Infrared Singularities in the Triple Collinear Limit

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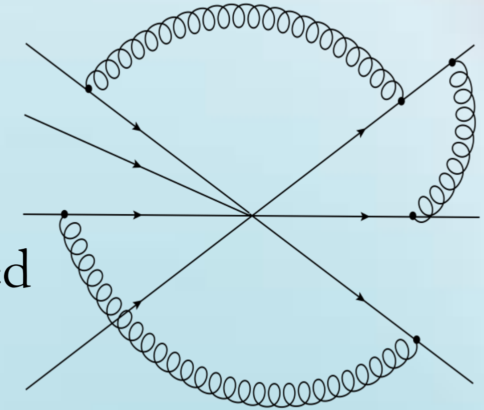
Contents

Soft Anomalous Dimension



- Consider the case of scattering of massless particles
- Infrared (soft and collinear) singularities can be factorised

as follows:

$$M_n(\{p_i\},\mu,\alpha(\mu^2),\epsilon_{IR}) = Z_n(\{p_i\},\mu_f,\alpha\left(\mu_f^2\right),\epsilon_{IR})H_n(\{p_i\},\mu,\mu_f,\alpha(\mu^2))$$

- And

$$Z_n(\{p_i\},\mu_f,\alpha\left(\mu_f^2\right),\epsilon_{IR}) = Pexp\left\{-\frac{1}{2}\int_0^{\mu_f^2}\frac{d\lambda^2}{\lambda^2}\Gamma_n(\{p_i\},\lambda,\alpha_s(\lambda^2))\right\}$$

- Γ_n is the soft anomalous dimension.

The Dipole Formula

$$\Gamma_{\mathrm{n}}(\{p_i\}, \mu_f, \alpha(\mu_f^2)) = \Gamma_n^{\mathrm{dip}}(\{p_i\}, \mu_f, \alpha(\mu_f^2)) + \Delta_n(\rho_{ijkl}, \alpha(\mu_f^2))$$

Using constraints, such as, rescaling symmetry and Bose symmetry we arrive at:

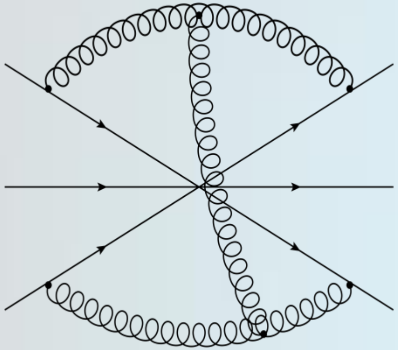
$$\Gamma_n^{\mathrm{dip}}(\{p_i\}, \mu_f, \alpha(\mu_f^2)) = -\frac{1}{2}\hat{\gamma}_K(\alpha_s) \sum_{i<j} \log\left(-\frac{s_{ij}}{\lambda^2}\right) T_i . T_j + \sum_{i=1}^{n} \gamma_{J_i}(\alpha_s)$$

Where $-s_{ij} = 2|p_i . p_j| e^{-i\pi\lambda_{ij}}$ and $\lambda_{ij} = 1$ if partons are both final or initial, or $\lambda_{ij} = 0$ otherwise.

$$\rho_{ijkl} = \frac{(p_i . p_j)(p_k . p_l)}{(p_i . p_k)(p_j . p_l)} = \frac{(-s_{ij})(-s_{kl})}{(-s_{ik})(-s_{jl})}$$

Three Loop Order Correction

$$\Delta(z,\bar{z}) = \frac{1}{(4\pi)^3} 16 f^{abe} f^{cde} \Bigg[\sum_{1\le i<j<k<l\le n} \Big\{ T_i^a T_j^b T_k^c T_l^d \left(F(1-1/z) - F\left({}^1/_z\right)\right) + T_i^a T_k^b T_j^c T_l^d \left(F(1-z) - F(z)\right) + T_i^a T_l^b T_j^c T_k^d \left(F\left({}^1/_{(1-z)}\right) - F\left(1 - {}^1/_{(1-z)}\right)\right) \Big\} - (\zeta_5 + 2\zeta_2\zeta_3) \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ j,k\neq i}} \{T_i^a, T_i^d\} T_j^b T_k^c \Bigg]$$



- $F(z) = \mathcal{L}_{10101}(z) + 2\,\zeta_2[\mathcal{L}_{100}(z) + \mathcal{L}_{001}(z)]$

where $\mathcal{L}_w(z)$ are Brown's single-valued harmonic polylogarithyms (SVHPLs)

- $\rho_{1234} = z\bar{z}$ and $\rho_{1432} = (1-z)(1-\bar{z})$

Colour Considerations

$$T_1^a T_2^b T_3^c T_4^d \left(f^{abe} f^{cde}(\dots) + f^{ade} f^{bce}(\dots)\right)$$

The first term yields:

$$-\frac{1}{8}\left[T_{s-u}^2, [T_t^2, T_{s-u}^2]\right] + \frac{1}{8}\left[\frac{T_t^2}{2}, [T_t^2, T_{s-u}^2]\right]$$

and second term is:

$$-\frac{1}{8}\left[T_t^2, [T_t^2, T_{s-u}^2]\right]$$

where

$$T_s = T_1 + T_2$$
$$T_t = T_2 + T_3$$
$$T_u = T_1 + T_3$$

$$T_{s-u}^2 = \frac{T_s^2 - T_u^2}{2}$$

Colour in the Constant

$$-\left[\frac{1}{4} f^{abe} f^{cde} \left(2\{T_t^a, T_t^c\}\left(\{T_s^b, T_s^d\} + \{T_u^b, T_u^d\}\right) + \frac{1}{2}\{T_s^b - T_u^b, T_s^d - T_u^d\}\{T_s^b + T_u^b, T_s^d + T_u^d\}\right) - \frac{5}{16} C_A^2 T_t . T_t\right]$$

s to u symmetry manifest

arxiv.org/pdf/1701.05241.pdf

Two-parton scattering in the high-energy limit

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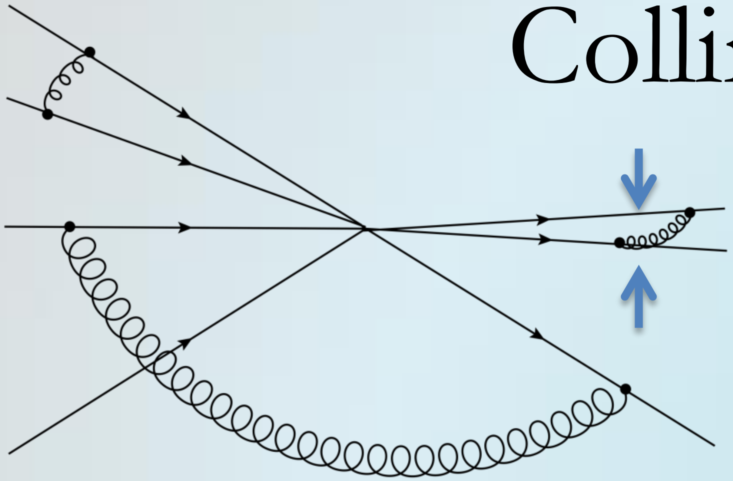
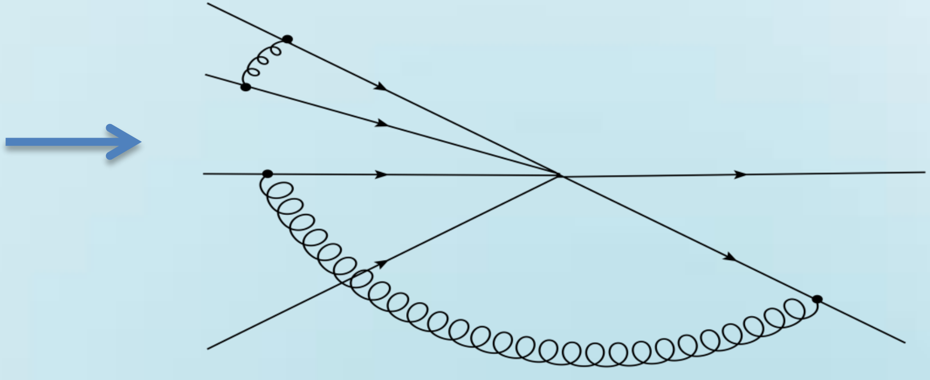
Collinear Limit





$$M_n(p_1, p_2, p_j; \mu, \epsilon_{IR}) \overset{1||2}{\rightarrow} \mathrm{Sp}(p_1, p_2; \mu, \epsilon_{IR}) M_{n-1}(P, p_j; \mu, \epsilon_{IR})$$

