Stringy T^3-fibrations, T-folds and Mirror Symmetry

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Overview

Introduction

Target space

Super string theory lives on a 10-dimensional pseudo-Riemannian manifold, e.g.

$$M = \mathbb{R}^4 \times T^6$$

with metric

$$G = \begin{pmatrix} \eta_{\mu\nu} & \\ & G_{T^6} \end{pmatrix}$$

In general we have $M = \Sigma_{\mu\nu} \times X$, with $\Sigma_{\mu\nu}$ a solution to Einsteins equation. M is a solution to the supergravity equations of motion.

Non-geometric backgrounds

X need not be a manifold. Exotic backgrounds can lead to non-commutative and non-associative gravity.

T-duality

Example

Somtimes different backgrounds yield the same physics

$$\begin{pmatrix} \text{IIA} \\ \mathbb{R}^9 \times S^1, R^2 dt^2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{IIB} \\ \mathbb{R}^9 \times S^1, \frac{1}{R^2} dt^2 \end{pmatrix}$$

More generally T-duality for torus compactifications is an $O(D, D; \mathbb{Z})$-transformation

$$(T^D, G, B, \Phi) \longleftrightarrow (\hat{T}^D, \hat{G}, \hat{B}, \hat{\Phi})$$

Mirror Symmetry

Hodge diamond of three-fold X and its mirror $\hat{X}$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{1,2} & & h^{1,2} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\longleftrightarrow
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,2} & & 0 & \\
1 & & h^{1,1} & & h^{1,1} & & 1 \\
 & 0 & & h^{1,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Mirror Symmetry

This operation induces the following symmetry

$$
\begin{pmatrix} \text{IIA} \\ X, g \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{IIB} \\ \hat{X}, \hat{g} \end{pmatrix}
$$

Mirror Symmetry is T-duality!?

SYZ-conjecture [Strominger, Yau, Zaslow, '96]

Consider singular bundles

$$\begin{array}{ccc} T^3 \longrightarrow & X & \\ & \downarrow & \\ & S^3 & \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} \hat{T}^3 \longrightarrow & \hat{X} & \\ & \downarrow & \\ & S^3 & \end{array}$$

Apply T-duality along the smooth fibers. Then fill in singular fibers 'as needed'.

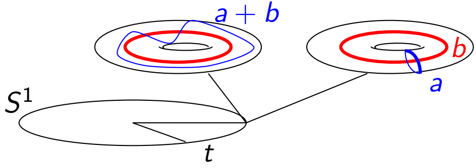
Weak SYZ-conjecture

Gross-Wilson/Kontsevich-Soibelman/Todorov

- Many technical issues with SYZ-conjecture, like existence of SLAG fibrations, etc.
- Proved for K3-surfaces in [Gross,Wilson '00]
- Weaker version still remains open problem for CY_3
- Key item: find Ooguri-Vafa type of metric near singular locus of fibration. Very hard!

Local model of the $K3$ surface

Consider the bundle with a Dehn twist $\phi \in SL(2;\mathbb{Z}) < O(2,2;\mathbb{Z})$ as monodromy:



Promote complex structure modulus τ to a function on the base

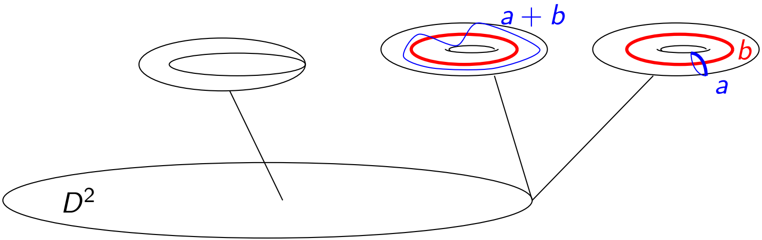
$$\tau \longrightarrow \tau(t) = \exp(\log(\phi)t) \cdot \tau_0,$$

and obtain metric on the total space, with $\tau_0 = i$:

$$ds^2 = dt^2 + dx^2 + (tdx + dy)^2$$

Local model of the $K3$ surface

Now extend the base:



Promote

$$\tau(t) \to \tau(r, t) = \exp(\log(\phi)t) \cdot \tau_0(r).$$

Local model for the K3 surface

Semi-flat metric

On $D^2 \setminus \{0\}$, set

$$\tau(r,t) = t + i\log(\mu/r), \quad \mu > 0.$$

Then

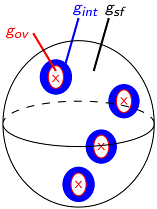
$$ds^2 = \log(\mu/r)(r^2 dt^2 + dr^2 + dx^2) + \frac{1}{\log(\mu/r)}(tdx + dy)^2$$

Semi-flat approximation of a KK-monopole smeared on a circle. Extend over singular fiber by means of the Ooguri-Vafa metric [Ooguri,Vafa, '96].

