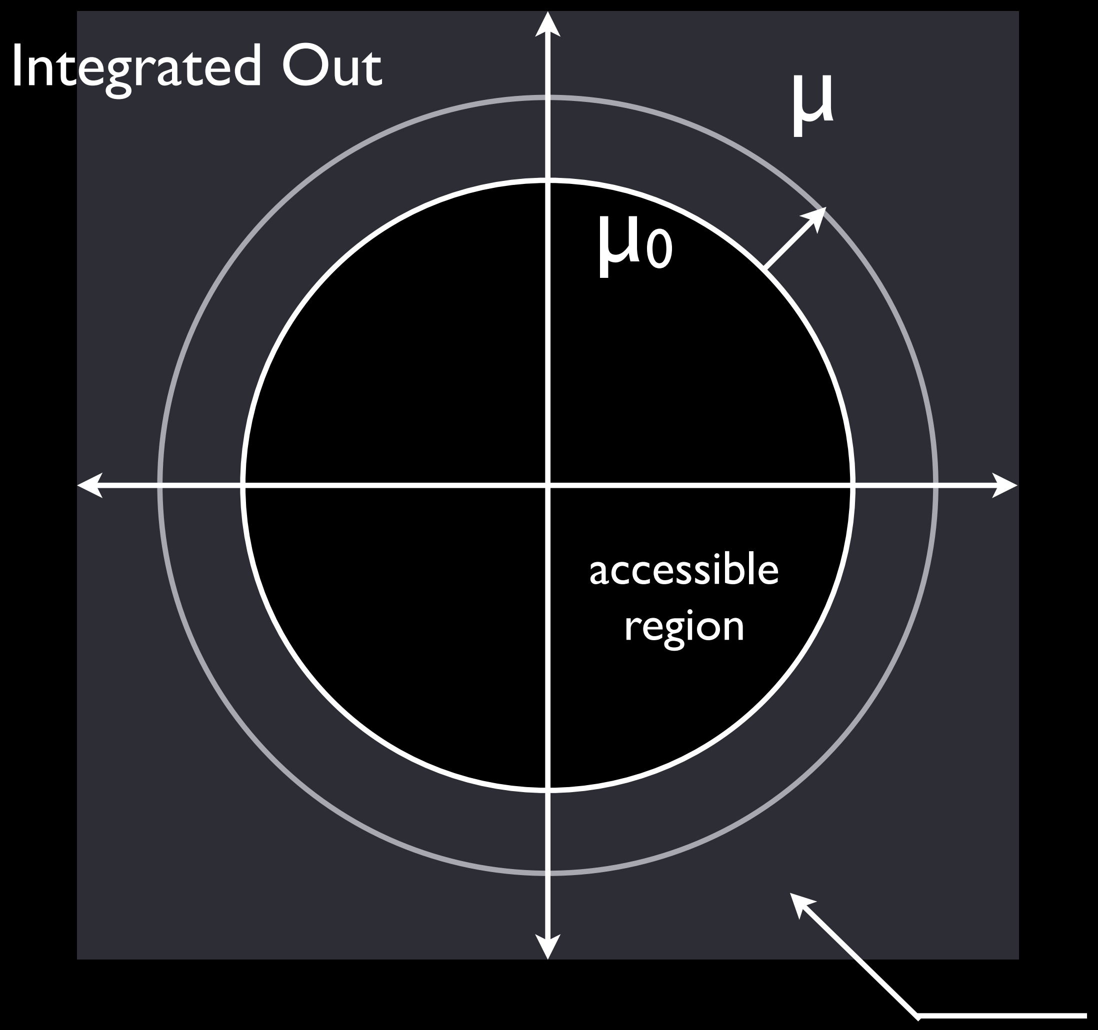
- $m^{\text{MSR}}(R = \overline{m}) \; = \; \overline{m}(\overline{m})$

Important

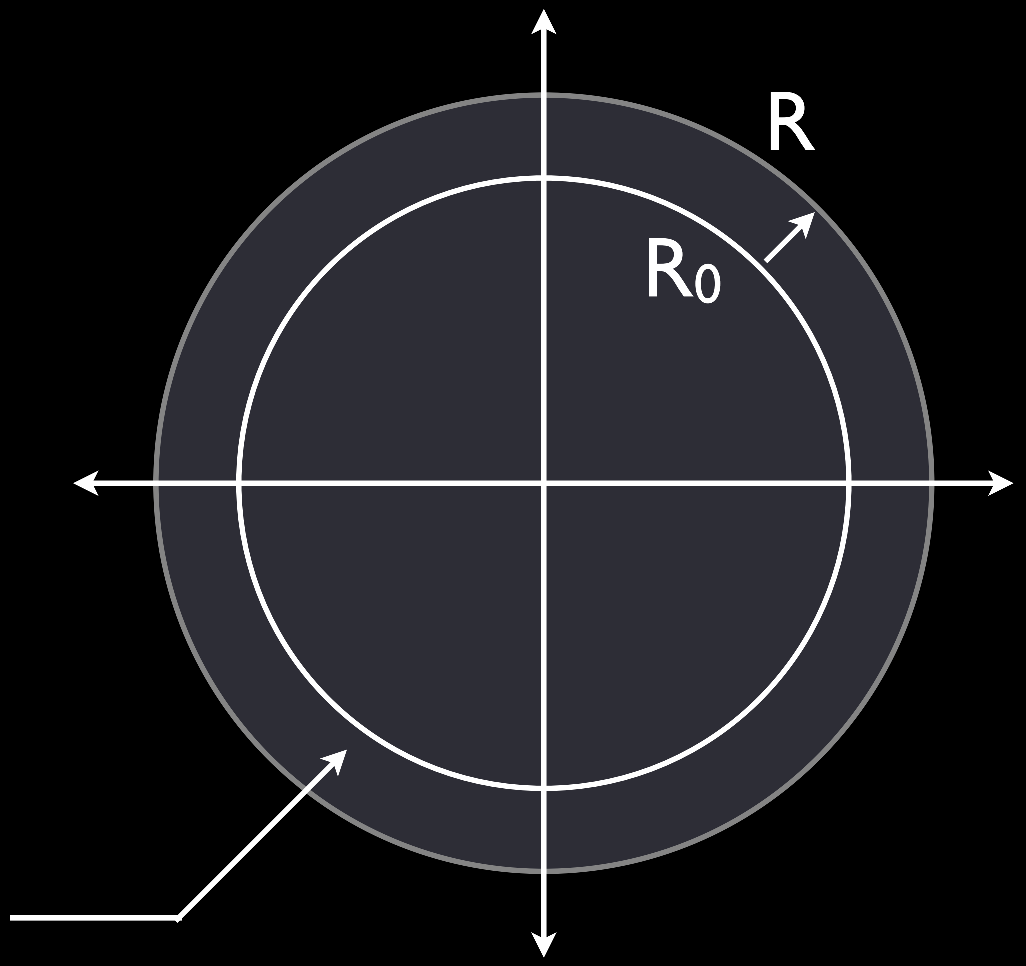
- threshold scheme for R << m
- known to three loops in perturbation theory
- smoothly connects to $\overline{\text{MS}}$ mass if we can smoothly vary R

