L_∞ algebras and conformal field theory

Matthias Traube[1]

[1]Max Planck Institut für Physik &
Ludwig Maximilians Universität München

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What?

Two dimensional conformal field theory with consistent extended $\mathcal{W}$-symmetry $\Leftrightarrow$ Theory has L_∞ symmetry

Outline

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Symmetries in physics

- Why should we care about symmetries at all?

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- Why should we care about symmetries at all?

$\Rightarrow$ Symmetries constrain physical theories!

Outline

Symmetries in the Standard Model

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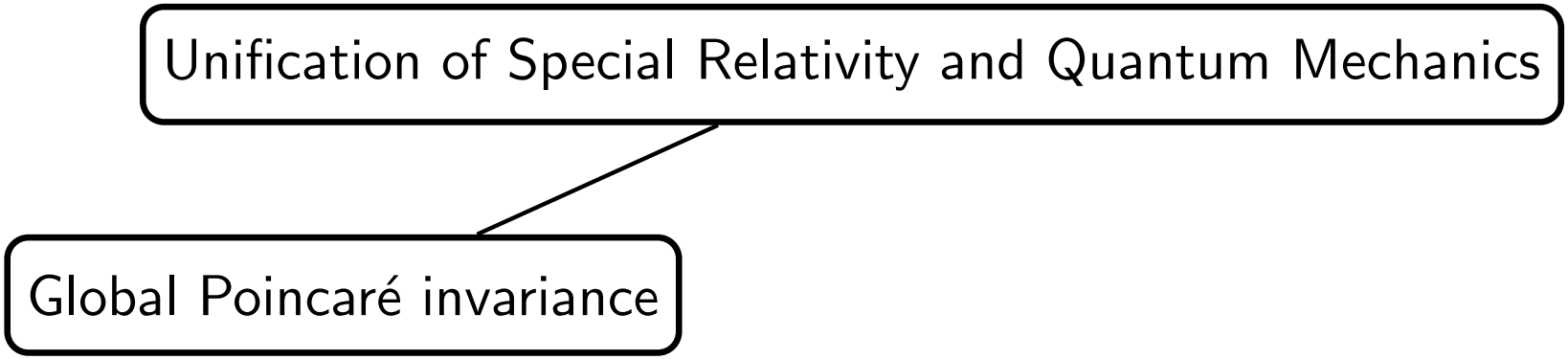
Unification of Special Relativity and Quantum Mechanics

