Topological superconductors

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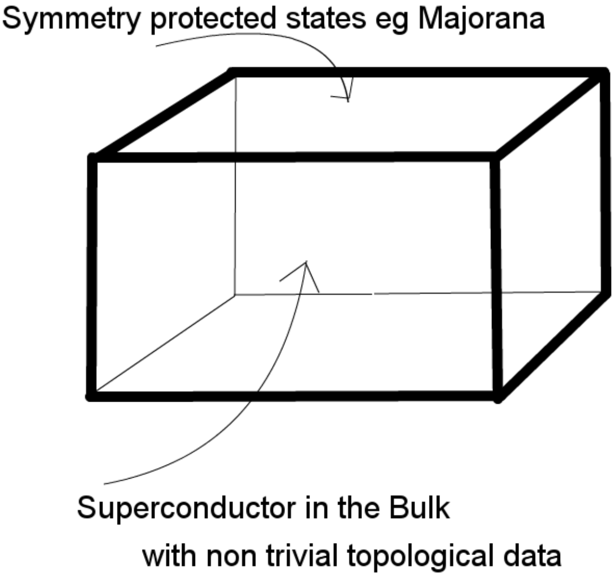
Overview

1. Superconductivity
   - BCS theory
2. Topology
3. Topological Superconductor
   - Bulk-boundary Correspondence
4. Experimental results

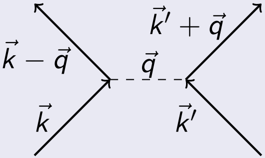
What is a topological superconductor?



BCS theory

Idea

An electron pair interacts via a phonon



Recall

Phonons are the quanta of background lattice oscillations.

BCS theory

Procedure

Start with the Hamiltonian describing electrons coupling to the background lattice

$$H = H_0 + H_{\text{int}}.$$

with

$$H_0 = \sum_{\vec{q}} \hbar\omega_{\vec{q}} b^\dagger_{\vec{q}} b_{\vec{q}} + \sum_{\vec{k}} \epsilon_{\vec{k}} a^\dagger_{\vec{k}} a_{\vec{k}}$$
$$H_{\text{int}} = \sum_{\vec{q},\vec{k}} M_{\vec{q}} a^\dagger_{\vec{k}+\vec{q}} a_{\vec{k}} (b_{\vec{q}} + b^\dagger_{-\vec{q}}).$$

BCS theory

Frohlichers Idea

Using the operator

$$S = \sum \left[\frac{b^\dagger_{-\vec{q}}}{\epsilon_{\vec{k}} - \epsilon_{\vec{k}+\vec{q}} - \hbar\omega_{\vec{q}}} + \frac{b_{\vec{q}}}{\epsilon_{\vec{k}} + \epsilon_{\vec{k}+\vec{q}} - \hbar\omega_{\vec{q}}} \right] M_{\vec{q}} a^\dagger_{\vec{q}+\vec{k}} a_{\vec{k}},$$

we make a unitary transformation

$$H' = e^{-S} H e^{S},$$

which yields

$$H' = H_0 + \sum_{k,k',q} |M_{\vec{q}}|^2 a^\dagger_{k+q} a^\dagger_{k'-q} a_k a_{k'} \frac{\hbar\omega_q}{(\epsilon_{k'} - \epsilon_{k'-q})^2 - (\hbar\omega_q)^2}$$

BCS theory

- Computation shows that electrons form Cooper pairs with opposite spin
- Bogoljubov approach $\langle a_{k,\uparrow} a_{-k,\downarrow} \rangle \neq 0$
- Condensate of Cooper pairs

Effective Hamiltonian

$$\begin{aligned} H = & \sum \epsilon_{\sigma_1,\sigma_2}(k) a^\dagger_{k,\sigma_1} a_{k,\sigma_2} + \\ & \frac{1}{2} \sum V_{\sigma_1,\ldots,\sigma_4}(k,k') a^\dagger_{k,\sigma_1} a^\dagger_{-k,\sigma_2} a_{k',\sigma_3} a_{-k',\sigma_4} \\ \sim & \sum \epsilon_{\sigma_1,\sigma_2}(k) a^\dagger_{k,\sigma_1} a_{k,\sigma_2} + \frac{1}{2} \sum \Delta_{\sigma_1,\sigma_2}(k) a^\dagger_{k,\sigma_1} a^\dagger_{-k,\sigma_2} + h.c. \end{aligned}$$

A calculation for later

Rewrite

$$H = (a^\dagger_{k,\sigma_1}, a_{-k,\sigma_1}) \begin{pmatrix} \epsilon_{\sigma_1,\sigma_2}(k) & \Delta_{\sigma_1,\sigma_2}(k) \\ \Delta^\dagger_{\sigma_1,\sigma_2}(k) & -\epsilon^T_{\sigma_1,\sigma_2}(-k) \end{pmatrix} \begin{pmatrix} a_{k,\sigma_2} \\ a^\dagger_{-k,\sigma_2} \end{pmatrix}$$

In the 2D case we can assume normalized eigenstates of the form

$$|u(k_x, k_y)\rangle = \begin{pmatrix} \cos(2\alpha(k_x, k_y)) \\ \sin(2\alpha(k_x, k_y)) \end{pmatrix},$$

where α is a certain function of $\epsilon(k)$ and $\Delta(k)$.