R-Scale and R-RGE

Renormalization Group Flow



absorbed in parameters
$\bar{m}(\mu)$



absorbed in parameters
m(R)

increase μ : less UV in mass

and more in matrix elements

increase R : less IR in ME

and more in mass

R scale in PS mass

$$m^{\mathrm{PS}}(R) \;=\; m_{\mathrm{pole}} \;-\; \delta m^{\mathrm{PS}}(R)$$

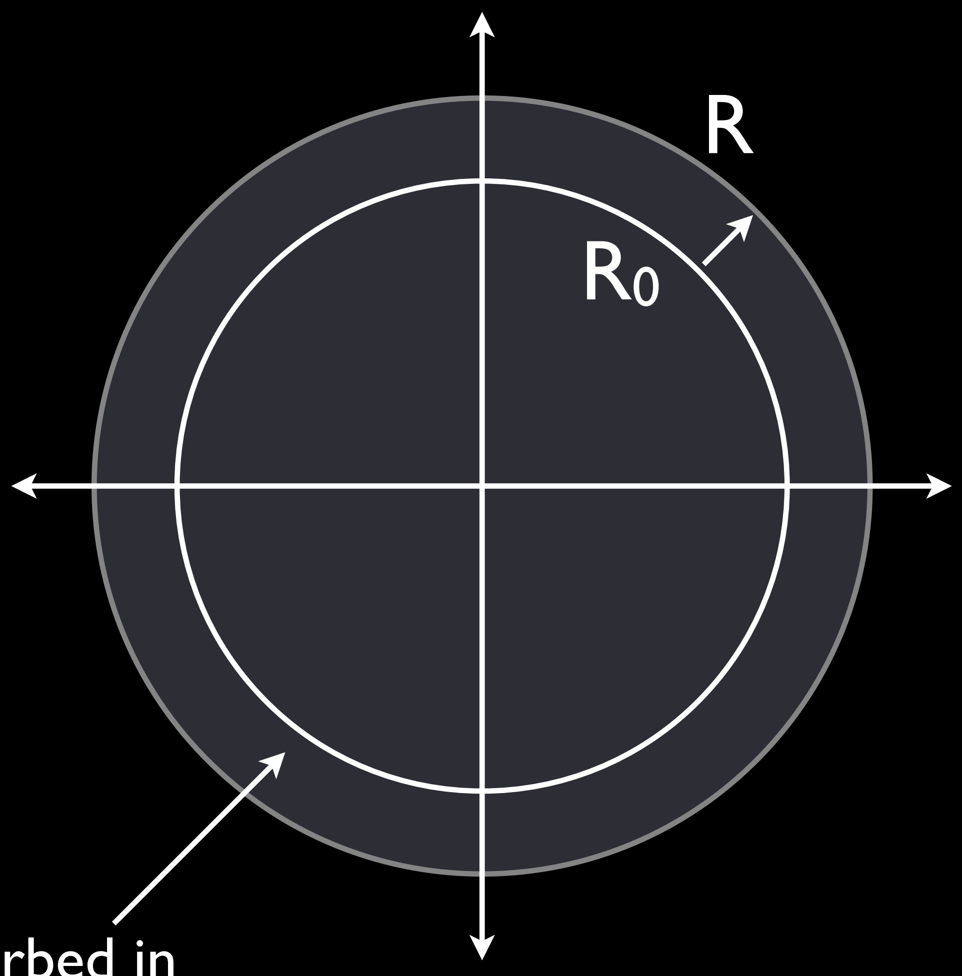
$$\delta m^{\mathrm{PS}}(R) \equiv -\frac{1}{2}\int_{|q|<R} \frac{d^3q}{(2\pi)^3}\, V(q)$$

removing the potential energy contained in the sphere of radius R

$$-\delta m^{\mathrm{PS}}(R_1) \;=\; \int_0^{R_0} dq\, \frac{q^2\, V(q)}{4\pi^2} \;+\; \int_{R_0}^{R_1} dq\, \frac{q^2\, V(q)}{4\pi^2}$$

fluctuations in the bulk

fluctuations in the shell



absorbed in parameters
m(R)

Absorb the IR fluctuations that cause instability in perturbative series for observables, into the mass parameter

R-RGE: Formulation

$$R\frac{d}{dR} \quad : \quad m(R) = m_{\text{pole}} - R\left[a_1\left(\frac{\alpha_s(R)}{4\pi}\right) + a_2\left(\frac{\alpha_s(R)}{4\pi}\right)^2 + \ldots\right] \qquad \& \qquad R\frac{d\alpha_s(R)}{dR} = \beta[\alpha_s(R)]$$

γ_0 , γ_1 , ... are linear in a_n and β_n

$$R\frac{dm(R)}{dR} = -R\left[\gamma_0\left(\frac{\alpha_s(R)}{4\pi}\right) + \gamma_1\left(\frac{\alpha_s(R)}{4\pi}\right)^2 + \ldots\right]$$

R-RGE

similar eqn. by
Bigi, Shifman, Uraltsev (1997)

R-anomalous dimension

scheme	γ_0
PS / MSR	$4C_F$
kinetic	$16C_F/3$
jet	$2C_F\, e^{\gamma}$

generates a continuous class of schemes

$$R \to R' = \lambda R \quad : \quad \begin{aligned} \gamma_0 \to \gamma_0' &= \lambda\gamma_0 \\ \gamma_1 \to \gamma_1' &= \lambda[\gamma_1 - 2\beta_0\gamma_0 \ln\lambda] \end{aligned}$$

μ-RGE vs. R-RGE

comparison at leading log

$$\mu\,\frac{d\,\overline{m}(\mu)}{d\mu} = \underline{\overline{m}\;\overline{\gamma}_0}\,\frac{\alpha_s(\mu)}{4\pi}$$

$$\int\frac{d\overline{m}}{\overline{m}} = \overline{\gamma}_0\int\frac{d\mu}{\mu}\,\frac{\alpha_s(\mu)}{4\pi}$$

$$\boxed{\frac{d\mu}{\mu} = \frac{d\alpha_s}{\beta[\alpha_s]}}$$

$$R\frac{d\,m(R)}{dR} = -\underline{R\;\gamma_0}\,\frac{\alpha_s(R)}{4\pi}$$

$$\int d\,m(R) = -\,\gamma_0\int dR\,\frac{\alpha_s(R)}{4\pi}$$

need solution of α_s - RGE !!!

α_s-RGE: All order Solution

α_s - RGE

$$\int_{R_0}^{R_1} \frac{dR}{R} = \int_{\alpha_1}^{\alpha_2} \frac{d\alpha_s(R)}{\beta[\alpha_s(R)]}$$

$$\beta[\alpha_s(R)] \sim -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 - \ldots$$

change of variable

$$t = -\frac{2\pi}{\beta_0\,\alpha_s(R)}$$

$$\begin{aligned} \hat{b}_1 &= \frac{\beta_1}{2\beta_0^2}, \\ \hat{b}_2 &= \frac{\beta_1^2 - \beta_0\beta_2}{4\beta_0^4} \\ &\vdots \end{aligned}$$

$$\Lambda_{\mathrm{QCD}} = R\,e^t(-t)^{\hat{b}_1}\ \exp\left(-\frac{\hat{b}_2}{t} - \frac{\hat{b}_3}{2t^2} - \ldots\right)$$

order by order inversion gives familiar expression of α_s which diverges at Λ_{QCD}