Considering the Hard scattering function and definition of Γ_n

$$\Gamma_{sp}\left(p_1, p_2, \mu_f, \alpha\left(\mu_f^2\right)\right) = \Gamma_n\left(p_i, \mu_f, \alpha\left(\mu_f^2\right)\right) - \Gamma_{n-1}\left(P, p_i, \mu_f, \alpha\left(\mu_f^2\right)\right)$$

Collinear Limit

Using previous definitions

$$\Gamma_{\text{sp}}\left(p_1, p_2, \mu_f, \alpha\left(\mu_f^2\right)\right) = \Gamma_{\text{sp}}^{\text{dip}}\left(p_1, p_2, \mu_f, \alpha\left(\mu_f^2\right)\right) + \Delta_n\left(\rho_{ijkl}, \alpha\left(\mu_f^2\right)\right) - \Delta_{n-1}\left(\rho_{ijkl}, \alpha\left(\mu_f^2\right)\right)$$

$$\Gamma_{\text{sp}}^{\text{dip}}\left(p_1, p_2, \mu_f, \alpha\left(\mu_f^2\right)\right) = \Gamma_n^{\text{dip}}\left(p_i, \mu_f, \alpha\left(\mu_f^2\right)\right) - \Gamma_{n-1}^{\text{dip}}\left(P, p_i, \mu_f, \alpha\left(\mu_f^2\right)\right)$$

Two Particle Collinear Limit

$$\Gamma_{\mathrm{sp}}^{\mathrm{dip}}(p_1, p_2, \mu_f, \alpha(\mu_f^2)) = \Gamma_n^{\mathrm{dip}}(p_i, \mu_f, \alpha(\mu_f^2)) - \Gamma_{n-1}^{\mathrm{dip}}(P, p_i, \mu_f, \alpha(\mu_f^2))$$

Parametrise the timelike splitting as $p_1 = zP$, $p_2 = (1-z)P$, $T = T_1 + T_2$

$$\Gamma_n^{\mathrm{dip}}(\{p_i\}, \mu_f, \alpha(\mu_f^2)) = -\frac{1}{2}\hat{\gamma}_K(\alpha_s)\sum_{i<j}\log\left(-\frac{s_{ij}}{\lambda^2}\right)T_i.T_j + \sum_{i=1}^{n}\gamma_{J_i}(\alpha_s)$$

$$\Gamma_{sp}^{dip}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = -\frac{2}{4}\gamma_k(\alpha_s(\lambda^2))\left[\log\left(\frac{2|p_1\cdot p_2|e^{-i\pi\lambda_{12}}}{\lambda^2}\right)T_1\cdot T_2 - \log\left(z\right)T_1\cdot(T_1+T_2) - \log\left((1-z)\right)T_2\cdot(T_1+T_2)\right] + \gamma_{J_1}(\alpha_s(\lambda^2)) + \gamma_{J_1}(\alpha_s(\lambda^2)) - \gamma_{J_P}(\alpha_s(\lambda^2))$$

Two Particle Collinear Limit

Recall beyond dipole corrections

$$\Delta_n(z,\bar{z}) = \frac{1}{(4\pi)^3} 16 f^{abe} f^{cde} \Big[\sum_{1\le i<j<k<l\le n} T_i^a T_j^b T_k^c T_l^d \left(F(1-1/z) - F(^1/_z)\right)$$
$$+T_i^a T_k^b T_j^c T_l^d \left(F(1-z) - F(z)\right)$$
$$+T_i^a T_l^b T_j^c T_k^d \left(F(^1/_{(1-z)}) - F(1-{}^1/_{(1-z)})\right)\}$$
$$-(\zeta_5 + 2\zeta_2\zeta_3) \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ j,k\ne i}} \{T_i^a, T_i^d\} T_j^b T_k^c \Big]$$

$$\Delta_{sp} = \Delta_n - \Delta_{n-1} = \Delta_3 = -\frac{1}{(4\pi)^3} 24 \left[f^{abe} f^{cde} \{T_1^a, T_1^c\}\{T_2^b, T_2^d\} + \frac{1}{2} C_A^2 T_1 . T_2 \right]$$

Two Particle Collinear Limit

Recall the answer for $F(z)$

$$F_{\text{actual}}(z) = \mathcal{L}_{10101}(z) + 2\,\zeta_2[\mathcal{L}_{100}(z) + \mathcal{L}_{001}(z)]$$

$$\mathscr{F}(\rho_{ijkl}, \rho_{ilkj}) = F(1 - z_{ijkl}) - F(z_{ijkl})$$

$$\mathscr{F}(\rho_{ikjl}, \rho_{iljk}) = F\left(1 - \frac{1}{z_{ijkl}}\right) - F\left(\frac{1}{z_{ijkl}}\right)$$

$$\mathscr{F}(\rho_{ijlk}, \rho_{ikjl}) = F\left(\frac{1}{1 - z_{ijkl}}\right) - F\left(\frac{z_{ijkl}}{z_{ijkl} - 1}\right)$$

Two Particle Collinear Limit

$$\Delta_n(z,\bar{z}) = \frac{1}{(4\pi)^3} 8 T_i^a T_j^b T_k^c T_l^d \sum_{1\le i<k<l\le n} \Big[\left(f^{ace} f^{bde} + f^{ade} f^{bce}\right) \left[\mathscr{F}\left(\rho_{ijkl}, \rho_{ilkj}\right) + \mathscr{F}\left(\rho_{ijlk}, \rho_{iklj}\right)\right]$$

$$+ \left(f^{ace} f^{bde} - f^{ade} f^{bce}\right) \left[\mathscr{F}\left(\rho_{ijkl}, \rho_{ilkj}\right) - \mathscr{F}\left(\rho_{ijlk}, \rho_{iklj}\right) + 2\mathscr{F}\left(\rho_{ikjl}, \rho_{iljk}\right)\right]\Big]$$

$$- \frac{1}{(4\pi)^3} 16(\zeta_5 + 2\zeta_2\zeta_3) f^{abe} f^{cde} \sum_{i=1}^{n} \sum_{\substack{1\le j<k\le n \\ j,k\ne i}} \{T_i^a, T_i^d\} T_j^b T_k^c$$

$$\Delta_n = \frac{1}{(4\pi)^3} (A_n + B_n)$$

Two Particle Collinear Limit

$$\Delta_n^{(3)} - \Delta_{n-1}^{(3)} = 8 f^{abe} f^{cde} \Bigg[\sum_{3\le k<l\le n} \Big(\big(\mathcal{F}(\rho_{12kl}, \rho_{1lk2}) - \mathcal{F}(\rho_{12lk}, \rho_{1kl2}) + 2\mathcal{F}(\rho_{1k2l}, \rho_{1l2k})\big) T_1^a T_2^b T_k^c T_l^d$$
$$+ \big(\mathcal{F}(\rho_{12kl}, \rho_{1lk2}) + \mathcal{F}(\rho_{12lk}, \rho_{1kl2}) - 4C\big) T_1^a T_k^b T_2^c T_l^d$$
$$+ \big(\mathcal{F}(\rho_{12kl}, \rho_{1lk2}) + \mathcal{F}(\rho_{12lk}, \rho_{1kl2}) - 4C\big) T_1^a T_l^b T_2^c T_k^d \Big)$$
$$+ 2C \sum_{i=3}^{n} \Big(\{T_i^a, T_i^c\} T_1^b T_2^d + \{T_1^a, T_1^c\} T_2^b T_i^d + \{T_2^a, T_2^c\} T_1^b T_i^d \Big) \Bigg]$$