Symmetries in the Standard Model



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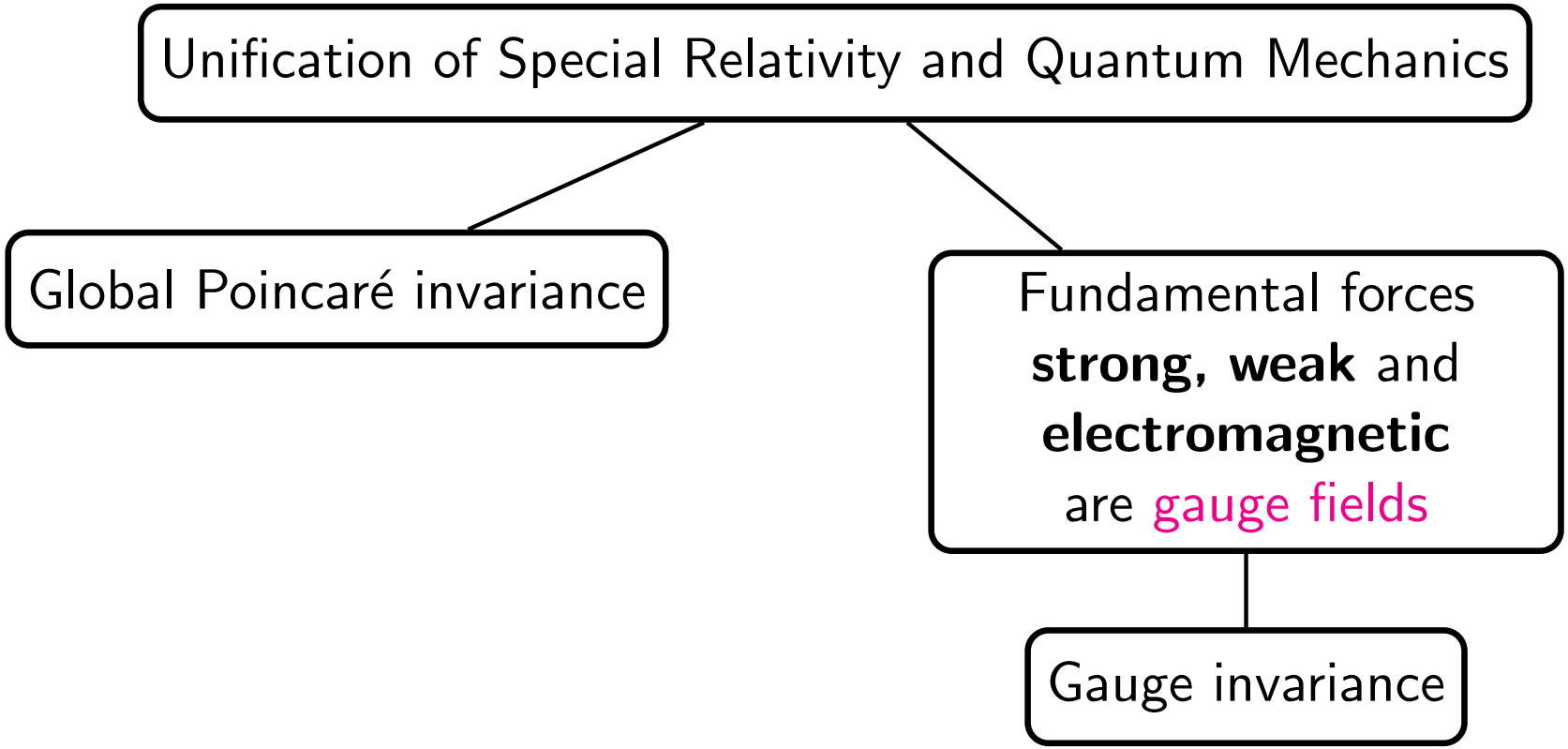

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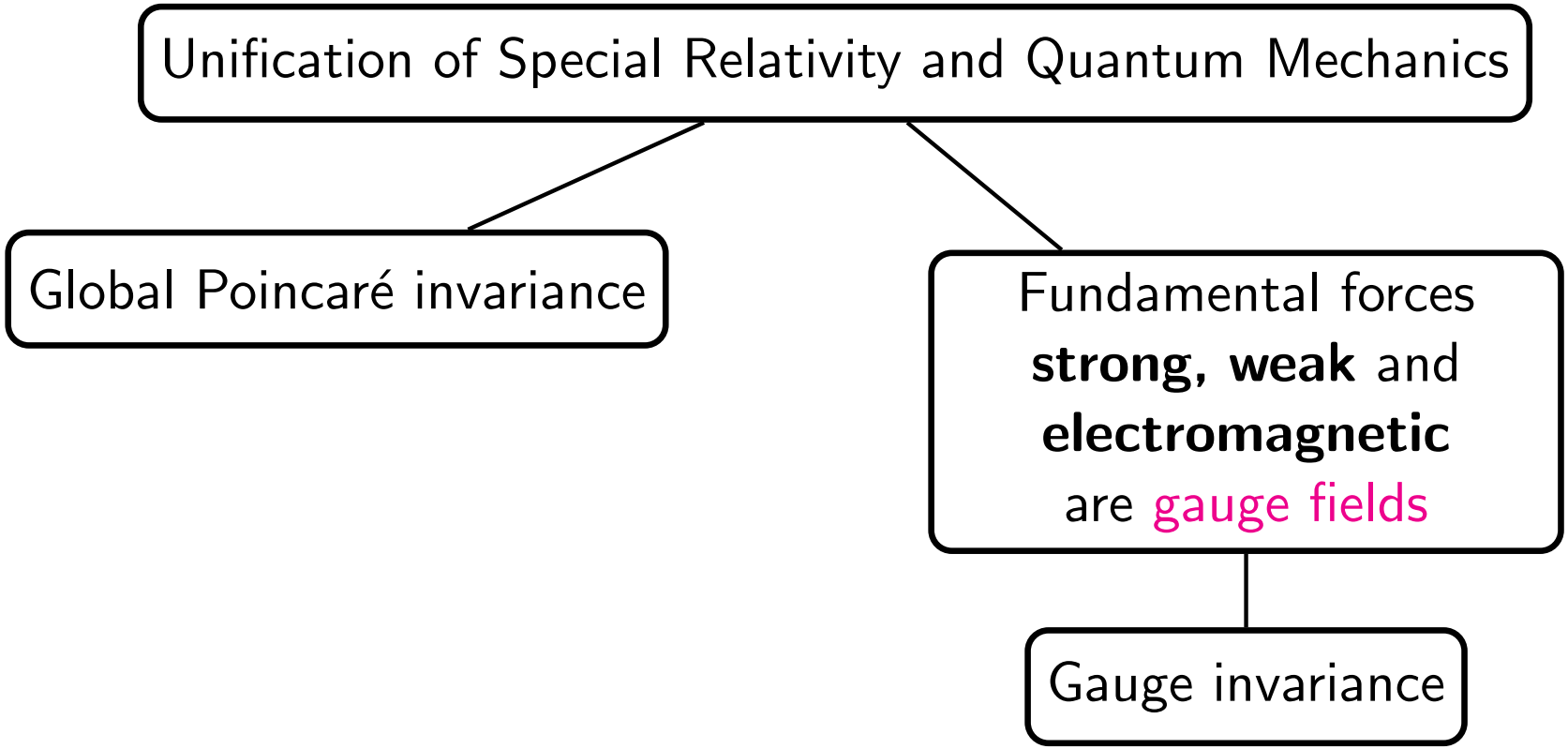