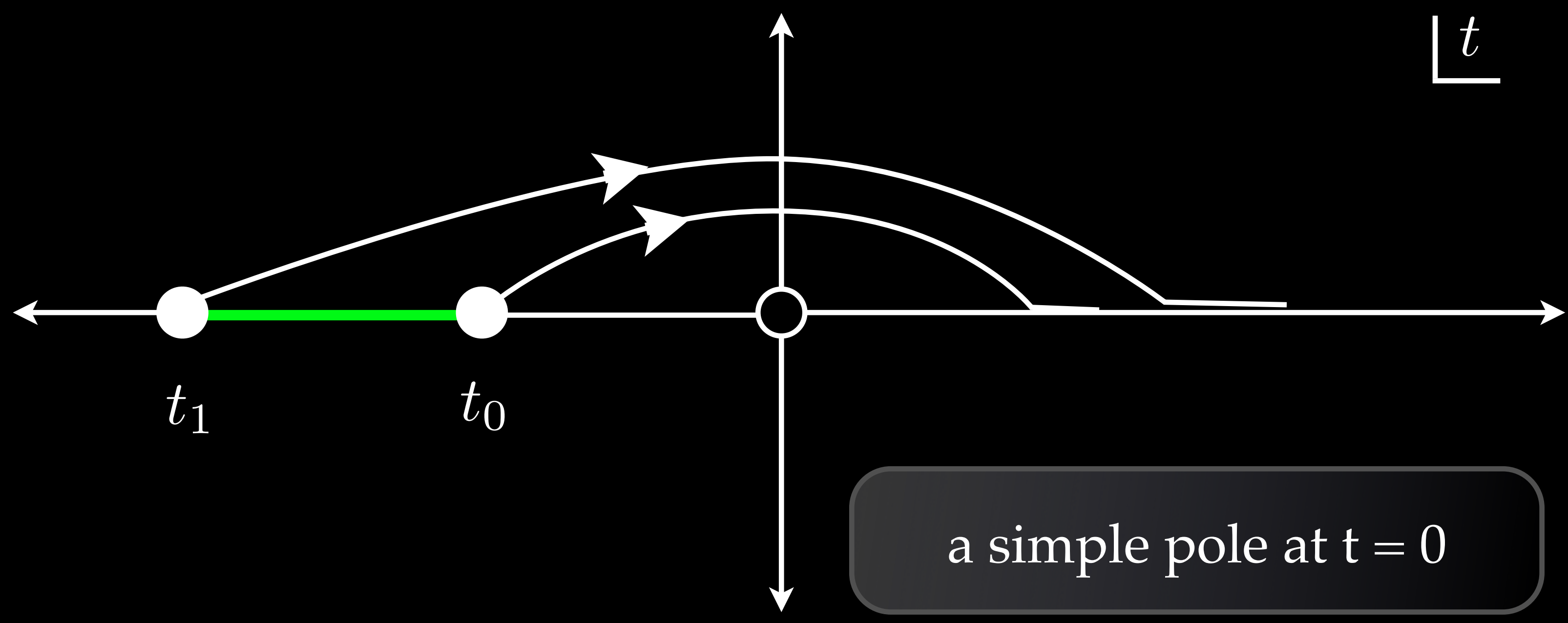
R-RGE : leading log solution

$$\Lambda^{(0)}_{\rm QCD} = R\, e^{t} \qquad\qquad t = -\,\frac{2\pi}{\beta_0\, \alpha_s(R)}$$

$$\int d\,m(R) = -\,\gamma_0 \int dR\, \frac{\alpha_s(R)}{4\pi}$$

$$= \boxed{\frac{\Lambda^{(0)}_{\rm QCD}\, \gamma_0}{2\beta_0} \int_{t_1}^{t_0} dt\, \frac{e^{-t}}{t}}$$

t - plane

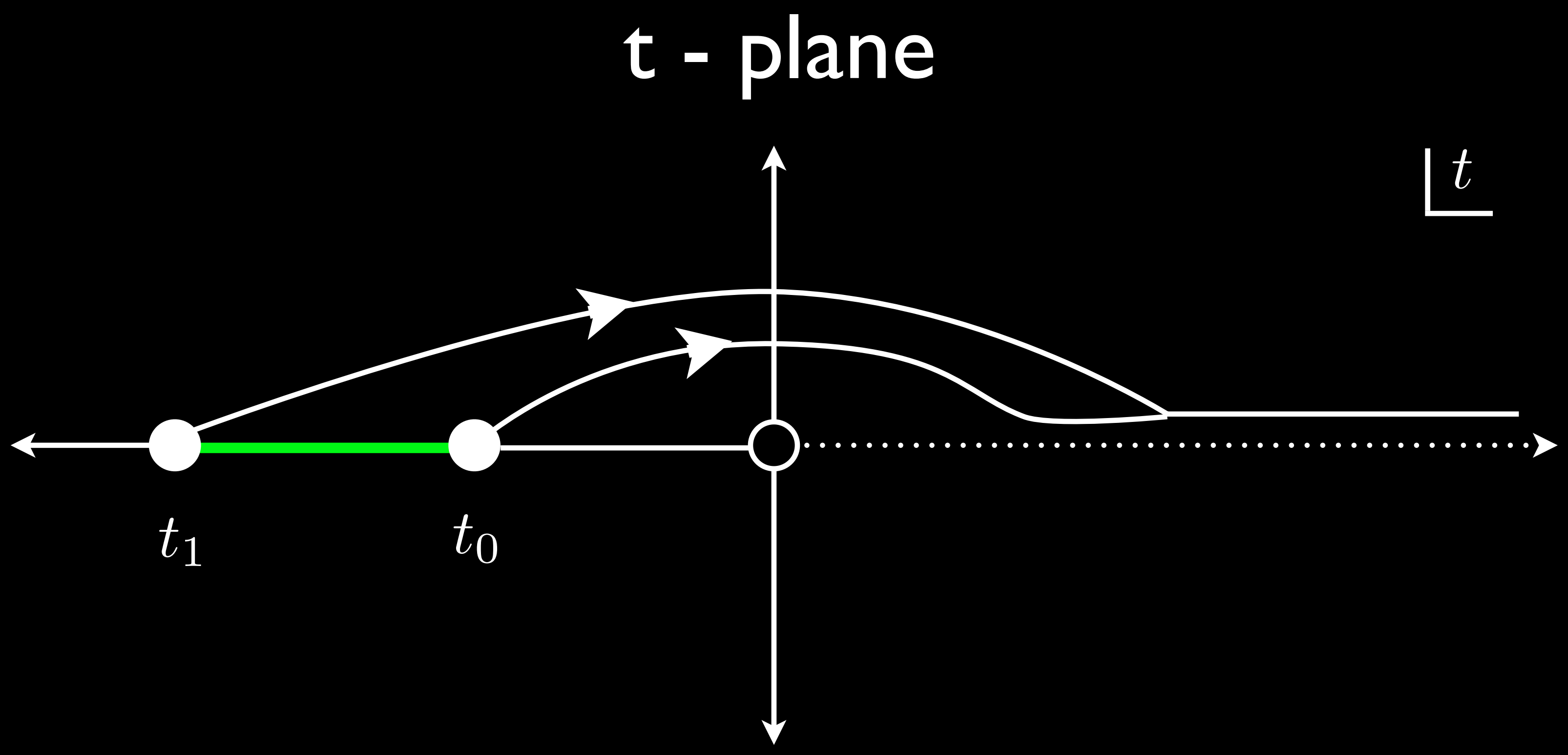


a simple pole at t = 0

$$m(R_1) - m(R_0) \;=\; \frac{\Lambda^{(0)}_{\rm QCD}\, \gamma_0}{2\beta_0}\left[\Gamma(0,t_1) - \Gamma(0,t_0)\right] \;\propto\; -\,\frac{\gamma_0^R R_0}{2\beta_0} \sum_{n=0}^{\infty} \left[\frac{\beta_0 \alpha_1}{2\pi}\right]^{n+1} \sum_{k=n+1}^{\infty} \frac{n!}{k!} \ln^k \frac{R_1}{R_0}$$

R-RGE : all order solution

$$\boxed{\int_{t_1}^{t_0} dt \, \frac{e^{-t}}{(-t)^{n+\hat{b}_1}}}$$



$$\begin{aligned} S_0 &= \frac{\gamma_0}{2\beta_0} \\ S_1 &= \frac{\gamma_1}{(2\beta_0)^2} - (\hat{b}_1 + \hat{b}_2)\,\frac{\gamma_0}{2\beta_0} \end{aligned}$$

n^{th} order pole at $t = 0$ and branch cut for $t > 0$

$$\Big[m(R_1) - m(R_0)\Big]^{\mathrm{N}^k\mathrm{LL}} = \Lambda_{\mathrm{QCD}}^{(k)} \sum_{j=0}^{k} S_j \, (-1)^j \; e^{i\pi \hat{b}_1} \Big[\Gamma(-\hat{b}_1 - j, t_1) - \Gamma(-\hat{b}_1 - j, t_0)\Big]$$