Gross-Wilson procedure

Setup

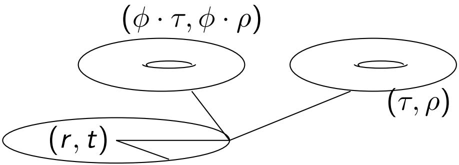
We are given an elliptically fibered $K3$-surface, i.e. the total space of a T^2-bundle over $\mathbb{P}^1_{\mathbb{C}}$.



- $Ric_{sf} = Ric_{ov} = 0$
- $Ric_{int} \sim 0$ error of $O(e^{-C/\epsilon})$
- $Ric_{int} \to 0$, as $\epsilon = \mathrm{vol}|_{T^2} \to 0$

T-folds

Generalize to $\phi \in O(2,2;\mathbb{Z}) \cap \exp(\mathfrak{o}(2,2;\mathbb{R}))$, see [Lüst, Massai, Vall Camell, '15]:



Again, promote moduli to function on D^2

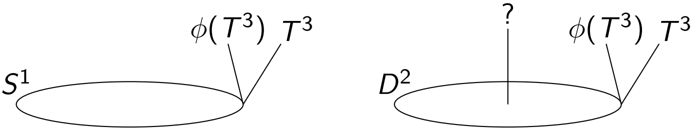
$$(\tau, \rho) \longrightarrow (\exp(\log(\phi)t) \cdot \tau_0(r),\ \exp(\log(\phi)t) \cdot \rho_0(r)).$$

Consequence

The total space need not have a well-defined Riemannian structure.
$\Rightarrow$ Notion of *T-fold*.

Stringy T^3-fibrations

Take a monodromy $\phi \in O(3,3;\mathbb{Z}) \cap \exp(\mathfrak{o}(3,3;\mathbb{R}))$.



Question:

Is there a convenient way to parametrize the metric and B-field in terms of moduli?

Ansatz for metric

Yes we can!

$$G = \frac{1}{\sqrt[2/3]{\tau_2\sigma_2}} \begin{pmatrix} 1 & \tau_1 & \sigma_1 \\ \tau_1 & |\tau|^2 & \tau_1\sigma_1 + \rho_1\tau_2^2 \\ \sigma_1 & \tau_1\sigma_1 + \rho_1\tau_2^2 & |\sigma|^2 + \rho_1^2\tau_2^2 \end{pmatrix}$$

$$\begin{aligned} \tau &= \tau_1 + i\tau_2, \\ \rho &= \rho_1 + i\rho_2, \\ \sigma &= \sigma_1 + i\sigma_2. \end{aligned}$$

Warning: ρ is not related to the Kähler modulus of the T^2 in any way!

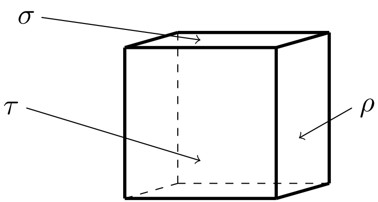


Figure: Fundamental cell of T^3

Embedding the KK-monopole

Firstly consider

$$\phi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$G = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 & 0 \\ \tau_1 & |\tau|^2 & 0 \\ 0 & 0 & \tau_2 \end{pmatrix}$$

together with

$$\begin{aligned} \tau &= t + i \log(\mu/r) \\ \sigma &= \frac{1}{2} i \log(\mu/r) \\ \rho &= -\frac{1}{2} i \log(\mu/r). \end{aligned}$$

Another version of the KK-monopole

Taking

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we can obtain

$$G = \frac{1}{\tilde{\tau}_2} \begin{pmatrix} 1 & \tilde{\tau}_1 + 1 & -\tilde{\tau}_1 \\ \tilde{\tau}_1 + 1 & |\tilde{\tau} + 1|^2 & -\tilde{\tau}_1{}^2 - \tilde{\tau}_1 - \tilde{\tau}_2{}^2 \\ -\tilde{\tau}_1 & -\tilde{\tau}_1{}^2 - \tilde{\tau}_1 - \tilde{\tau}_2{}^2 & \tilde{\tau}_1{}^2 + \tilde{\tau}_2{}^2 + \tilde{\tau}_2 \end{pmatrix}.$$

with $\tilde{\tau} = t + i \log(\mu/r)$.