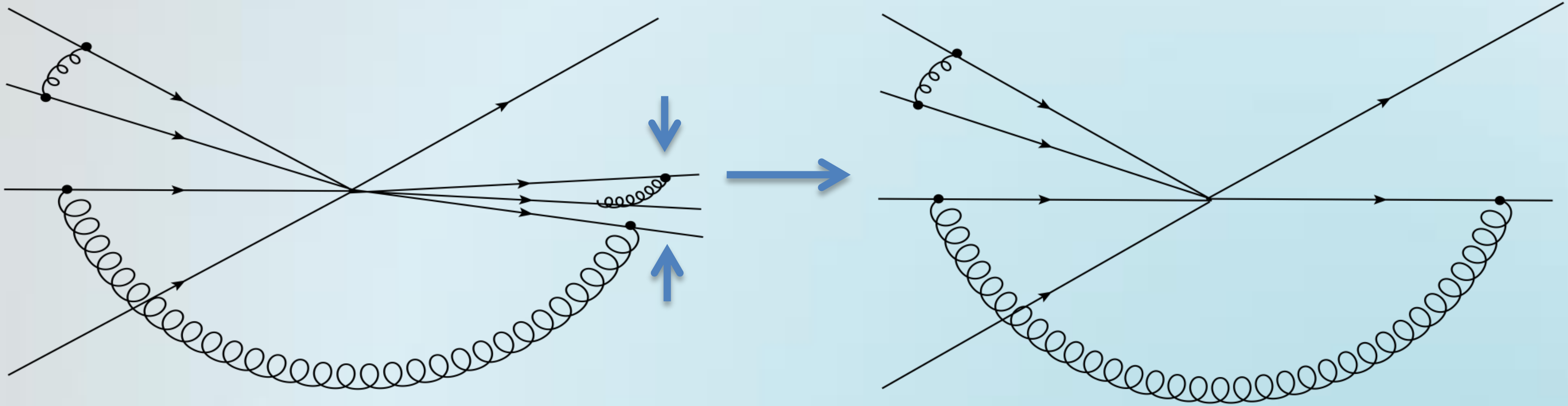
$$\Delta_{Sp}^{(3)} = \Delta_n^{(3)} - \Delta_{n-1}^{(3)} = 16C \Big[-\frac{3}{2} f^{abe} f^{cde} \{T_1^a, T_1^c\}\{T_2^d, T_2^b\} - \frac{3}{4} C_A^2 T_1 \cdot T_2 \Big]$$

$$\big(\mathcal{F}(\rho_{12kl}, \rho_{1lk2}) - \mathcal{F}(\rho_{12lk}, \rho_{1kl2}) + 2\mathcal{F}(\rho_{1k2l}, \rho_{1l2k})\big) = 0$$

$$\frac{1}{2}\big(\mathcal{F}(\rho_{12kl}, \rho_{1lk2}) + \mathcal{F}(\rho_{12lk}, \rho_{1kl2}) - 4C\big) = 2C$$

$$\rho_{12kl} = \frac{P^2\,(p_k.p_l)}{(P.p_k)(P.p_l)} \to 0$$

Three Particle Collinear Limit



$$p_1 = zwP\,, p_2 = (1-z)wP\,, p_3 = (1-w)P \qquad T_P = T_1 + T_2 + T_3$$

$$\Gamma_{\mathrm{sp3}}\left(p_1, p_2, \mu_f, \alpha\left(\mu_f^2\right)\right) = \Gamma_{\mathrm{sp3}}^{\mathrm{dip}}\left(p_1, p_2, \mu_f, \alpha\left(\mu_f^2\right)\right) + \Delta_n\left(\rho_{ijkl}, \alpha\left(\mu_f^2\right)\right) - \Delta_{n-2}\left(\rho_{ijkl}, \alpha\left(\mu_f^2\right)\right)$$

$$\Gamma_{\mathrm{sp3}}^{\mathrm{dip}}\left(p_1, p_2, \mu_f, \alpha\left(\mu_f^2\right)\right) = \Gamma_n^{\mathrm{dip}}\left(p_i, \mu_f, \alpha\left(\mu_f^2\right)\right) - \Gamma_{n-2}^{\mathrm{dip}}\left(P, p_i, \mu_f, \alpha\left(\mu_f^2\right)\right)$$

Three Particle Collinear Limit

$$
\begin{aligned}
\Gamma^{dip}_{sp3}\Big(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\Big) = -\frac{2}{4}\gamma_k(\alpha_s(\lambda^2))\Bigg[& -\log\Big(zw\Big) T_1 \cdot (T_1 + T_2 + T_3) \\
-\log\Big((1-z)w\Big) T_2 \cdot (T_1 + T_2 + T_3) & - \log\Big((1-w)\Big) T_3 \cdot (T_1 + T_2 + T_3) \\
+\log\Big(\frac{2|p_1 \cdot p_2| e^{-i\pi\lambda_{12}}}{\lambda^2}\Big) T_1 \cdot T_2 & + \log\Big(\frac{2|p_1 \cdot p_3| e^{-i\pi\lambda_{13}}}{\lambda^2}\Big) T_1 \cdot T_3 \\
& + \log\Big(\frac{2|p_2 \cdot p_3| e^{-i\pi\lambda_{23}}}{\lambda^2}\Big) T_2 \cdot T_3 \Bigg] \\
+\gamma_{J_1}(\alpha_s(\lambda^2)) + \gamma_{J_2}(\alpha_s(\lambda^2)) & + \gamma_{J_3}(\alpha_s(\lambda^2)) - \gamma_{J_P}(\alpha_s(\lambda^2))
\end{aligned}
$$

Three Particle Collinear Limit

$$
\begin{aligned}
\Delta_n - \Delta_{n-2} = 8 f^{abe} f^{cde} \Bigg[\sum_{4 \le k < l \le n} \Big[& T_1^a T_2^b T_k^c T_l^d \left(\mathcal{F}(\rho_{12kl}, \rho_{1lk2}) - \mathcal{F}(\rho_{12lk}, \rho_{1kl2}) + 2\mathcal{F}(\rho_{1k2l}, \rho_{1l2k}) \right) \\
& + T_1^a T_3^b T_k^c T_l^d \left(\mathcal{F}(\rho_{13kl}, \rho_{1lk3}) - \mathcal{F}(\rho_{13lk}, \rho_{1kl3}) + 2\mathcal{F}(\rho_{1k3l}, \rho_{1l3k}) \right) \\
& + T_2^a T_3^b T_k^c T_l^d \left(\mathcal{F}(\rho_{23kl}, \rho_{2lk3}) - \mathcal{F}(\rho_{23lk}, \rho_{2kl3}) + 2\mathcal{F}(\rho_{2k3l}, \rho_{2l3k}) \right) \\
& + \Big(\left(\mathcal{F}(\rho_{12kl}, \rho_{1lk2}) + \mathcal{F}(\rho_{12lk}, \rho_{1kl2}) \right) - 4C \Big) T_1^a T_k^b T_2^c T_l^d \\
& + \Big(\left(\mathcal{F}(\rho_{12kl}, \rho_{1lk2}) + \mathcal{F}(\rho_{12lk}, \rho_{1kl2}) \right) - 4C \Big) T_1^a T_l^b T_2^c T_k^d \\
& + \Big(\left(\mathcal{F}(\rho_{13kl}, \rho_{1lk3}) + \mathcal{F}(\rho_{13lk}, \rho_{1kl3}) \right) - 4C \Big) T_1^a T_k^b T_3^c T_l^d \\
& + \Big(\left(\mathcal{F}(\rho_{13kl}, \rho_{1lk3}) + \mathcal{F}(\rho_{13lk}, \rho_{1kl3}) \right) - 4C \Big) T_1^a T_l^b T_3^c T_k^d \\
& + \Big(\left(\mathcal{F}(\rho_{23kl}, \rho_{2lk3}) + \mathcal{F}(\rho_{23lk}, \rho_{2kl3}) \right) - 4C \Big) T_2^a T_k^b T_3^c T_l^d \\
& + \Big(\left(\mathcal{F}(\rho_{23kl}, \rho_{2lk3}) + \mathcal{F}(\rho_{23lk}, \rho_{2kl3}) \right) - 4C \Big) T_2^a T_l^b T_3^c T_k^d \Big] \\
& + 2C \sum_{i=4}^{n} \left[\{T_i^a, T_i^c\} T_1^b T_2^d + \{T_i^a, T_i^c\} T_1^b T_3^d + \{T_i^a, T_i^c\} T_2^b T_3^d \right] \\
+ \sum_{4 \le l \le n} \Big[& T_1^a T_2^b T_3^c T_l^d \left(\mathcal{F}(\rho_{123l}, \rho_{1l32}) - \mathcal{F}(\rho_{12l3}, \rho_{13l2}) + 2\mathcal{F}(\rho_{132l}, \rho_{1l23}) \right) \\
+ T_1^a T_3^b T_2^c T_l^d & \left(\mathcal{F}(\rho_{123l}, \rho_{1l32}) + \mathcal{F}(\rho_{12l3}, \rho_{13l2}) \right) + T_1^a T_l^b T_2^c T_3^d \left(\mathcal{F}(\rho_{123l}, \rho_{1l32}) + \mathcal{F}(\rho_{12l3}, \rho_{13l2}) \right) \Big] \\
& + 2C \sum_{k=4}^{n} \Big[\{T_1^a, T_1^c\} T_2^b T_k^d + \{T_1^a, T_1^c\} T_3^b T_k^d + \{T_3^a, T_3^c\} T_1^b T_k^d \\
& \qquad + \{T_2^a, T_2^c\} T_1^b T_k^d + \{T_2^a, T_2^c\} T_3^b T_k^d + \{T_3^a, T_3^c\} T_2^b T_k^d \Big] \\
& + 2C \left[\{T_1^a, T_1^c\} T_2^b T_3^d + \{T_2^a, T_2^c\} T_1^b T_3^d + \{T_3^a, T_3^c\} T_1^b T_2^d \right] \Bigg]
\end{aligned}
$$