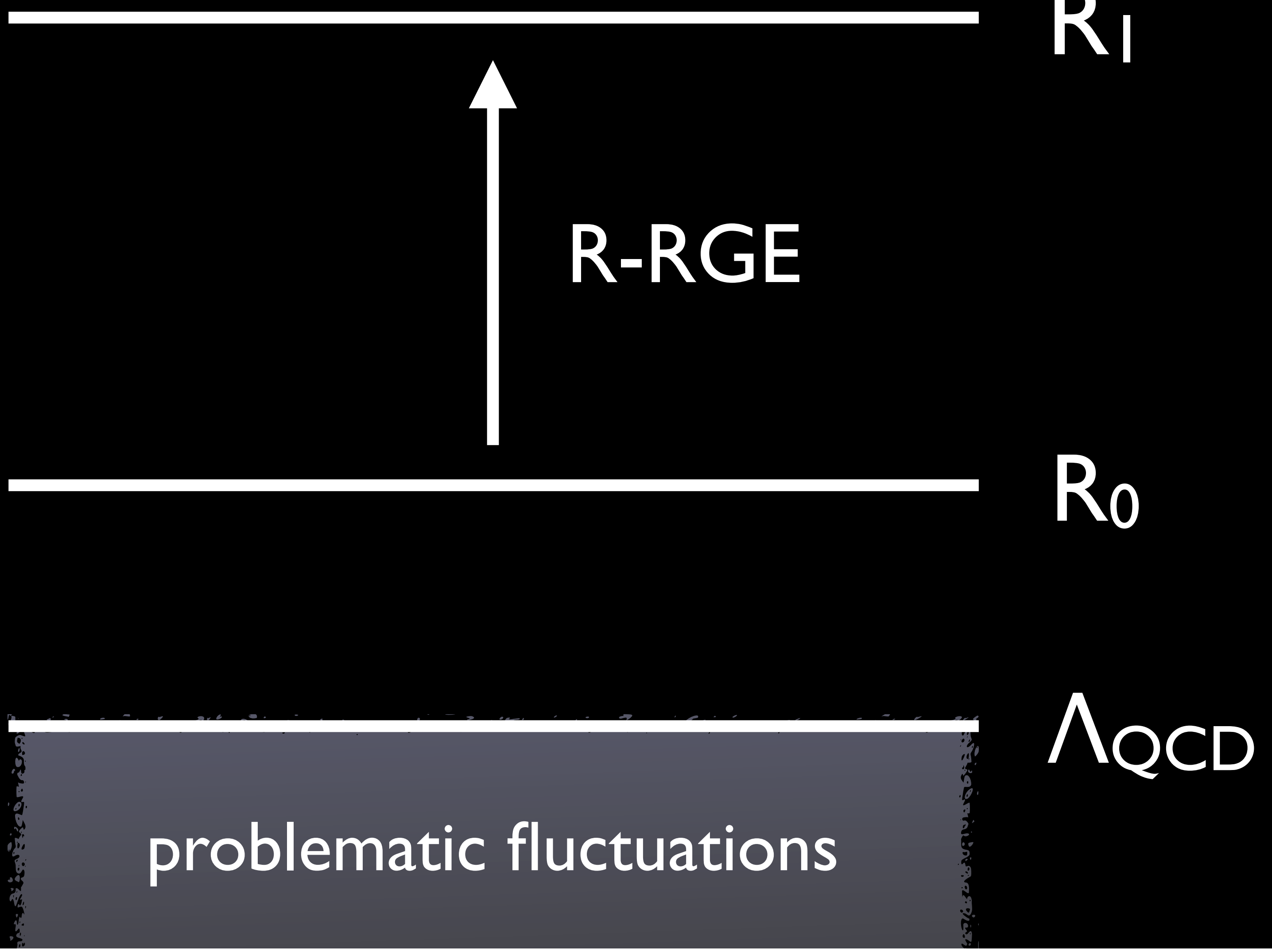
R-evolution : Message

Old Story
with large logs

$$m^A(R_0) - m^B(R_1) = R_1\left[a_1'\left(\frac{\alpha_s(R_1)}{4\pi}\right) + a_2'\left(\frac{\alpha_s(R_1)}{4\pi}\right)^2 + \ldots\right] - R_0\left[a_1\left(\frac{\alpha_s(R_1)}{4\pi}\right) + (a_2 + \beta_0 \log(R_1/R_0))\left(\frac{\alpha_s(R_1)}{4\pi}\right)^2 + \ldots\right]$$

Yuck!!

New Method



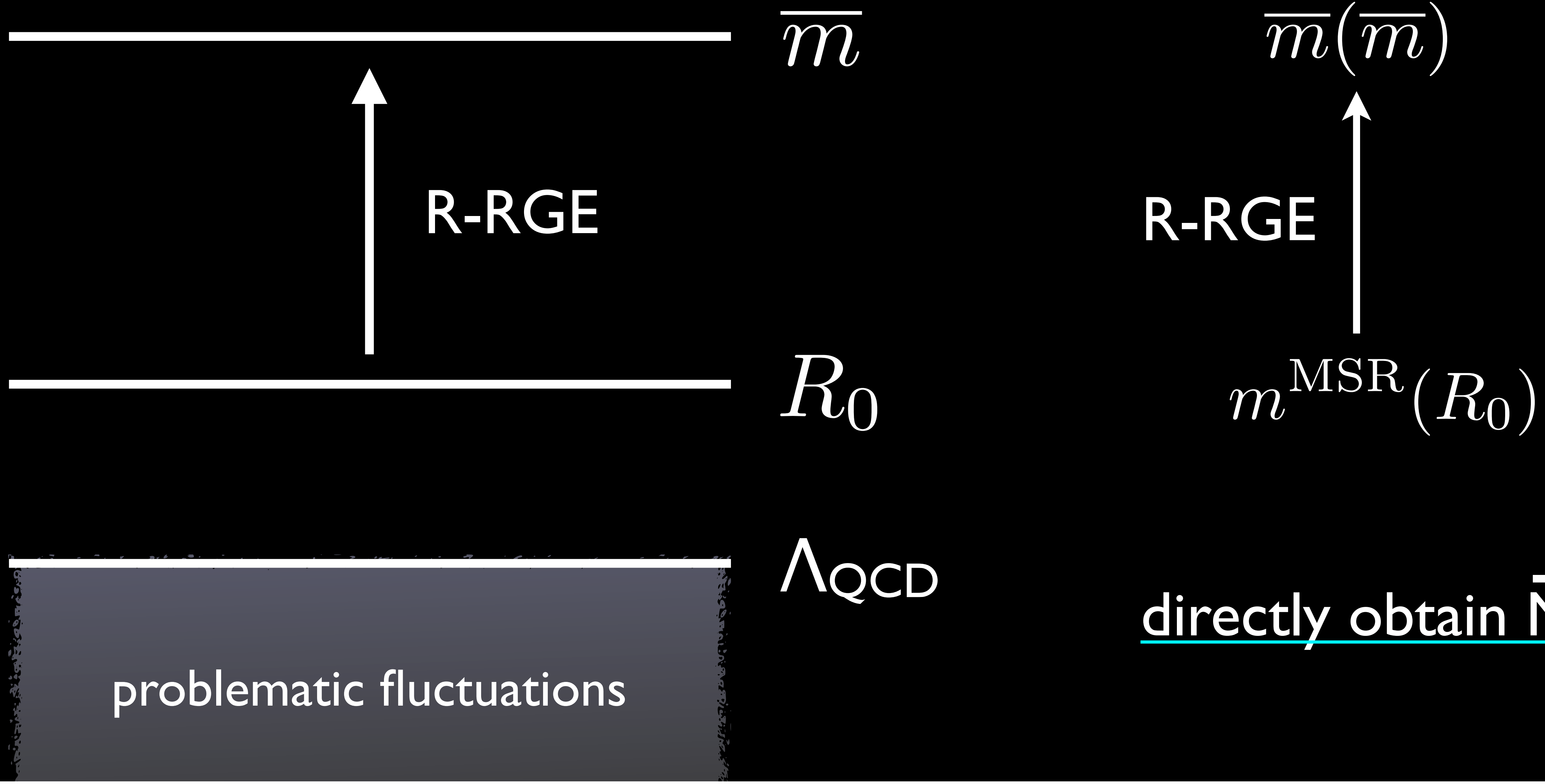
$$\begin{tikzcd} m^A(R_1) \arrow[r, "FO"'] & m^B(R_1) \\ m^A(R_0) \arrow[u, "R\text{-}RGE"] & \end{tikzcd}$$

Yum!

smooth renormalon free
large log free
evolution

R-evolution with MSR mass

Important for top precision era !