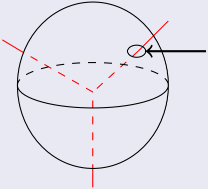
But...

This monodromy is conjugate in $SL(3;\mathbb{Z})$ to the monodromy of the KK-monopole. Why care about this?

Higher dimensional Ooguri-Vafa metric?

Local model

$$T_1 = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$T_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Figure: *The base is a connected open subset of* $\mathbb{R}^3$; S^2 *intersects the singular locus in three points*

Gross-Wilson program for the quintic CY_3?

Now we can write down an approximately Ricci-flat metric on the thrice-punctured S^2 around a vertex:

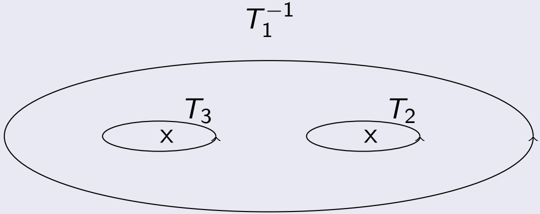


Figure: Deformation of a punctured S^2 around vertex

Program

Consider a foliation of $\mathbb{R}^3 \setminus \Delta$, where Δ is the singular locus. Then possible to prove Gross-Kontsevich-Soibelman-Wilson ('weak' SYZ) conjecture?

Stringy T^3 as geometric T^4

Idea

Use a known map $SO(3,3) \to SL(4)$ to interpret stringy T^3-bundles, as geometric T^4-bundles. Connection to the correspondence

$$\text{IIA on } T^3 \longleftrightarrow \text{M-theory on } T^4$$

as in [McGreevy,Vegh, '08].

Data transfer

$$(G_{T^3}, B_{T^3}) \longleftrightarrow (G_{T^4}, vol_{T^4} = 1).$$

The global model

In analogy to $K3$-surface there is a construction:

$$\begin{array}{ccc} T^4 & \longrightarrow & CY_3 \\ & & \downarrow \\ & & \mathbb{P}^1_{\mathbb{C}} \cong S^2 \end{array}$$

a singular bundle, with monodromies given by $SL(4;\mathbb{Z})$-matrices [Donagi, Gao, Schulz, '08].

Consistency Condition

$$A^{16-4mn} B_1 C_2 B_2 C_1 B_3 C_3 B_4 C_4 = id$$

The global model

Gross-Wilson procedure

- The singular locus is given by $24 - 4mn$ points on the base
- For $m = n = 0$ we recover a $K3 \times T^2$
- On their own all monodromies are conjugate in $SL(4;\mathbb{Z})$ to

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

- Therefore the topology of any singular fiber is $T^2 \times I_1$
- There is no global conjugation achieving this simultaneously for all monodromies
- Twisted version of Gross-Wilson? (For any (m, n))

Recap

- T-folds glued together from T^3-bundles by $O(3,3;\mathbb{Z})$-transformations
- Studied T^3-bundles over D^2 with a B-field
- Already geometric case, i.e. transition functions only diffeomorphisms and gauge transformations non-trivial
- Non-geometric bundles are studied by studying geometric T^4-bundles
- In this way construct global model for non-geometric T^3-fibration over $\mathbb{P}^1_{\mathbb{C}}$

Open problems

- Using the fact that IIA on T^3 is related to M-theory on T^4 investigate manifolds with G_2-holonomy, e.g. Joyce manifolds
- Explore Heterotic/F-theory duality: Is the Jacobian of Σ_2 related to $T^2_\tau \times T^2_\rho$ bundles?
- What is the shape of a singular fiber in a stringy T^3-fibration? Other topological questions.

Thanks everyone!