Three Particle Collinear Limit

$$\Delta_n - \Delta_{n-2} = 8f^{abe}f^{cde}\Bigg[-4C\Big[\{T_1^a,T_1^c\}T_2^bT_3^d + \{T_2^a,T_2^c\}T_1^bT_3^d + \{T_3^a,T_3^c\}T_1^bT_2^d\Big]$$

$$+\frac{1}{2}\big(\{T_1^a,T_1^c\}T_2^bT_3^d - \{T_2^a,T_2^c\}T_1^bT_3^d\big)\big(\mathcal{F}(\rho_{123l},\rho_{1l32}) - \mathcal{F}(\rho_{12l3},\rho_{13l2}) + 2\mathcal{F}(\rho_{132l},\rho_{1l23})\big)$$

$$+\Big(\frac{1}{2}\{T_1^a,T_1^c\}T_2^bT_3^d + \frac{1}{2}\{T_2^a,T_2^c\}T_1^bT_3^d - \{T_3^a,T_3^c\}T_1^bT_2^d\Big)\big(\mathcal{F}(\rho_{123l},\rho_{1l32}) + \mathcal{F}(\rho_{12l3},\rho_{13l2})\big)\Bigg]$$

$$+16C\Big[-\frac{3}{4}C_A^2T_1\cdot T_2 - \frac{3}{2}f^{abe}f^{cde}\{T_1^a,T_1^c\}\{T_2^b,T_2^d\}$$

$$-\frac{3}{4}C_A^2T_1\cdot T_3 - \frac{3}{2}f^{abe}f^{cde}\{T_1^a,T_1^c\}\{T_3^b,T_3^d\}$$

$$-\frac{3}{4}C_A^2T_2\cdot T_3 - \frac{3}{2}f^{abe}f^{cde}\{T_2^a,T_2^c\}\{T_3^b,T_3^d\}\Big]$$

Four Particle Collinear Limit

$$A_n = 8 f^{abe} f^{cde} \Bigg[\sum_{5\leq i<j<k<l\leq n} + \sum_{\substack{5\leq j<k<l\leq n \\ i=1}} + \sum_{\substack{5\leq j<k<l\leq n \\ i=2}} + \sum_{\substack{5\leq j<k<l\leq n \\ i=3}} + \sum_{\substack{5\leq j<k<l\leq n \\ i=4}} + \sum_{\substack{5\leq k<l\leq n \\ (i,j)=(1,2)}}$$

$$+ \sum_{\substack{5\leq k<l\leq n \\ (i,j)=(1,3)}} + \sum_{\substack{5\leq k<l\leq n \\ (i,j)=(1,4)}} + \sum_{\substack{5\leq k<l\leq n \\ (i,j)=(2,3)}} + \sum_{\substack{5\leq k<l\leq n \\ (i,j)=(2,4)}} + \sum_{\substack{5\leq k<l\leq n \\ (i,j)=(3,4)}}$$

$$+ \sum_{\substack{5\leq l\leq n \\ (i,j,k)=(1,2,3)}} + \sum_{\substack{5\leq l\leq n \\ (i,j,k)=(1,2,4)}} + \sum_{\substack{5\leq l\leq n \\ (i,j,k)=(1,3,4)}} + \sum_{\substack{5\leq l\leq n \\ (i,j,k)=(2,3,4)}}$$

$$+ \Big[\big(\mathcal{F}(\rho_{1234}, \rho_{1432}) - \mathcal{F}(\rho_{1243}, \rho_{1342}) + 2\mathcal{F}(\rho_{1324}, \rho_{1423})\big) T_1^a T_2^b T_3^c T_4^d$$

$$- \big(\mathcal{F}(\rho_{1234}, \rho_{1432}) + \mathcal{F}(\rho_{1243}, \rho_{1342})\big) T_1^a T_3^b T_4^c T_2^d + \big(\mathcal{F}(\rho_{1234}, \rho_{1432}) + \mathcal{F}(\rho_{1243}, \rho_{1342})\big) T_1^a T_4^b T_2^c T_3^d \Big] \Bigg]$$

Four Particle Collinear Limit

$$
\begin{aligned}
\Delta_n - \Delta_{n-3} = 8f^{abe}f^{cde}\Big[4C\Big[&f^{abe}f^{cde}T_1^aT_2^c(T_4^bT_3^d + T_3^bT_4^d) + f^{abe}f^{cde}T_1^aT_3^c(T_4^bT_2^d + T_2^bT_4^d) \\
&+f^{abe}f^{cde}T_1^aT_4^c(T_3^bT_2^d + T_2^bT_3^d) + f^{abe}f^{cde}T_2^aT_3^c(T_1^bT_4^d + T_4^bT_1^d) \\
&+f^{abe}f^{cde}T_2^aT_4^c(T_1^bT_3^d + T_3^bT_1^d) + f^{abe}f^{cde}T_3^aT_4^c(T_1^bT_2^d + T_2^bT_1^d)\Big] \\
-4C\Big[&\{T_1^a,T_1^c\}T_2^bT_3^d + \{T_1^a,T_1^c\}T_2^bT_4^d + \{T_1^a,T_1^c\}T_3^bT_4^d + \{T_2^a,T_2^c\}T_1^bT_3^d + \{T_2^a,T_2^c\}T_1^bT_4^d + \{T_2^a,T_2^c\}T_3^bT_4^d \\
&+\{T_3^a,T_3^c\}T_1^bT_2^d + \{T_3^a,T_3^c\}T_1^bT_4^d + \{T_3^a,T_3^c\}T_2^bT_4^d + \{T_4^a,T_4^c\}T_1^bT_2^d + \{T_4^a,T_4^c\}T_1^bT_3^d + \{T_4^a,T_4^c\}T_2^bT_3^d\Big]\Big] \\
+16C\Big[&-\frac{3}{4}C_A^2T_1\cdot T_2 - \frac{3}{2}f^{abe}f^{cde}\{T_1^a,T_1^c\}\{T_2^b,T_2^d\} - \frac{3}{4}C_A^2T_1\cdot T_3 - \frac{3}{2}f^{abe}f^{cde}\{T_1^a,T_1^c\}\{T_3^b,T_3^d\} \\
&-\frac{3}{4}C_A^2T_2\cdot T_3 - \frac{3}{2}f^{abe}f^{cde}\{T_2^a,T_2^c\}\{T_3^b,T_3^d\} - \frac{3}{4}C_A^2T_1\cdot T_4 - \frac{3}{2}f^{abe}f^{cde}\{T_1^a,T_1^c\}\{T_4^b,T_4^d\} \\
&-\frac{3}{4}C_A^2T_2\cdot T_4 - \frac{3}{2}f^{abe}f^{cde}\{T_2^a,T_2^c\}\{T_4^b,T_4^d\} - \frac{3}{4}C_A^2T_3\cdot T_4 - \frac{3}{2}f^{abe}f^{cde}\{T_3^a,T_3^c\}\{T_4^b,T_4^d\}\Big]
\end{aligned}
$$

