Abelian Yang-Baxter-Deformations

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based on ongoing work with Stijn van Tongeren

Motivation

- Type IIb Green-Schwarz superstring in $AdS_5 \times S^5$ intensively studied in last years in context of $AdS\,/\,CFT$-correspondence

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- Type IIb Green-Schwarz superstring in $AdS_5 \times S^5$ intensively studied in last years in context of $AdS\,/\,CFT$-correspondence
- Motivations to consider deformations
  - for use in context of $AdS\,/\,CFT$-correspondence: Extension of the conjecture to less symmetric theories?
  - from point of view of *integrable structures*:
    - Other integrable type IIb string σ-models?
    - Symmetries behind integrable structures in the deformed models?
  - from *(super)gravity* perspective: generating new solutions

Overview

1. Basic Notions
    - T-Duality and TsT-Transformations
    - Strings in $AdS_5 \times S^5$
2. Abelian Yang-Baxter-Deformations
    - Yang-Baxter-Deformations
    - Equivalence to TsT-Transformations

The Notion of T-Duality

- **World-sheet point of view**:
  - e.g. closed string compactified on circle:
    spectrum invariant under $R \leftrightarrow 1/R$ (R: radius of S^1)
  - Open string: Dirichlet $\leftrightarrow$ Neumann boundary conditions
  - Superstring: IIa $\leftrightarrow$ IIb

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- **World-sheet point of view**:
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  - Superstring: IIa $\leftrightarrow$ IIb
- **Space-time perspective**:
  Coordinate-dependent notion in σ-model approach:

$$S \propto \int \mathrm{d}^2\sigma \mathcal{E}_{MN}(X^i)\partial_+ X^M \partial_- X^N \equiv \int \mathrm{d}^2\sigma \mathcal{L} \qquad M, N = 1, ..., D; \;\; i = 2, ..., D$$

Rewrite $\mathcal{L}$: $\quad \partial_\pm X^1 \to k_\pm \qquad \mathcal{L} \to \mathcal{L} - \bar{X}^1(\partial_+ k_- - \partial_- k_+)$,

Integrating out $k_\pm$ yields equivalent geometry:

$$\bar{\mathcal{E}}_{11} = \frac{1}{\mathcal{E}_{11}}, \quad \bar{\mathcal{E}}_{1i} = \frac{\mathcal{E}_{1i}}{\mathcal{E}_{11}}, \quad \bar{\mathcal{E}}_{1i} = -\frac{\mathcal{E}_{i1}}{\mathcal{E}_{11}}, \quad \bar{\mathcal{E}}_{ij} = \mathcal{E}_{ij} - \frac{\mathcal{E}_{i1}\mathcal{E}_{1j}}{\mathcal{E}_{11}}$$

TsT-Transformations

- For two isometry directions X^1, X^2:
  T-duality on X^1 $\rightarrow$ shift: $\bar{\bar{X}}^1 = \bar{X}^1, \ \bar{\bar{X}}^2 = \bar{X}^2 - \gamma \bar{X}^1$ $\rightarrow$ T-duality back on $\bar{\bar{X}}^1$
  $\curvearrowright$ Transformation of background:

$$\bar{\mathcal{E}} = \mathcal{E}(\mathbb{1} + \tilde{\Gamma}\mathcal{E})^{-1} \qquad \text{with } \tilde{\Gamma} = \left(\begin{array}{cc|c} 0 & \gamma & \\ -\gamma & 0 & \\ \hline & & 0_{D-2} \end{array}\right)$$

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- TsT-transformations of $AdS_5 \times S^5$, e.g.
  - on S^5, Lunin-Maldacena-backgrounds, dual to β-deformation of field theory
  - on AdS_5, gravity dual of noncommutative SYM
- Generalization to d (commuting) isometries:

$$\bar{\mathcal{E}} = \mathcal{E}(\mathbb{1} + \tilde{\Gamma}\mathcal{E})^{-1} \qquad \text{with } \tilde{\Gamma} = \begin{pmatrix} \Gamma & \\ & 0_{D-d} \end{pmatrix}, \quad \Gamma_{kl} = -\Gamma_{lk} \text{ for } k, l = 1, \dots, d$$