$\overline{m}$

R-RGE

R_0

Λ_{QCD}

problematic fluctuations

$\overline{m}(\overline{m})$

R-RGE

$m^{\mathrm{MSR}}(R_0)$

$R_0 \sim \Gamma_t$

$\sim \alpha_s m_t$

directly obtain $\overline{\mathrm{MS}}$ mass

R-RGE is generalizable for quantities with higher power IR sensitivity

An Application:
Renormalon Sum Rules

Renormalon Sum Rule

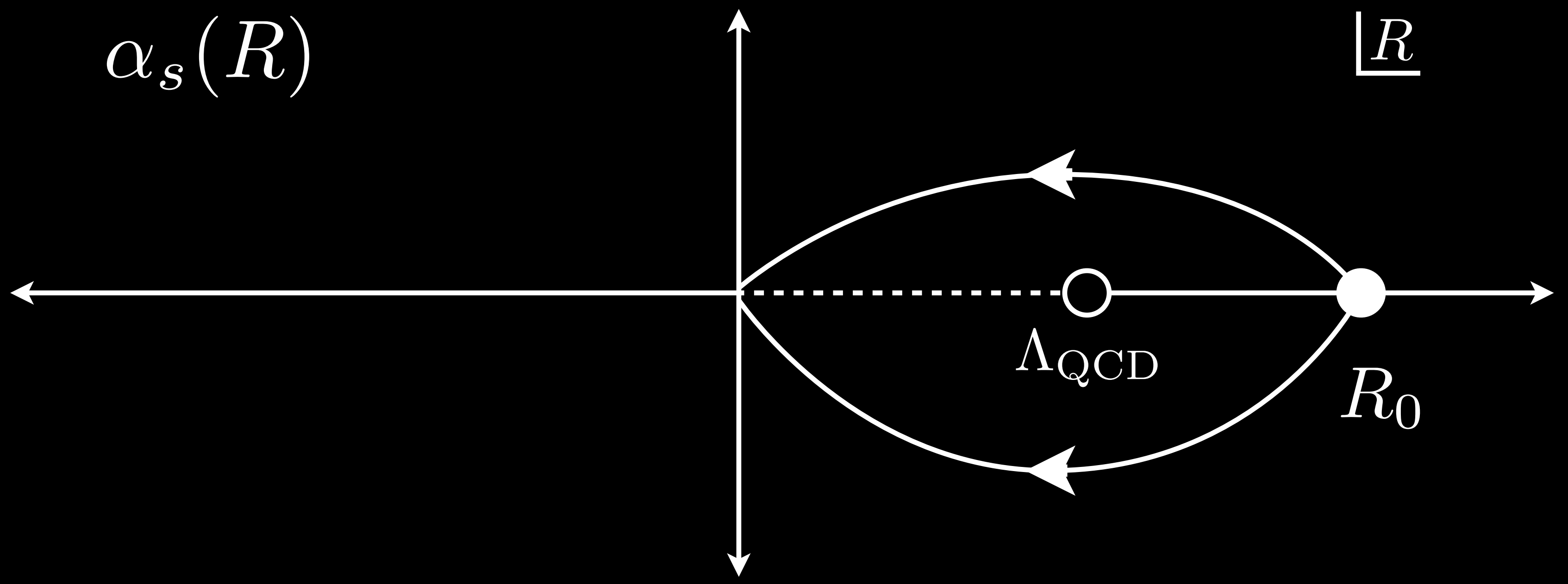
$$\Lambda^{(0)}_{\rm QCD} = R\, e^{t} (-t)^{\hat b_1} \quad \Rightarrow \quad \boxed{\lim_{R\to 0} \alpha_s(R) = 0}$$

$$\delta m(R) \;=\; R \left[a_1 \left(\frac{\alpha_s}{4\pi}\right) + a_2 \left(\frac{\alpha_s}{4\pi}\right)^2 + \ldots \right] \xrightarrow{R \to 0} 0$$

$$\lim_{R_0 \to 0} m(R_0) = m_{\rm pole}$$

- ✦ limiting process restores the ambiguity back in $m_{\rm pole}$
- ✦ limiting process is always perturbative

R - plane



Renormalon Sum Rule

R → 0 Limit + Asymptotic expansion + Borel Transform

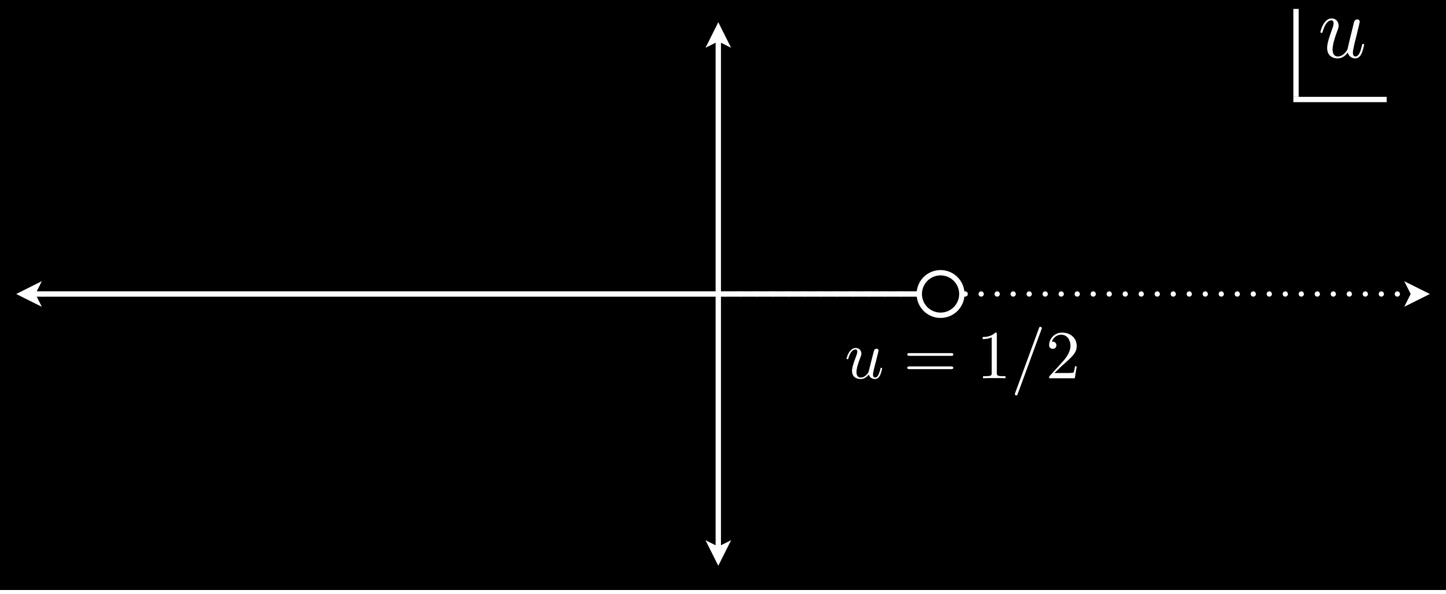
$$B[m_{\text{pole}} - m(R_1)](u) = R\left(\sum_{k=0}^{\infty}\frac{S_k}{\Gamma(1+\hat{b}_1+k)}\right)\times\left(\sum_{\ell=0}^{\infty} g_\ell \frac{\Gamma[1+\hat{b}_1-\ell]}{(1-2u)^{1+\hat{b}_1-\ell}}\right) + (\text{ non singular near } u = 1/2)$$