Four Particle Collinear Limit

$$
\begin{aligned}
\Delta_n - \Delta_{n-3} = 8 f^{abe} f^{cde} \Bigg[-4C \Big[&\{T_1^a, T_1^c\} T_2^b T_3^d + \{T_1^a, T_1^c\} T_2^b T_4^d + \{T_1^a, T_1^c\} T_3^b T_4^d \\
&+ \{T_2^a, T_2^c\} T_1^b T_3^d + \{T_2^a, T_2^c\} T_1^b T_4^d + \{T_2^a, T_2^c\} T_3^b T_4^d \\
+ \{T_3^a, T_3^c\} T_1^b T_2^d + \{T_3^a, T_3^c\} T_1^b T_4^d + \{T_3^a, T_3^c\} T_2^b T_4^d &+ \{T_4^a, T_4^c\} T_1^b T_2^d + \{T_4^a, T_4^c\} T_1^b T_3^d + \{T_4^a, T_4^c\} T_2^b T_3^d \Big] \Bigg] \\
+16C \Big[-\frac{3}{4} C_A^2 T_1 \cdot T_2 - \frac{3}{2} f^{abe} f^{cde} \{T_1^a, T_1^c\} \{T_2^b, T_2^d\} &- \frac{3}{4} C_A^2 T_1 \cdot T_3 - \frac{3}{2} f^{abe} f^{cde} \{T_1^a, T_1^c\} \{T_3^b, T_3^d\} \\
-\frac{3}{4} C_A^2 T_2 \cdot T_3 - \frac{3}{2} f^{abe} f^{cde} \{T_2^a, T_2^c\} \{T_3^b, T_3^d\} &- \frac{3}{4} C_A^2 T_1 \cdot T_4 - \frac{3}{2} f^{abe} f^{cde} \{T_1^a, T_1^c\} \{T_4^b, T_4^d\} \\
-\frac{3}{4} C_A^2 T_2 \cdot T_4 - \frac{3}{2} f^{abe} f^{cde} \{T_2^a, T_2^c\} \{T_4^b, T_4^d\} &- \frac{3}{4} C_A^2 T_3 \cdot T_4 - \frac{3}{2} f^{abe} f^{cde} \{T_3^a, T_3^c\} \{T_4^b, T_4^d\} \Big]
\end{aligned}
$$

Summary

- Soft singularities and the soft anomalous dimension
- Three loop correction
- Explored the colour structure in the high energy limit
- Collinear limits
  - Splitting amplitude computed for three and four collinear partons
  - Only dependant on partons going collinear as expected.

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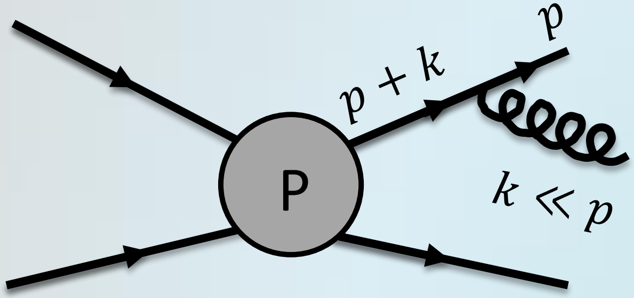
Eikonal Feynman Rules



- Simplified Feynman rules
- Rescaling Invariance

$$\bar{u}(p)[-ig_s T^a \gamma^\mu] \frac{i(\not{p}+\not{k}+m)}{(p+k)^2 - m^2 + i\epsilon} \underset{k \ll p}{\Longrightarrow} \bar{u}(p) g_s T^a \frac{p^\mu}{p.k + i\epsilon} \rightarrow \quad g_s T^a \frac{\beta^\mu}{\beta.k + i\epsilon}$$

$$\Phi_{\beta_i}(0,\infty) = Pexp\left[ig_s \int_0^\infty d\lambda \beta^\mu A_\mu(\lambda\beta^\mu) T^a\right]$$

Soft Anomalous Dimension

Z is renormalized multiplicatively hence we define Γ_n as

$$\frac{d}{d\ln\mu_f} Z_n(\{p_i\},\mu_f,\alpha(\mu_f^2),\epsilon_{IR}) = -Z_n(\{p_i\},\mu_f,\alpha(\mu_f^2),\epsilon_{IR})\Gamma_\mathrm{n}(\{p_i\},\mu_f,\alpha(\mu_f^2))$$

And solving

$$Z_n(\{p_i\},\mu_f,\alpha(\mu_f^2),\epsilon_{IR}) = Pexp\left\{-\frac{1}{2}\int_0^{\mu_f^2}\frac{d\lambda^2}{\lambda^2}\Gamma_n(\{p_i\},\lambda,\alpha_s(\lambda^2))\right\}$$

Γ_n is the soft anomalous dimension.

Colour Considerations

- First consider the colour structure in the part of the correction that contains the kinematic functions.

$$T_1^a T_2^b T_3^c T_4^d (f^{abe} f^{cde}(\dots) + f^{ace} f^{bde}(\dots) + f^{ade} f^{bce}(\dots))$$

$$T_s^2 + T_t^2 + T_u^2 = \sum_{i=1}^{4} T_i^2$$

$$f^{ace} f^{bde} = f^{abe} f^{cde} + f^{ade} f^{bce}$$

$$\rightarrow \quad T_1^a T_2^b T_3^c T_4^d (f^{abe} f^{cde}(\dots) + f^{ade} f^{bce}(\dots))$$

$$[T_1^a, T_1^b] = i f^{abc} T_1^c$$

- Consider the first term.

$$\left(\sum_{i=1}^{4} T_i^a\right) H_n = 0$$

$$\rightarrow \quad [T_1^b, T_1^c] T_2^b [T_3^c, T_3^d] T_4^d$$

$$\rightarrow \quad [T_3 . T_3, [T_1 . T_2, T_1 . T_3]] + [T_2 . T_3, [T_1 . T_2, T_1 . T_3]] + [T_1 . T_3, [T_1 . T_2, T_1 . T_3]]$$

Colour in the Constant

- The constant term has the form

$$B_4 = f^{abe} f^{cde} \sum_{i=1}^{4} \sum_{\substack{1\leq j<k\leq n \\ j,k\neq i}} \{T_i^a, T_i^c\} T_j^b T_k^d$$

$$T_A = T_1 + T_2 \qquad T_C = T_3 - T_4$$

$$T_B = T_1 - T_2 \qquad T_D = T_3 + T_4$$

$$B_4' = -\left[\frac{1}{4} f^{abe} f^{cde} \left(\{T_A^a, T_A^c\}\{T_B^b, T_B^d\} + \{T_C^a, T_C^c\}\{T_D^b, T_D^d\} + \frac{1}{2}\{T_B^a, T_B^c\}\{T_C^b, T_C^d\} \right) - \frac{5}{16} C_A^2 T_A . T_A \right]$$

$T_1^a T_2^c T_1^b T_2^d$ and $T_1^a T_2^c T_3^b T_4^d$ → Can bring to a form where contains only

$$T_1^a T_1^b = \frac{1}{2}\{T_1^a, T_1^b\} + \frac{1}{2}[T_1^a, T_1^b]$$

$$f^{abe} f^{cde} \{T_1^a, T_1^c\} T_3^b T_4^d$$

$$f^{abe} f^{cde} \{T_1^a, T_1^c\}\{T_2^b, T_2^d\}$$

$$T_i . T_j$$

Colour in the Constant

$$f^{abe}f^{cde}\{T_1^a,T_1^c\}\{T_2^b,T_2^d\} = -2af^{abe}f^{cde}\{T_1^a,T_1^c\}T_2^bT_3^d - 2af^{abe}f^{cde}\{T_1^a,T_1^c\}T_2^bT_4^d$$

$$-2(1-a)f^{abe}f^{cde}\{T_2^a,T_2^c\}T_1^bT_3^d - 2(1-a)f^{abe}f^{cde}\{T_2^a,T_2^c\}T_1^bT_4^d - \frac{1}{2}C_A^2T_1.T_2$$

Dipole contributions cancel.

$$B_4' = -\left[f^{abe}f^{cde}\left\{\left(\frac{1}{2}-2a-\frac{1}{4}c\right)\{T_1^a,T_1^c\}T_2^bT_3^d + \left(-2a-\frac{1}{4}d\right)\{T_1^a,T_1^c\}T_2^bT_4^d + \ \ldots\right\}\right]$$