Strings in $AdS_5 \times S^5$

- Coset σ-model for $G = \frac{\mathrm{SO}(2,4)\times\mathrm{SO}(6)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)} \simeq AdS_5 \times S^5$
- Symmetric space: $\mathbb{Z}_2$-grading of $\mathfrak{g} = \mathfrak{so}(2,4) \times \mathfrak{so}(6) = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(2)}$.
  identifications: $\mathfrak{g}^{(0)} \equiv \mathfrak{so}(1,4) \oplus \mathfrak{so}(5)$ isotropy group, $\mathfrak{g}^{(2)}$ coset algebra
- Bosonic action (With fermions: [Metsaev and Tseytlin, 1998])

$$S \propto \int \mathrm{d}^2\sigma \ \mathrm{Tr}(A_+ P^{(2)}(A_-))$$

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- A notion of Integrability:
  - equations of motion equivalent to
    $\partial_+ L_- - \partial_- L_+ - [L_+, L_-] = 0$

$$\text{with} \quad L_\pm(\lambda) = A_\pm^{(0)} + \lambda^{\mp 2} A_\pm^{(2)} \in \mathfrak{g}$$

  - Dependence on spectral parameter λ allows for an infinite tower of conserved charges: eigenvalues of $\mathfrak{P}\exp\left(\int \mathrm{d}\sigma L_\sigma(\lambda)\right)$

Classical Yang-Baxter-Equation (CYBE)

- Common form:

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \qquad \text{for } r \in \mathfrak{g} \otimes \mathfrak{g}$$

Transition from a skew-symmetric r-matrix to a R-operator:

$$r = a \wedge b := \frac{1}{2}(a \otimes b - b \otimes a) \quad \rightarrow \quad R(M) := \mathrm{Tr}_2(r \cdot (1 \otimes M))$$

- There are deformations based on solutions R of
  - CYBE: $[R(M), R(N)] - R\left([R(M), N] + [M, R(N)]\right) = 0$
  - mCYBE: $[R(M), R(N)] - R\left([R(M), N] + [M, R(N)]\right) = [M, N]$

Yang-Baxter-Deformations

- Integrable deformations of symmetric space coset σ-models, [Klimčík, 2009] & [Delduc et al., 2014]

$$S \propto \int \mathrm{d}^2\sigma \, \mathrm{Tr}\left(A_+ P^{(2)}(J_-)\right) \qquad \text{with } J_\pm = \frac{1}{\mathbb{1} \pm \eta R_g \circ P^{(2)}}(A_\pm)$$

for $\eta \in [0,1)$ and with $R_g = \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g$.

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  for $\eta \in [0,1)$ and with $R_g = \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g$.
- Lax Pair exists - e.g. for CYBE:

$$L_\pm = J_\pm^{(0)} + \lambda^{\mp 2} J_\pm^{(2)}$$

- Different classes of CYBE-based deformations, e.g.:
  - *Abelian*: of type $a, b \in \mathfrak{g}$ and $[a, b] = 0 \to r = a \wedge b$
  - *Jordanian*: e.g. $e, h \in \mathfrak{g}$ and $[h, e] = e \to r = e \wedge h$
- mCYBE-deformation:
  - σ-model with q-deformed $\mathrm{SO}(2,4) \times \mathrm{SO}(6)$-symmetry
  - Not a supergravity solution

Abelian Yang-Baxter-Deformations and Equivalence to TsT-Transformations

- Abelian solution to CYBE:
  $[h_i, h_j] = 0 \rightarrow r = -\Gamma^{ij} h_i \wedge h_j, \ \{h_i\} \in \mathfrak{g}$.
- Consider a natural group parameterisation: $g = \exp(X^i h_i)\bar{g}(Y)$.
  Manifest isometries in Maurer-Cartan-form
  $$A = -\mathrm{Ad}_g^{-1}(h_i)\mathrm{d}X^i + \bar{A}(Y) \equiv A_i(Y)\mathrm{d}X^i + \bar{A}(Y).$$
- Then *abelian* Yang-Baxter-Deformation with
  $R_g(M) = -\Gamma^{ij} A_i \mathrm{Tr}(A_j M)$ and after a little algebra:
  $$\Rightarrow \quad \bar{\mathcal{E}} = \mathcal{E}(\mathbb{1} + \Gamma\mathcal{E})^{-1}. \qquad \rightarrow TsT\text{-transformation}.$$

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  $$\Rightarrow \quad \bar{\mathcal{E}} = \mathcal{E}(\mathbb{1} + \Gamma\mathcal{E})^{-1}. \qquad \to TsT\text{-transformation.}$$
- Starting from another parameterisation
  $$X = (X^i, Y^A) \to \underline{X}(X) \quad \Rightarrow \quad \mathrm{d}X^M \equiv J^M{}_N \mathrm{d}\underline{X}^N,$$
  the background transforms like
  $$\underline{\bar{\mathcal{E}}} = \underline{\mathcal{E}}(\mathbb{1} + J^{-1}\Gamma(J^{-1})^T\, \underline{\mathcal{E}})^{-1} = J^T \mathcal{E}(\mathbb{1} + \Gamma\mathcal{E})^{-1} J.$$