Residue

Renormalon Singularity

$$S_0 = \frac{\gamma_0}{2\beta_0}$$

$$S_1 = \frac{\gamma_1}{(2\beta_0)^2} - (\hat{b}_1+\hat{b}_2)\frac{\gamma_0}{2\beta_0}$$



Renormalon Sum Rule

R → 0 Limit + Asymptotic expansion + Borel Transform

$$B[m_{\rm pole} - m(R_1)](u) = R\underbrace{\left(\sum_{k=0}^{\infty} \frac{S_k}{\Gamma(1+\hat{b}_1+k)}\right)}_{P_{1/2}} \times \left(\sum_{\ell=0}^{\infty} g_\ell \frac{\Gamma[1+\hat{b}_1-\ell]}{(1-2u)^{1+\hat{b}_1-\ell}}\right) + (\text{ non singular near } u=1/2)$$

Renormalon Singularity

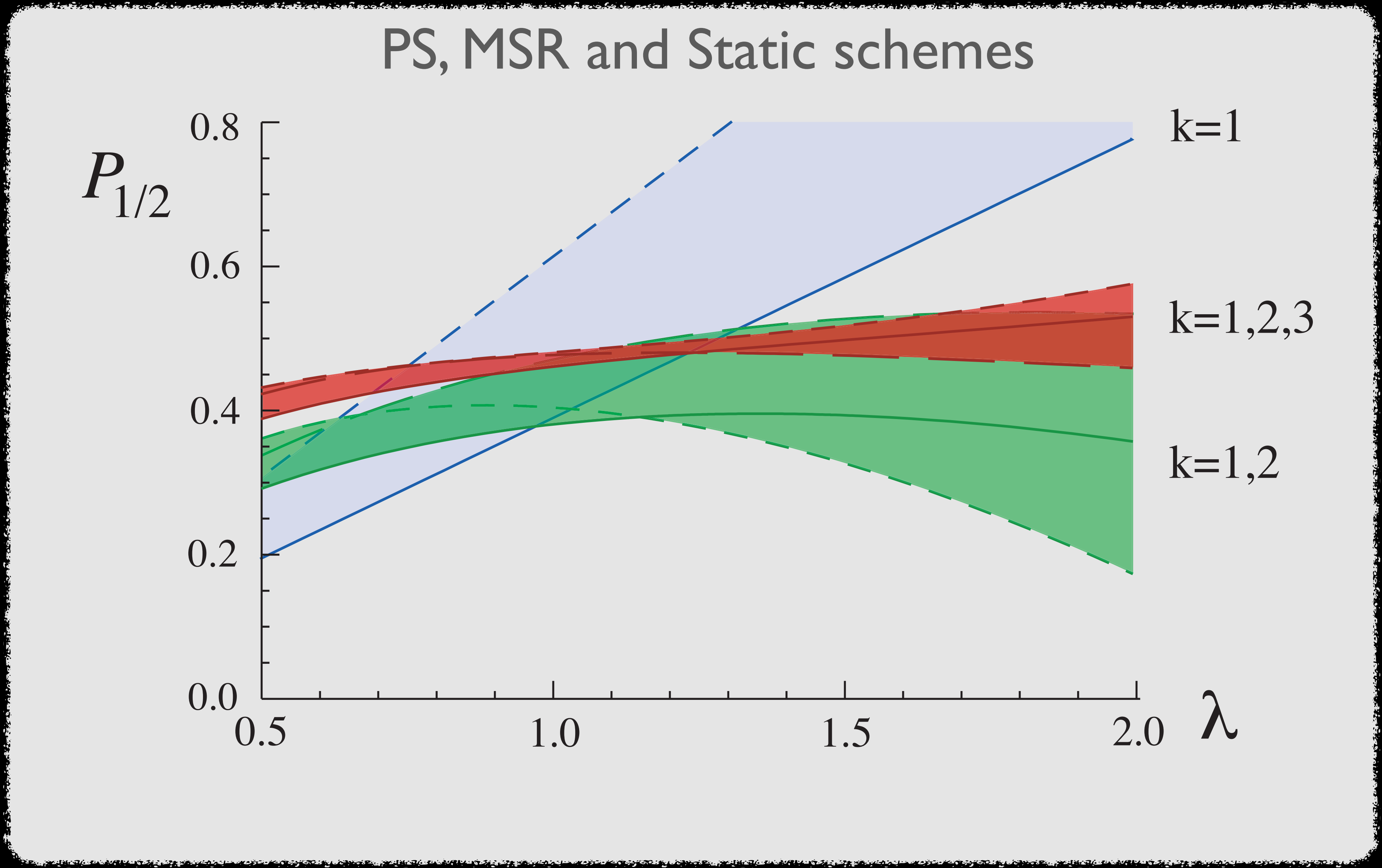
A new way to test for renormalons ! no approximations !!

$P_{1/2} \longrightarrow 0$ ⇒ no renormalon

$P_{1/2} \longrightarrow \text{number} \neq 0$ ⇒ renormalon exists

Normalization of renormalon in pole mass

$$P_{1/2} = \sum_{k=0}^{\infty} \frac{S_k}{\Gamma(1+\hat{b}_1+k)}$$



n_f	NNA	$P_{1/2}$
3	0.68	0.45
4	0.74	0.47
5	0.80	0.48
6	0.88	0.48
7	0.97	0.43

Analyzing Renormalons in OPE

Renormalons in OPE and $\overline{\text{MS}}$

Example: Chromomagnetic operator

$$m_B = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2}{2m_b} + \ldots$$

$$\bar{\Lambda} \sim \langle B|\bar{b}_v\, iv.D\, b_v|B\rangle$$

$$\lambda_1 \sim \langle B|\bar{b}_v D_\perp^2\, b_v|B\rangle$$

$$m_{B^*} = m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} + \frac{\lambda_2}{2m_b} + \ldots$$

$$\lambda_2 \sim C(m_b,\mu)\,\langle B|\bar{b}_v\, g\sigma_{\mu\nu}G^{\mu\nu}\, b_v|B\rangle(\mu)$$

$$m_{B^*}^2 - m_B^2 = \frac{4}{3}C(m_b,\mu)\,\mu_G^2(\mu) + \ldots$$

Similar relation can be written for D and D* mesons

$$\frac{m_{B^*}^2 - m_B^2}{m_{D^*}^2 - m_D^2} = \frac{C(m_b,\mu)}{C(m_c,\mu)} + \ldots$$

C(m,μ) known to three loops
Grozin et. al. (2007)