We can solve these equations, when coefficient is of each term is equal to -1 to match B_4

$$a = \frac{1}{2}, b = \frac{5}{8}, c = 2, d = 0, e = 2, f = 0$$

Colour in the Constant

Identify $T_A = T_s$ $T_B = T_u - T_t$ $T_C = T_u + T_t$ $T_D = -T_s$

→

$$B_4' = -\left[\frac{1}{4} f^{abe} f^{cde} \left(\{T_s^a, T_s^c\}\{T_u^b - T_t^b, T_u^d - T_t^d\} + \{T_u^b + T_t^b, T_u^d + T_t^d\}\{T_s^a, T_s^c\} \right.\right.$$

$$\left.\left. + \frac{1}{2}\{T_u^b - T_t^b, T_u^d - T_t^d\}\{T_u^b + T_t^b, T_u^d + T_t^d\} \right) - \frac{5}{16} C_A^2 T_s . T_s \right]$$

$s \to t \to u \to s$

$$B_4' = -\left[\frac{1}{4} f^{abe} f^{cde} \left(\{T_t^a, T_t^c\}\{T_s^b - T_u^b, T_s^d - T_u^d\} + \{T_t^a, T_t^c\}\{T_s^b + T_u^b, T_s^d + T_u^d\} \right.\right.$$

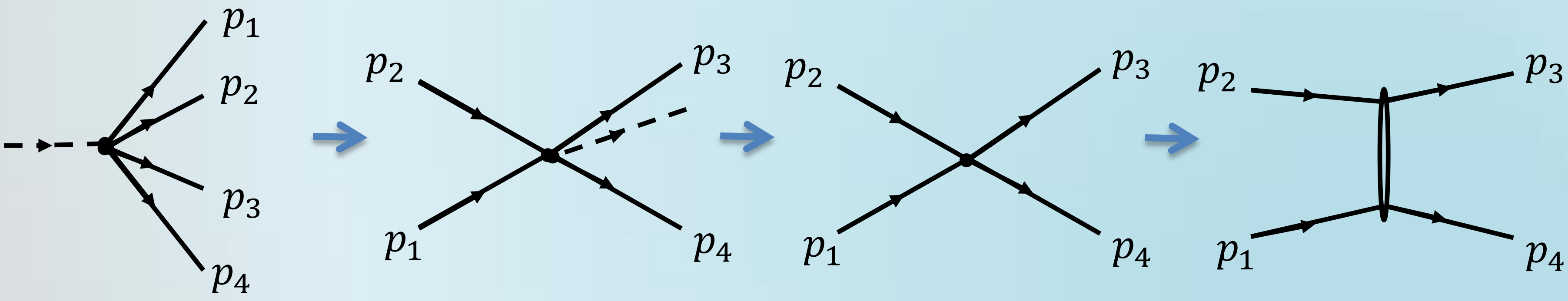
$$\left.\left. + \frac{1}{2}\{T_s^b - T_u^b, T_s^d - T_u^d\}\{T_s^b + T_u^b, T_s^d + T_u^d\} \right) - \frac{5}{16} C_A^2 T_t . T_t \right]$$

Kinematics

- Recall earlier we had the correction written in terms of functions $F(z)$ defined as:

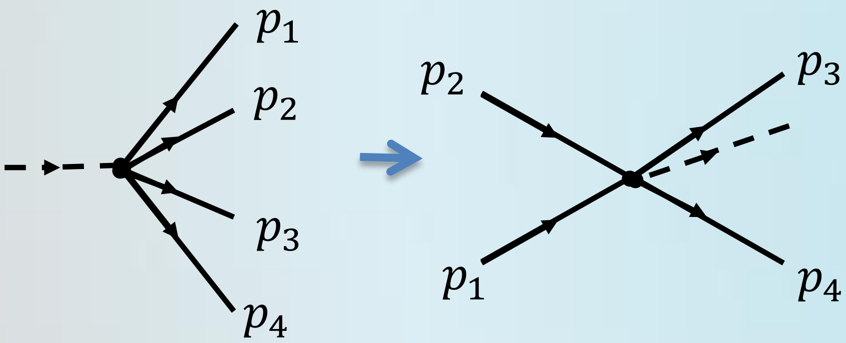
$$F(z) = \mathcal{L}_{10101}(z) + 2\,\zeta_2[\mathcal{L}_{100}(z) + \mathcal{L}_{001}(z)]$$

- $\rho_{1234} = z\bar{z}$ and $\rho_{1432} = (1 - z)(1 - \bar{z})$



Euclidean to Physical Scattering Region

- Recall we defined the invariants $-s_{ij} = 2|p_i.p_j|e^{-i\pi\lambda_{ij}}$
- In the Euclidean region all the invariants are spacelike $p_i.p_j < 0$, λ_{ij}=0 for all i, j.



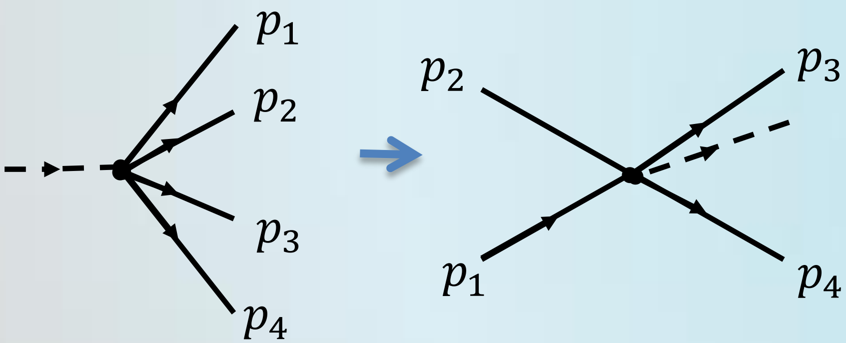
- In the Physical Scattering Region if the partons are both initial or final state then $p_i.p_j > 0$, so λ_{ij}=1 for these and λ_{ij}=0 otherwise.

Euclidean to Physical Scattering Region

- Any pair can be chosen, then CICRs acquire phases. For (1,2) incoming:

$$\rho_{1234} = \frac{(2|p_1.p_2|)(2|p_3.p_4|)}{(2|p_1.p_3|)(2|p_2.p_4|)} \rightarrow \frac{(2|p_1.p_2|e^{-i\pi})(2|p_3.p_4|e^{-i\pi})}{(2|p_1.p_3|)(2|p_2.p_4|)}$$

$$\rho_{1432} = \frac{(2|p_1.p_4|)(2|p_2.p_3|)}{(2|p_1.p_3|)(2|p_2.p_4|)} \rightarrow \frac{(2|p_1.p_4|)(2|p_2.p_3|)}{(2|p_1.p_3|)(2|p_2.p_4|)}$$



Euclidean to Physical Scattering Region

$$z = \frac{1}{2}\left[1 + \rho_{1234} - \rho_{1432} + \sqrt{\lambda(1, \rho_{1234}, \rho_{1234})}\right]$$

$$\bar{z} = \frac{1}{2}\left[1 + \rho_{1234} - \rho_{1432} - \sqrt{\lambda(1, \rho_{1234}, \rho_{1234})}\right]$$

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2bc - 2ac$$

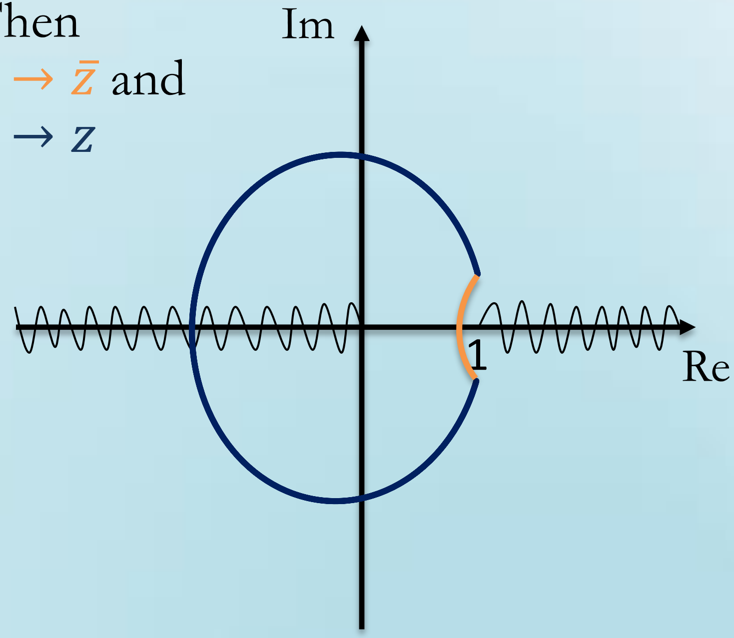
$$F^{cont}(z, \bar{z}) = F(z, \bar{z}) + \text{Disc}$$