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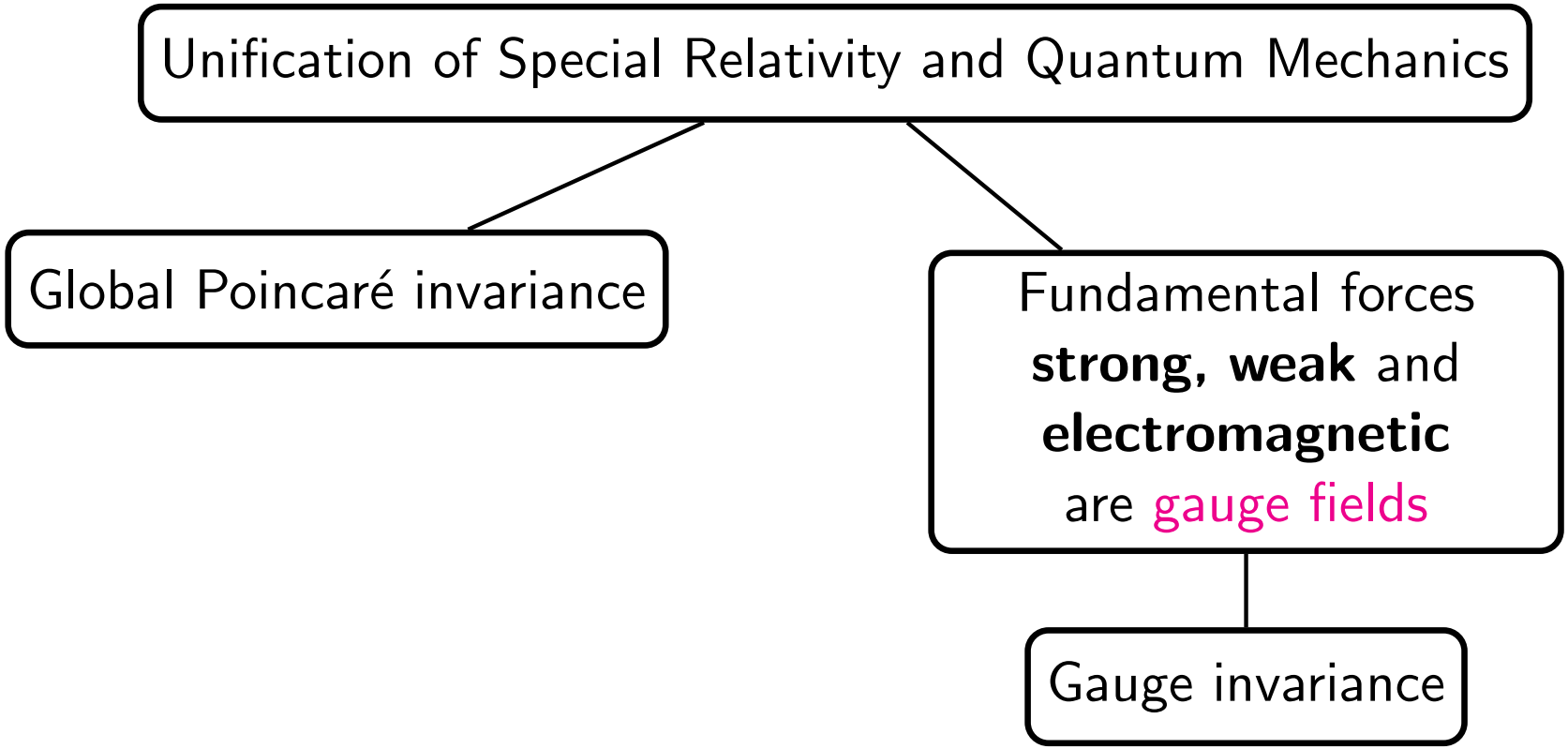