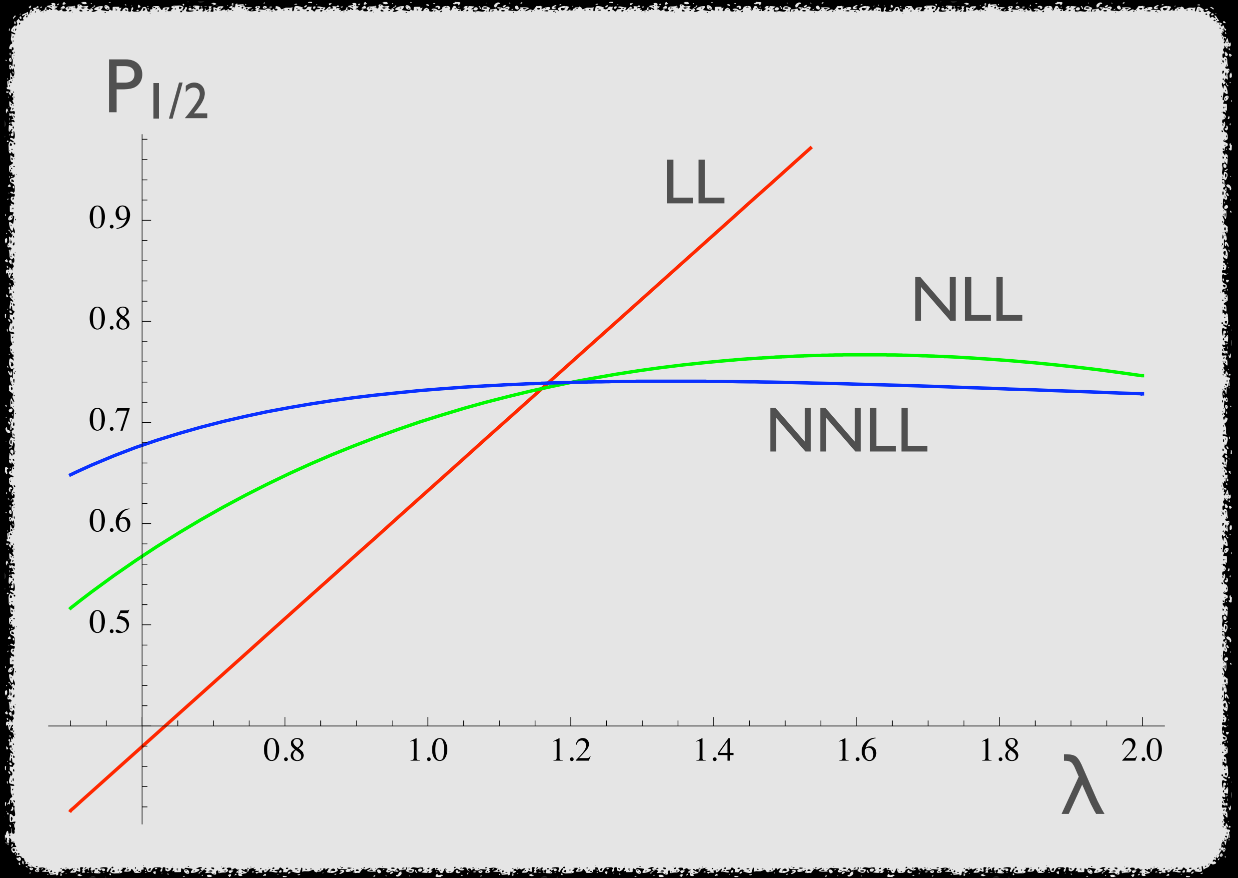
B-D mass splitting ratio

$$\left.\frac{m_{B*}^2 - m_B^2}{m_{D*}^2 - m_D^2}\right|_{\overline{MS}}^{LP} = 0.8517 - 0.0696 - 0.0908 - [0.1285]\ldots$$

Grozin et. al. (2007)

$$\left.\frac{m_{B*}^2 - m_B^2}{m_{D*}^2 - m_D^2}\right|_{\mathrm{expt.}} = 0.88$$

perturbation theory seems to fail !!!



our renormalon sum rule for C(m,μ)

Observable

$$\mathcal{O} = C_1(M,\mu)Q_1(\mu) + C_2(M,\mu)\frac{Q_2(\mu)}{M} + \ldots$$

OPE

Define:

$$C_1^{(R)}(M,\mu) = C_1(M,\mu) - \frac{R}{M}\delta C_1(M,\mu,R)$$

$R \gtrsim \Lambda_{\rm QCD}$

Generics: Reminder

$$\mathcal{O} = C_1^{(R)}(M,\mu)Q_1(\mu) + C_2(M,\mu)\frac{Q_2^{(R)}(\mu)}{M} + \ldots$$

$$\delta Q_2(\mu,R) = -R\frac{\delta C_1(M,\mu,R)}{C_2(M,\mu)}Q_1(\mu)$$

How to define δC_1?
Will see later!

For simplicity μ = R

(1) $$C^{(R)}(m,R) = C(m,R) - \frac{R}{m}[C(R,R) - 1]$$

(2) $$\log\, C^{(R)}(m,R) = \log\, C(m,R) - \frac{R}{m}\log\, C(R,R)$$

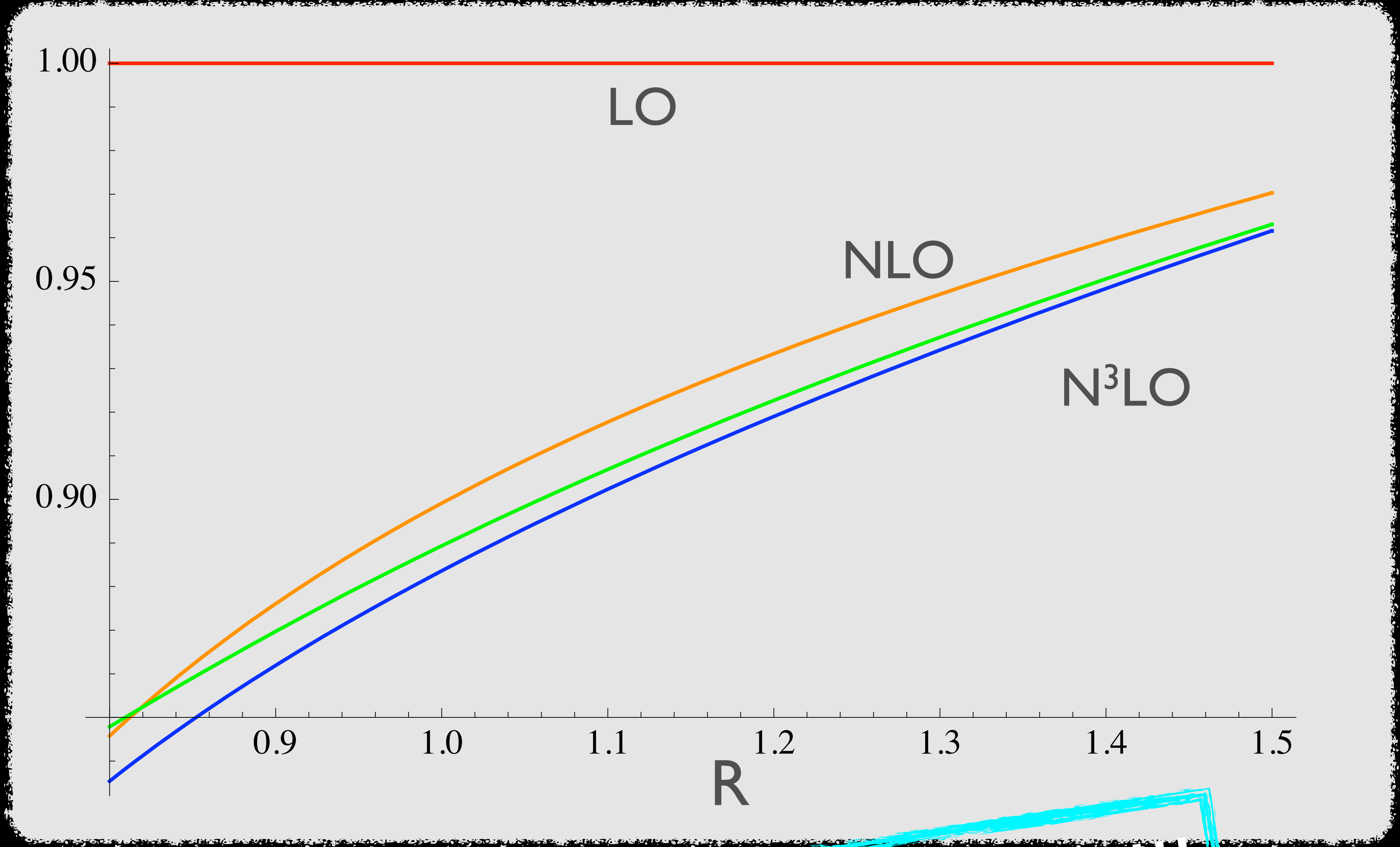
Can do combined
μ-R RGE

B-D mass splitting ratio

$$\log\, C^{(R)}(m,R) = \log\, C(m,R) - \frac{R}{m}\log\, C(R,R)$$

$$\frac{m_{B*}^2 - m_B^2}{m_{D*}^2 - m_D^2} = \frac{C^{(R)}(m_b,R)}{C^{(R)}(m_c,R)} + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{m_{b,c}}\right)$$