Make use of G functions to calculate the Disc.

$$G(a_1, \ldots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots a_n; z)$$

$$\Delta\big(Disc(F)\big) = (Disc \otimes 1). \Delta F$$

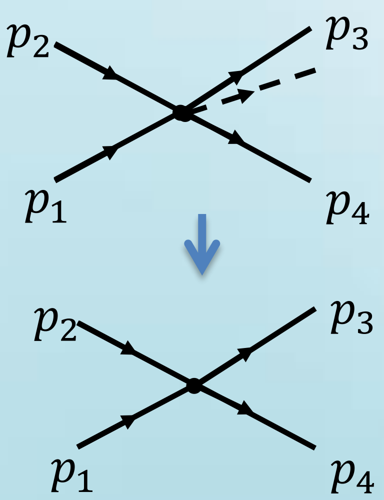
- Then
  $z \rightarrow \bar{z}$ and
  $\bar{z} \rightarrow z$



Imposing Momentum Conservation

- Imposing $p_1 + p_2 = p_3 + p_4$ means that now

$$\rho_{1234} = \frac{((p_1+p_2))^4}{((p_1-p_3))^4} \qquad \rho_{1432} = \frac{((p_1-p_4))^4}{((p_1-p_3))^4}$$



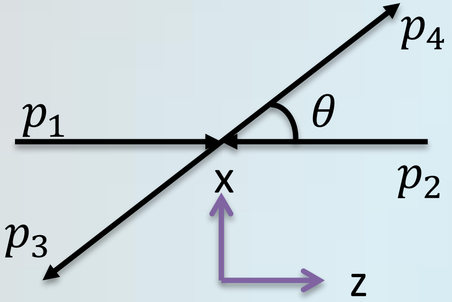
- Then

$$\lambda(1, \rho_{1234}, \rho_{1234}) = 0$$

- Hence $z = \bar{z}$ and using Mandelstam invariants :

$$\rho_{1234} = \frac{s^2}{(s+t)^2} = z\bar{z} \qquad \rho_{1432} = \frac{t^2}{(s+t)^2} = (1-z)(1-\bar{z})$$

High Energy Scattering



$$p_1 = \frac{\sqrt{s}}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \quad p_2 = \frac{\sqrt{s}}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \quad p_4 = \frac{\sqrt{s}}{2}\begin{pmatrix}1\\\sin(\theta)\\0\\\cos(\theta)\end{pmatrix} \quad p_3 = \frac{\sqrt{s}}{2}\begin{pmatrix}1\\-\sin(\theta)\\0\\-\cos(\theta)\end{pmatrix}$$

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1.p_2 = \frac{2s(1+1)}{4} = s$$

$$t = (p_1 - p_4)^2 = p_1^2 + p_4^2 - 2p_1.p_4 = \frac{-2s(1-\cos(\theta))}{4}$$

$$u = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = \frac{-2s(1+\cos(\theta))}{4}$$

Forward Limit: $\theta \ll 1 \Rightarrow s \gg t$ and $s \approx -u$

Backward Limit: $\theta \approx \pi \Rightarrow s \gg u$ and $s \approx -t$

Forward High Energy Scattering

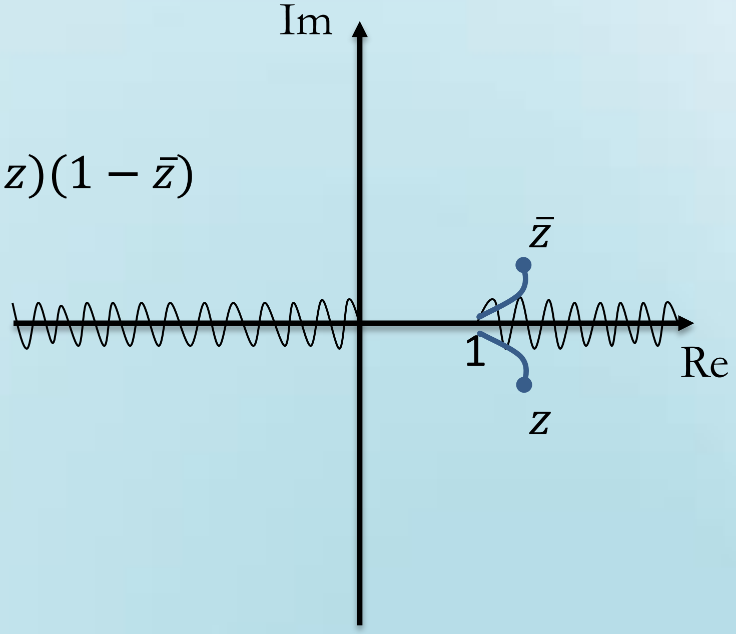
recall

$$\rho_{1234} = \frac{s^2}{(s+t)^2} = z\bar{z}$$

$$\rho_{1432} = \frac{t^2}{(s+t)^2} = (1-z)(1-\bar{z})$$

So

$$z = \bar{z} = \frac{s}{(s+t)} \to 1$$



Forward High Energy Scattering

The final results:

$$f^{ace}f^{bde} = f^{abe}f^{cde} + f^{ade}f^{bce}$$

$$\frac{2i\pi^5}{15} - \frac{4}{3}i\pi\log\left[\frac{s}{-t}\right]^2 + \frac{20}{3}\pi^2\zeta_3 + 8i\pi\log\left[\frac{s}{-t}\right]\zeta_3 - 4\zeta_5$$

$$-\frac{2i\pi^5}{15} + \frac{4}{3}i\pi\log\left[\frac{s}{-t}\right]^2 - \frac{4}{3}\pi^2\zeta_3 - 8i\pi\log\left[\frac{s}{-t}\right]\zeta_3 - 4\zeta_5$$

$$-\frac{38i\pi^5}{45} - \frac{4}{3}i\pi^3\log\left[\frac{s}{-t}\right]^2$$

$$\rightarrow \quad 32\zeta_2\zeta_3 - 8\zeta_5$$

$$-8\zeta_2\zeta_3 - 88\pi\zeta_4 i - 4\zeta_5 + 8\pi\zeta_3\log\left[\frac{-t}{s}\right]i$$

$$\Delta(z,\bar{z}) = \frac{1}{(4\pi)^3}16T_1^aT_2^bT_3^cT_4^d[\,f^{abe}f^{cde}(32\zeta_2\zeta_3 - 8\zeta_5) + f^{ade}f^{bce}(-8\zeta_2\zeta_3 - 88\pi\zeta_4 i - 4\zeta_5 + 8\pi\zeta_3\log\left[\frac{-t}{s}\right]i)]$$

$$-\frac{16}{(4\pi)^3}(\zeta_5 + 2\zeta_2\zeta_3)\sum_{\substack{(i,j,k)\in(1,2,3,4,)\\ j<k}} f^{abe}f^{cde}\{T^a{}_i, T^d{}_i\}T^b{}_jT^c{}_k$$

Backward High Energy Scattering

Recall in this limit: $s \gg u$ and $s \approx -t$

$$\rho_{1234} = \frac{s^2}{u^2} = z\bar{z} \quad \rho_{1432} = \frac{t^2}{u^2} = (1-z)(1-\bar{z}) \qquad z = \bar{z} = \frac{s}{u} \to \infty$$

$$-\frac{2i\pi^5}{15} + \frac{4}{3} i\pi \log\left[\frac{s}{-u}\right]^2 - \frac{20}{3}\pi^2\zeta_3 - 8i\pi\log\left[\frac{s}{-u}\right]\zeta_3 + 4\zeta_5$$

$$-\frac{38i\pi^5}{45} - \frac{4}{3} i\pi^3 \log\left[\frac{s}{-u}\right]^2$$

$$-\frac{2i\pi^5}{15} + \frac{4}{3} i\pi \log\left[\frac{s}{-u}\right]^2 - \frac{4}{3}\pi^2\zeta_3 - 8i\pi\log\left[\frac{s}{-u}\right]\zeta_3 - 4\zeta_5$$