→ Action for Standard Model is Poincaré and gauge invariant.

- Standard Model is not enough, as it doesn't include General Relativity.

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String theory

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Unification of General Relativity and Quantum Mechanics

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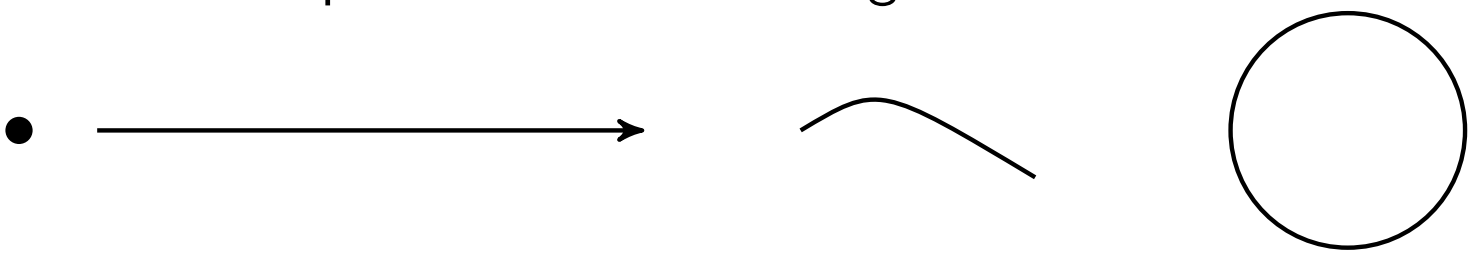
Idea: Point particle becomes string



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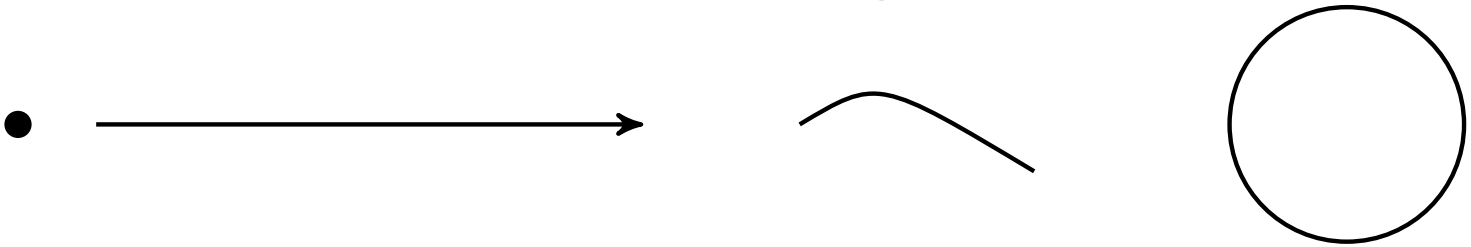