$$\left.\frac{m_{B*}^2 - m_B^2}{m_{D*}^2 - m_D^2}\right|_R^{\rm LP} = 0.90^{+0.05}_{-0.06} + \mathcal{O}\left(\frac{\Lambda_{\rm QCD}}{m_{b,c}}\right)$$



- convergent perturbation series at LP
- error bars give size of scheme dependence

perturbation theory saved!!!

Conclusions

- R-RGE new way to convert between schemes — avoiding large logs and renormalon at the same time
- MSR scheme connects smoothly to $\overline{\text{MS}}$ mass — great for precision measurement
- R-RGE is generalizable for higher power law sensitivities to IR
- new way to test for renormalons in QCD — using renormalon sum rule
- We find: α_s in $\overline{\text{MS}}$ scheme has u=1 renormalon.
- method to stabilize prediction for observables in OPE — removing renormalons in Lagrangian parameters and $\overline{\text{MS}}$ matrix elements

Back up stuff

strong coupling in $\overline{\text{MS}}$

Q. Does the strong coupling in $\overline{\text{MS}}$ scheme have a renormalon ?

Q. Does the β function in $\overline{\text{MS}}$ scheme have a renormalon ?

Suslov (2004)

Naive non-abelianization and bubble chain calculation does not work as a probe

Use renormalon sum rule to probe it

we see some signature of $u = 1$ renormalon !!!

Renormalon in $\overline{\rm MS}$ α_s

$\beta^{\overline{MS}}$ acts as a probe in the sum rule !
how do we probe the probe itself ?

$$R\frac{d\bar{\alpha}_s}{dR} = -\frac{\bar{\alpha}_s^2}{2\pi}\sum_{n=0}^{\infty}\beta_n\left(\frac{\bar{\alpha}_s}{4\pi}\right)^n$$

α_s RGE in $\overline{\rm MS}$ scheme

$$R\,\frac{d(\alpha_s^{tH})}{dR} = -\frac{(\alpha_s^{tH})^2}{2\pi}\left(\beta_0+\beta_1\frac{\alpha_s^{tH}}{4\pi}\right)$$

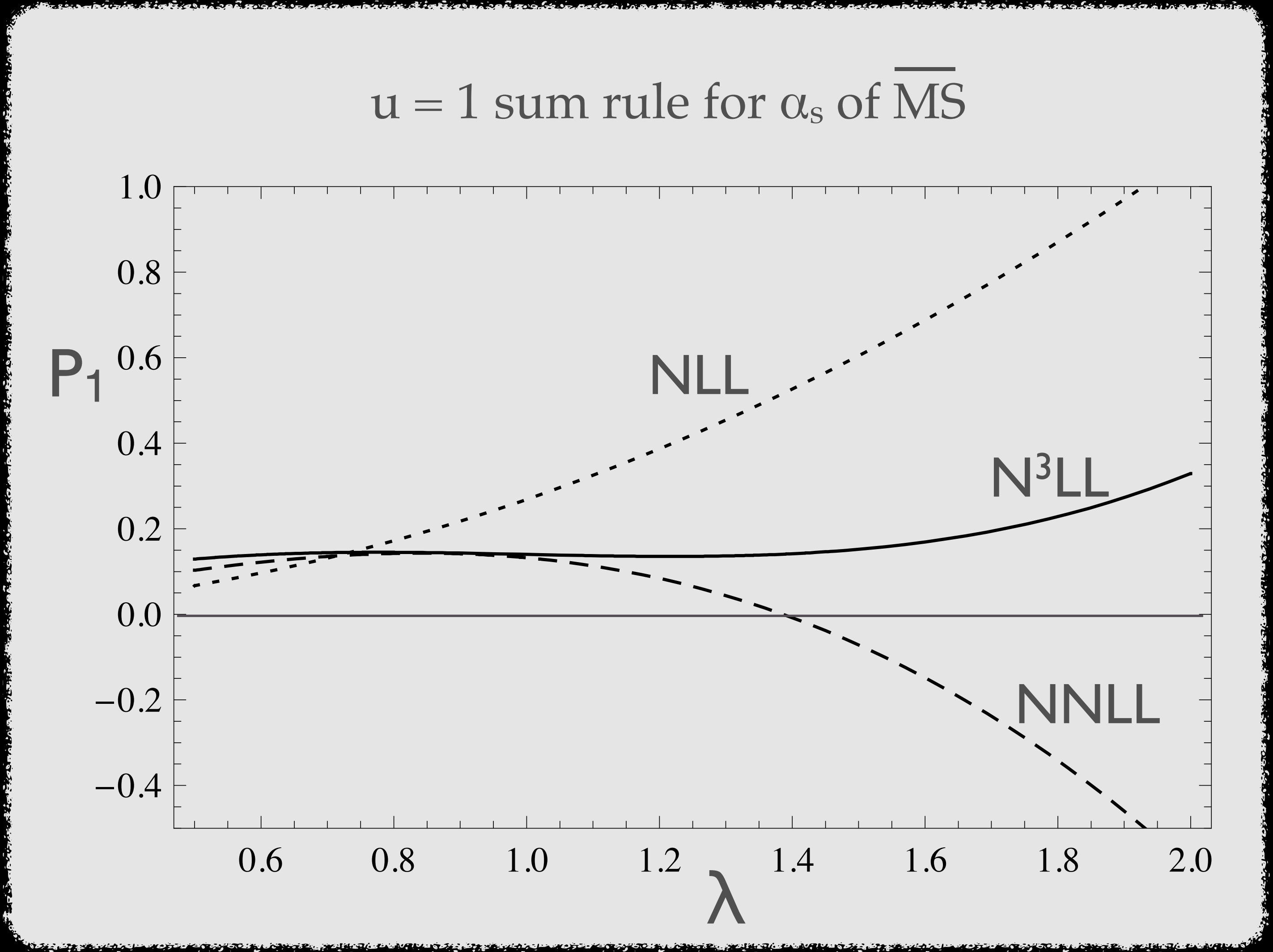
α_s RGE in tHooft scheme

$$\frac{\bar{\alpha}_s-\alpha_s^{tH}}{\alpha_s^{tH}} = \sum_{n=1}^{\infty}h_n\left(\frac{\alpha_s^{tH}}{4\pi}\right)^n$$

relation between two schemes

$$\mathrm{Amb}\left[\frac{\bar{\alpha}_s-\alpha_s^{tH}}{\alpha_s^{tH}}\right](\mu) \quad\approx\quad i\;0.2\,\frac{\Lambda_{\rm QCD}^2}{\mu^2}$$

N^{3}LL prediction need Padé approximation for β_4



u = 1 sum rule for α_s of $\overline{\rm MS}$