Exploring the Symmetries

$s \gg t$ and $s \approx -u$ ⟷ $s \gg u$ and $s \approx -t$

$$\frac{2i\pi^5}{15} - \frac{4}{3} i\pi \log\left[\frac{s}{-t}\right]^2 + \frac{20}{3}\pi^2\zeta_3 + 8i\pi \log\left[\frac{s}{-t}\right]\zeta_3 - 4\zeta_5$$

$$-\frac{2i\pi^5}{15} + \frac{4}{3} i\pi \log\left[\frac{s}{-t}\right]^2 - \frac{4}{3}\pi^2\zeta_3 - 8i\pi \log\left[\frac{s}{-t}\right]\zeta_3 - 4\zeta_5$$

$$-\frac{38i\pi^5}{45} - \frac{4}{3} i\pi^3 \log\left[\frac{s}{-t}\right]^2$$

$$-\frac{2i\pi^5}{15} + \frac{4}{3} i\pi \log\left[\frac{s}{-u}\right]^2 - \frac{20}{3}\pi^2\zeta_3 - 8i\pi \log\left[\frac{s}{-u}\right]\zeta_3 + 4\zeta_5$$

$$-\frac{38i\pi^5}{45} - \frac{4}{3} i\pi^3 \log\left[\frac{s}{-u}\right]^2$$

$$-\frac{2i\pi^5}{15} + \frac{4}{3} i\pi \log\left[\frac{s}{-u}\right]^2 - \frac{4}{3}\pi^2\zeta_3 - 8i\pi \log\left[\frac{s}{-u}\right]\zeta_3 - 4\zeta_5$$

$$T_1^a T_2^b T_3^c T_4^d \left(f^{abe} f^{cde}(\ldots) + f^{ace} f^{bde}(\ldots) + f^{ade} f^{bce}(\ldots)\right)$$

Recall $t = (p_1 - p_4)^2 = (p_2 - p_3)^2$ and $u = (p_1 - p_3)^2 = (p_2 - p_4)^2$

Derivation of Γ^{sp}

$$M_n(p_1, p_2, p_j; \mu, \epsilon_{IR}) \overset{1||2}{\to} \text{Sp}(p_1, p_2; \mu, \epsilon_{IR}) M_{n-1}(P, p_j; \mu, \epsilon_{IR})$$

Expect some singularities in Sp to come from the hard function. Denote collinear behaviour of hard functions by

$$H_n(p_1, p_2, p_j; \mu, \mu_f) \overset{1||2}{\to} \text{Sp}_H(p_1, p_2; \mu, \mu_f) H_{n-1}(P, p_j; \mu, \mu_f)$$

Recall

$$M_n(p_i, \mu, \alpha(\mu^2), \epsilon_{IR}) = Z_n(p_i, \mu_f, \alpha(\mu_f^2), \epsilon_{IR}) H_n(p_i, \mu, \mu_f, \alpha(\mu^2))$$

$$M_{n-1}(P, p_i, \mu, \alpha(\mu^2), \epsilon_{IR}) = Z_{n-1}(P, p_i, \mu_f, \alpha(\mu_f^2), \epsilon_{IR}) H_{n-1}(P, p_i, \mu, \mu_f, \alpha(\mu^2))$$

Hence

$$M_n(p_1, p_2, p_i; \mu, \epsilon_{IR}) \overset{1||2}{\to} \text{Sp}(p_1, p_2; \mu, \epsilon_{IR}) Z_{n-1}(P, p_i, \mu_f, \alpha(\mu_f^2), \epsilon_{IR}) H_{n-1}(P, p_i, \mu, \mu_f, \alpha(\mu^2))$$

$$M_n(p_i, \mu, \alpha(\mu^2), \epsilon_{IR}) \overset{1||2}{\to} Z_n(p_i, \mu_f, \alpha(\mu_f^2), \epsilon_{IR}) \text{Sp}_H(p_1, p_2; \mu) H_{n-1}(P, p_j; \mu, \mu_f)$$

Derivation of Γ^{sp}

Combining the two equations

$$\text{Sp}_H(p_1, p_2; \mu, \mu_f) = Z_n(p_i, \mu_f, \alpha(\mu_f^2), \epsilon_{IR})^{-1}\text{Sp}(p_1, p_2; \mu, \epsilon_{IR})Z_{n-1}(P, p_i, \mu_f, \alpha(\mu_f^2), \epsilon_{IR})$$

Z is renormalized multiplicatively hence we define Γ as

$$\frac{d}{d\ln\mu_f}Z_n(\{p_i\}, \mu_f, \alpha(\mu_f^2), \epsilon_{IR}) = -Z_n(\{p_i\}, \mu_f, \alpha(\mu_f^2), \epsilon_{IR})\Gamma_\text{n}(\{p_i\}, \mu_f, \alpha(\mu_f^2))$$

Taking the logarithmic derivative of top equation

$$\frac{d}{d\ln\mu_f}\text{Sp}_H(p_1, p_2; \mu, \mu_f) = \Gamma_n(p_i, \mu_f, \alpha(\mu_f^2))\text{Sp}_H(p_1, p_2; \mu, \mu_f)$$

$$-\text{Sp}_H(p_1, p_2; \mu, \mu_f)\Gamma_{n-1}(P, p_i, \mu_f, \alpha(\mu_f^2))$$

Derivation of Γ^{sp}

Then

$$\frac{d}{d\ln\mu_f}\mathrm{Sp}_H(p_1,p_2;\mu,\mu_f) = \Gamma_{sp}(p_1,p_2,\mu_f,\alpha(\mu_f^2))\mathrm{Sp}_H(p_1,p_2;\mu,\mu_f)$$

Where

$$\Gamma_{sp}(p_1,p_2,\mu_f,\alpha(\mu_f^2)) = \Gamma_n(p_i,\mu_f,\alpha(\mu_f^2)) - \Gamma_{n-1}(P,p_i,\mu_f,\alpha(\mu_f^2))$$

Two Particle Collinear Limit

$$\lim_{1\|2} A_n - A_{n-1} = 8 f^{abe} f^{cde} \sum_{\substack{3\leq k<l\leq n \\ (i,j)=(1,2)}} \Big[\big(\mathcal{F}(\rho_{ijkl},\rho_{ilkj}) - \mathcal{F}(\rho_{ijlk},\rho_{iklj}) + 2\mathcal{F}(\rho_{ikjl},\rho_{iljk})\big) T_i^a T_j^b T_k^c T_l^d$$

$$-\big(\mathcal{F}(\rho_{ijkl},\rho_{ilkj}) + \mathcal{F}(\rho_{ijlk},\rho_{iklj})\big) T_i^a T_k^b T_l^c T_j^d + \big(\mathcal{F}(\rho_{ijkl},\rho_{ilkj}) + \mathcal{F}(\rho_{ijlk},\rho_{iklj})\big) T_i^a T_l^b T_j^c T_k^d$$

$$B_n - B_{n-1} = 16 C f^{abe} f^{cde} \Big[-2 \sum_{3\leq k<l\leq n} \big(T_1^a T_k^b T_2^c T_l^d + T_1^a T_l^b T_2^c T_k^d \big)$$

$$\sum_{i=3}^{n} \big(\{T_i^a, T_i^c\} T_1^b T_2^d + \{T_1^a, T_1^c\} T_2^b T_i^d + \{T_2^a, T_2^c\} T_1^b T_i^d \big) \Big]$$

Three Particle Collinear Limit

$$A_{n-2} = 8 f^{abe} f^{cde} \Bigg[\sum_{4\leq i<j<k<l\leq n} \Big[\big(\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) - \mathcal{F}(\rho_{ijlk}, \rho_{iklj}) + 2\mathcal{F}(\rho_{ikjl}, \rho_{iljk})\big) T_i^a T_j^b T_k^c T_l^d$$

$$- \big(\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) + \mathcal{F}(\rho_{ijlk}, \rho_{iklj})\big) T_i^a T_k^b T_l^c T_j^d + \big(\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) + \mathcal{F}(\rho_{ijlk}, \rho_{iklj})\big) T_i^a T_l^b T_j^c T_k^d \Big]$$

$$+ \sum_{\substack{4\leq j<k<l\leq n \\ i=P}} \Big[\big(\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) - \mathcal{F}(\rho_{ijlk}, \rho_{iklj}) + 2\mathcal{F}(\rho_{ikjl}, \rho_{iljk})\big) T_i^a T_j^b T_k^c T_l^d$$

$$- \big(\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) + \mathcal{F}(\rho_{ijlk}, \rho_{iklj})\big) T_i^a T_k^b T_l^c T_j^d + \big(\mathcal{F}(\rho_{ijkl}, \rho_{ilkj}) + \mathcal{F}(\rho_{ijlk}, \rho_{iklj})\big) T_i^a T_l^b T_j^c T_k^d \Big] \Bigg]$$