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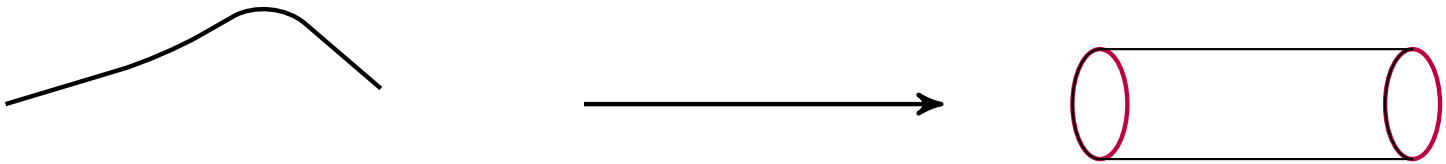

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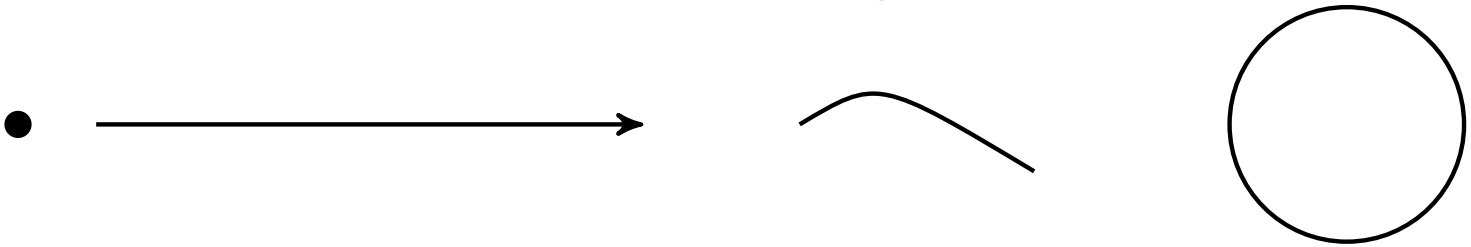

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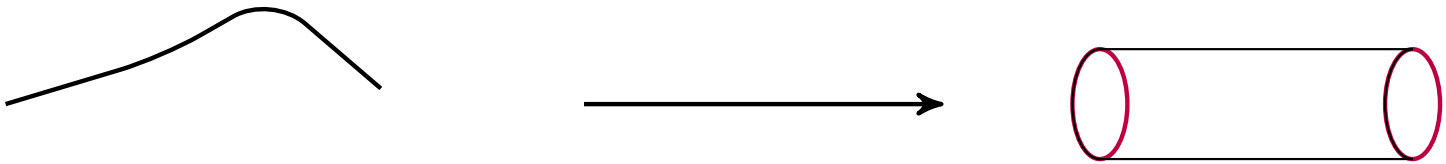

$\rightarrow$ Oscillation modes of the string describe different particles

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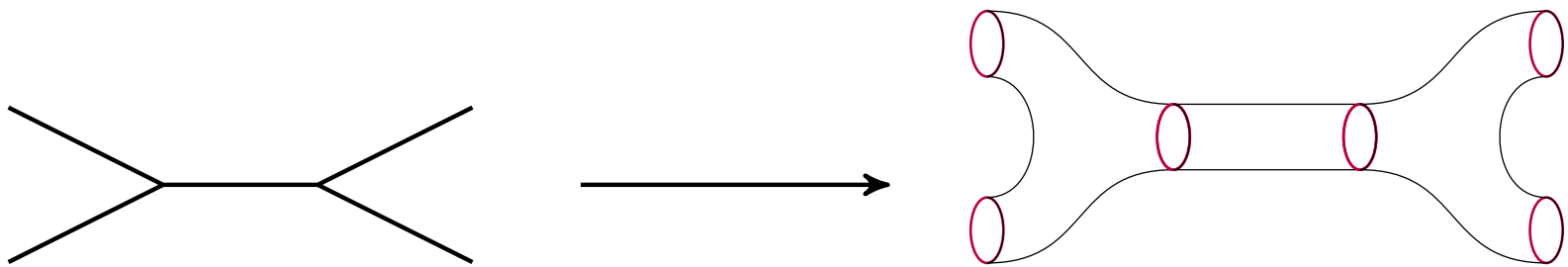