Conclusion

- Abelian Yang-Baxter-Deformations
  $\equiv$ coordinate-independent notion of TsT-transformations.
- Lax-pair for generic TsT-deformed models (now with fermions)

$$L_\pm = J_\pm^{(0)} + \lambda J_\pm^{(1)} + \lambda^{\mp 2} J_\pm^{(2)} + \lambda^{-1} J_\pm^{(3)} .$$

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- Comments on the corresponding supergravity deformation:
  - Abelian Yang-Baxter-deformed model is a supergravity solution
  - TsT-transformation for metric, B-field and for the RR-forms known $\rightarrow$ field redefinitions to understand deformation of the supergravity fields.

Open Questions

- Deformations based on r-matrices built from supercharges
- Full list of inequivalent TsT-transformations for AdS-spaces (currently under investigation)
- Understanding the other Yang-Baxter-Deformations.

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Thank you for your attention!

Appendix I: Green-Schwarz Superstring in $AdS_5 \times S^5$

- Supercoset σ-model for $G = \frac{\mathrm{PSU}(2,2|4)}{\mathrm{SO}(1,4)\times\mathrm{SO}(5)}$
- $\mathbb{Z}_4$-grading of $\mathfrak{g} = \mathfrak{su}(2,2|4) = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)}$ with:

$$[M^{(i)}, N^{(j)}] \in \mathfrak{g}^{(i+j)\mathrm{mod}4} \quad \text{for } M^{(k)}, N^{(k)} \in \mathfrak{g}^{(k)}$$
$$\mathrm{STr}(M^{(i)} N^{(j)}) = 0 \quad \text{for } m + n \neq 0 \text{ mod } 4.$$

identifications: $\mathfrak{g}^{(1)}$, $\mathfrak{g}^{(3)}$ odd parts; $\mathfrak{g}^{(0)} \equiv \mathfrak{so}(1,4) \oplus \mathfrak{so}(5)$, $\mathfrak{g}^{(2)}$ bosonic parts

- Action in conformal gauge:

$$S \propto \int \mathrm{d}^2\sigma \; \mathrm{STr}(A_+ d_-(A_-))$$

with projector $d_- = \mathfrak{P}^{(1)} + 2\mathfrak{P}^{(2)} - \mathfrak{P}^{(3)}$

- Lax-pair

$$\text{with} \quad L_\pm(\lambda) = A_\pm^{(0)} + \lambda A_\pm^{(1)} + \lambda^{\mp 2} A_\pm^{(2)} + \lambda^{-1} A_\pm^{(3)}$$

Appendix II: Yang-Baxter-Deformations for a supercoset σ-model

- Based on mCYBE:
  - For $\eta \in [0,1)$, R fulfilling mCYBE

    $$: S \propto \int \mathrm{d}^2\sigma \ \mathrm{STr}\left(A_+ d_-^\eta(J_-)\right) \qquad \text{with } J_\pm = \frac{1}{\mathbb{1} \pm \eta R_\phi \circ d_\pm^\eta}(A_\pm)$$

    with $R_g = \mathrm{Ad}_g^{-1} \circ R \circ \mathrm{Ad}_g$ and $d_\mp^\eta = \pm P^{(1)} + \frac{2}{1-\eta^2} P^{(2)} \mp P^{(3)}$.
  - Lax pair:

    $$L_\pm(\lambda) = J_\pm^{(0)} + \lambda\sqrt{1+\eta^2} J_\pm^{(1)} + \lambda^{\mp 2}\frac{1+\eta^2}{1-\eta^2} J_\pm^{(2)} + \lambda^{-1}\sqrt{1+\eta^2} J_\pm^{(3)}$$

- For $\mathcal{R}$ fulfilling CYBE
  - $S \propto \int \mathrm{d}^2\sigma \ \mathcal{L} = \int \mathrm{d}^2\sigma \ \mathrm{STr}\left(A_+ d_- \circ \frac{1}{\mathbb{1} - \mathcal{R}\circ d_-}(A_-)\right)$.
  - Corresponding Lax pair:

    $$L_\pm = J_\pm^{(0)} + \lambda J_\pm^{(1)} + \lambda^{\mp 2} J_\pm^{(2)} + \lambda^{-1} J_\pm^{(3)}$$

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