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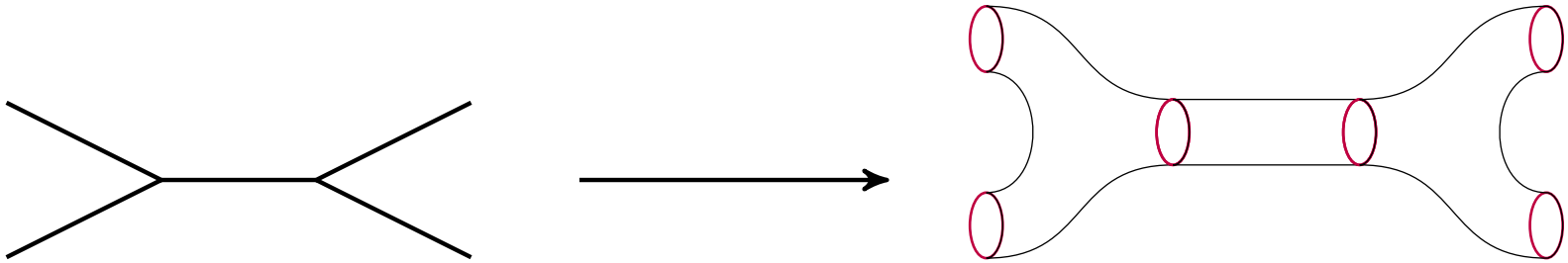

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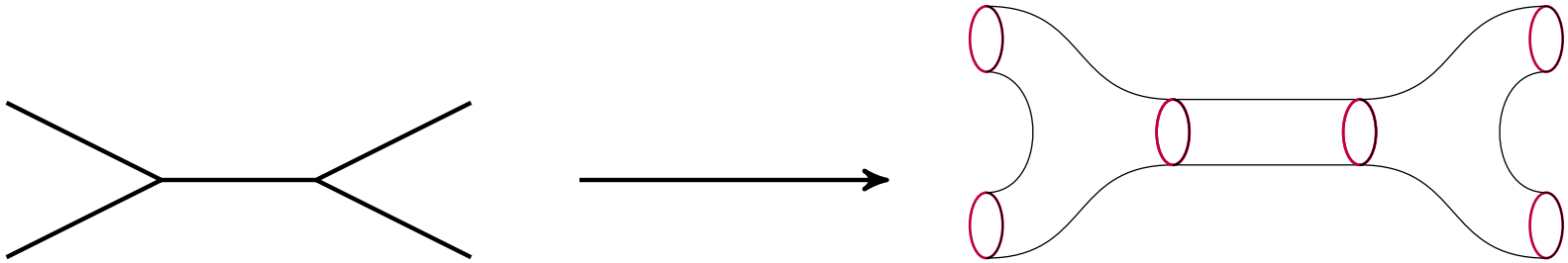

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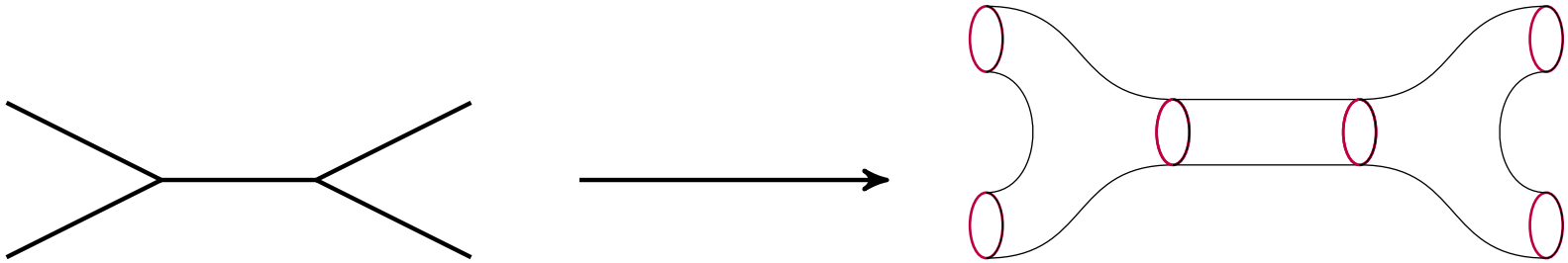

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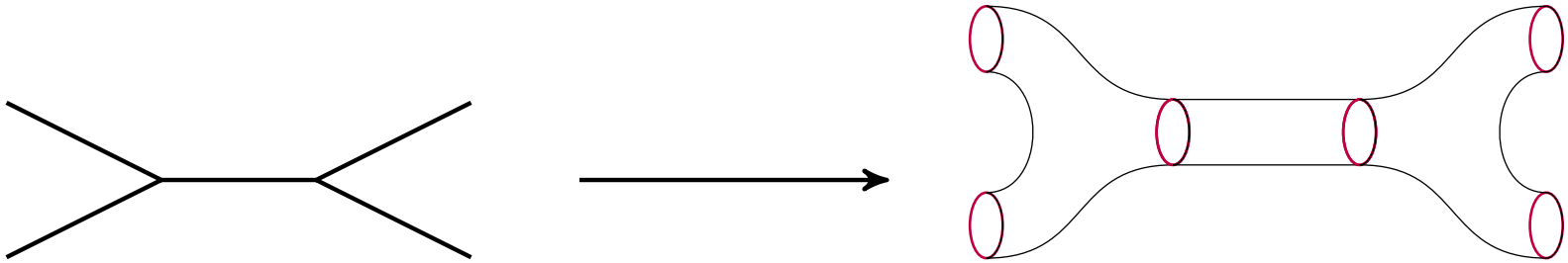

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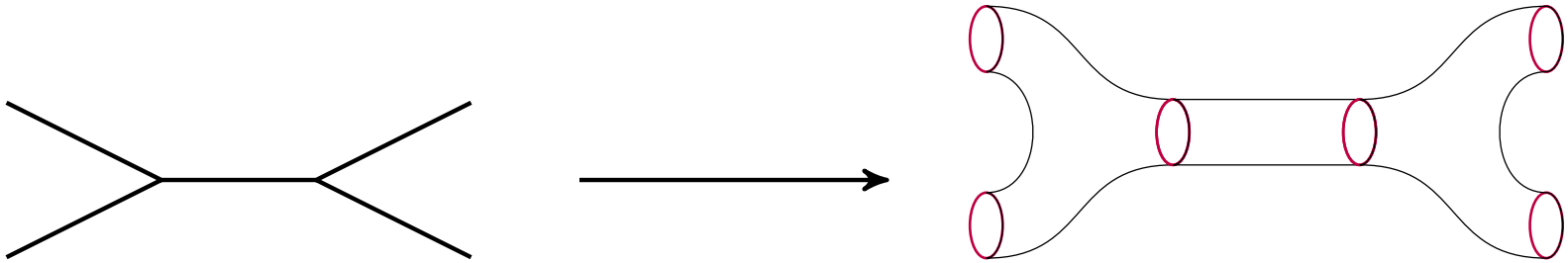

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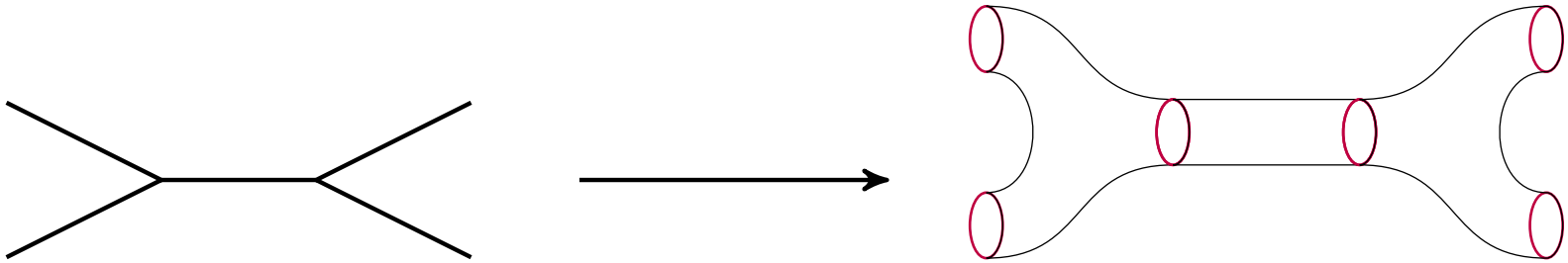

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- L_∞ algebra is generalization of Lie algebra

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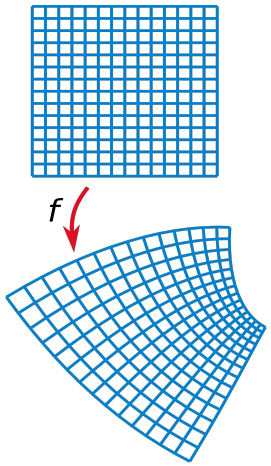

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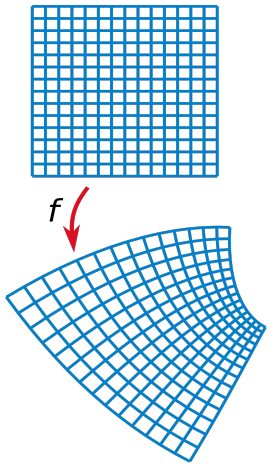

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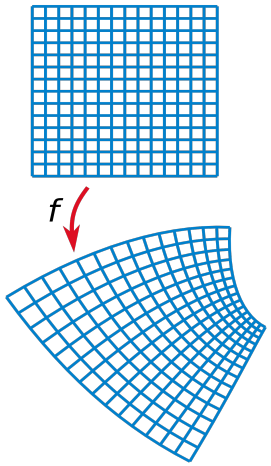

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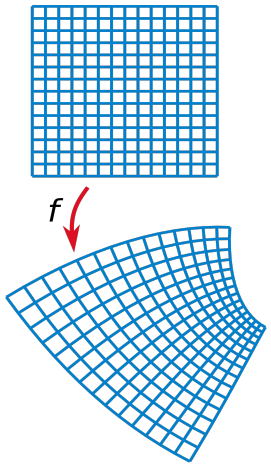

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- Upon **crossing symmetry** for the transformations one gets 2d CFT with additional $W_3, W_4, \ldots$ symmetry

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- L_∞ symmetry in classical Yang-Mills gauge theory, Einstein gravity

(O.Hohm, B.Zwiebach arXiv:1701.08824)

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2) Constraints for quantum field theory with W_N-symmetry are satisfied $\Leftrightarrow$ W_N-theory has Quantum L_∞ symmetry

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Thank you!

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$$\delta_\Lambda \Phi \sim \ell_1(\Lambda) + \ell_2(\Lambda, \Phi) + \ell_3(\Lambda, \Phi, \Phi) + \ell_4(\Lambda, \Phi^3) + \dots$$

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$\Rightarrow$ L_∞ algebra

L_∞ algebra

Definition

A L_∞ algebra consists of a graded vector space X and multilinear maps $\{\ell_n\}_{n\geq 1}$, $\ell_n : \underbrace{X \otimes \cdots \otimes X}_{n} \to X$, satisfying generalized Jacobi identities.

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Theorem (classical)

$\{T, W_3, \ldots, W_N\}$ form a classical $\mathcal{W}_N$ algebra iff their symmetry transformations form a L_∞ algebra.

(R.Blumenhagen, M.Fuchs, MT JHEP07(2017)060)

Classical $\mathcal{W}_3$ algebra

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$$\delta_\eta W = \tfrac{c}{360}\partial^5\eta + \alpha\left(\tfrac{1}{6}\partial^3\eta T + \tfrac{1}{4}\partial^2\eta\partial T + \tfrac{3}{20}\partial\eta\partial^2 T + \tfrac{1}{30}\eta\partial^3 T\right) + \beta\left(\partial\eta(TT) + \tfrac{1}{2}\eta\partial(TT)\right)$$

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\end{aligned}$$

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$\rightarrow$ Demand closure

$$[\delta_{\boldsymbol{\varepsilon}_1}, \delta_{\boldsymbol{\varepsilon}_2}] = \delta_{-\ell_2(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2)-\ell_3(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\mathbf{W})+\frac{1}{2}\ell_4(\boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\mathbf{W},\mathbf{W})+\ldots}$$

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- E.g. $[\delta_{\eta_1}, \delta_{\eta_2}] = \delta_{-\ell_2(\eta_1,\eta_2)-\ell_3(\eta_1,\eta_2,T)}$
- L_∞ relations fix $\alpha = 2$, $\beta = \frac{32}{5c}$

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$\Rightarrow$ Quantum L_∞ algebra (R.Blumenhagen, M.Fuchs, MT JHEP10(2017)163)