Bogoljubov quasi particles

Particle-hole duality allows diagonalization of H, yielding Bogoljubov modes

$$UHU^{\dagger} = \sum E_i \gamma^{\dagger}_{k,\sigma}\gamma_{k,\sigma},$$

with

$$\begin{aligned}\gamma_{k\uparrow} &= u_k a_{k\uparrow} - v_k a^{\dagger}_{-k\downarrow} \\ \gamma_{k\downarrow} &= u_k a_{k\downarrow} + v_k a^{\dagger}_{-k\uparrow}.\end{aligned}$$

Important point: Possibility for Majorana excitations

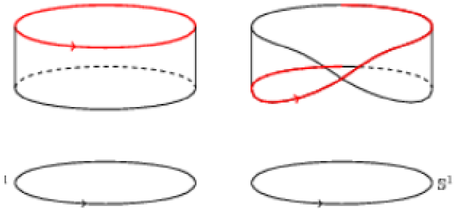
$$\gamma^{\dagger}_0 = \gamma_0.$$

A little bit of math

What is topology?

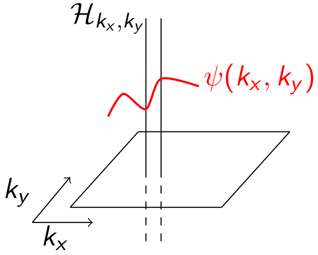
Roughly topology is the study of shapes. It is one of the 'craziest' parts of mathematics.

We are going to study vector bundles over smooth manifolds and their characteristic classes, or rather the first Chern class c_1 and the first Chern number $\int c_1^n$.



The Berry Connection

Consider the Brillouin zone of an effectively 2D solid



Berry Connection

$$\vec{\mathcal{A}}(\vec{k}) = i\langle\psi(\vec{k})|\nabla_{\vec{k}}\psi(\vec{k})\rangle$$

$$\mathcal{A} \longrightarrow \mathcal{A} - \nabla\phi$$

$$F_{ij} = \epsilon_{ij}\partial_i A_j$$

The Berry Connection

First Chern number

$$\nu = \frac{1}{2\pi}\int_{2DBZ} d\mathcal{A} = \frac{1}{2\pi}\int_{2DBZ} \epsilon_{ij}\partial_i A_j dk_x dk_y = \int_{2DBZ} \frac{F}{2\pi} \in \mathbb{Z}.$$

- Appearance of field strength F
- Thm: $c_1(\mathcal{L}) = F/2\pi$
- For 2D insulators $\sigma_{xy} = -\frac{e^2}{h}\nu$
- Time reversal: $\mathcal{A}(k) \mapsto \mathcal{A}(-k)$ implies $F(k) \mapsto -F(-k)$

$$\int F(k) \mapsto \int -F(-k) = -\int F(k) \Rightarrow \nu = 0.$$

Closer look at time reversal

Kramers rule

Consider a fermion in 2+1 dimensions

$$T\psi(t, x_1, x_2) = i\sigma_1\psi(-t, x_1, x_2) \Rightarrow T^2 = -1.$$

Further by anti-unitarity of T, i.e. $\langle u|v\rangle = \langle Tv|Tu\rangle$

$$\langle u|Tu\rangle = \langle T^2 u|Tu\rangle = -\langle u|Tu\rangle.$$

Thus

$$\langle u|Tu\rangle = 0,$$

and we get a 2-fold degenerate energy levels.

Closer look at time reversal

Now we write a basis for our Hilbert space at momentum (k_x, k_y)

$$\mathcal{H}_{k_x,k_y} = \text{span}\langle u_1(k), e^{i\phi_1} T u_1(-k), \ldots\rangle.$$

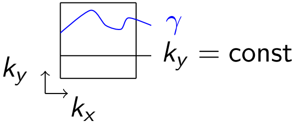
We obtain to different Berry connections per energy level $\mathcal{A}_n^I(k)$ and $\mathcal{A}_n^{II}(k)$. Then

$$2\pi\nu = \sum_n \int F_n^I + F_n^{II} = \nu_I + \nu_{II} = 0,$$

by T symmetry. However

$$\nu_I = -\nu_{II} \Rightarrow (-1)^{\nu_I} = (-1)^{\nu_{II}} \in \mathbb{Z}_2$$

Topological numbers



1D Example

Take a one-dimensional slice in the 2D Brillouin zone of the SCD example at the beginning

$$\begin{aligned} S^1 &\to S^1 \\ k_x &\mapsto \begin{pmatrix} \cos(2\alpha(k_x)) \\ \sin(2\alpha(k_x)) \end{pmatrix} \end{aligned}$$

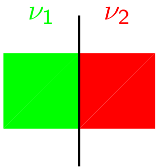
Winding number

This map gives a winding number as

$$\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \partial_{k_x}(2\alpha(k_x)) \sim \sum_{k_x:\epsilon(k_x)=0} \text{sign}(d(k))\text{sign}(\partial_{k_x}\epsilon(k_x))$$

Because the path γ is homotopy equivalent to $k_y = \text{const}$, the corresponding gapless mode has a flat energy dispersion.

Gapless modes on the boundary



For gapped systems and $E_n < E_F$ the following map is continuous

$$\begin{aligned} BZ &\longrightarrow \mathcal{H} \\ \vec{k} &\mapsto \psi_n(\vec{k}) \end{aligned}$$

This implies that the Chern number is constant. How then do we model discontinuities?
By allowing bands to intersect at the boundary. This gives rise to gapless modes.

Gapless modes on the boundary

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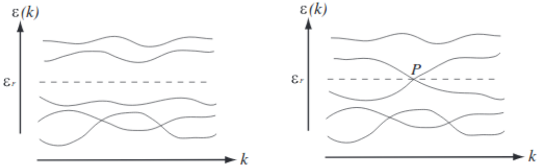
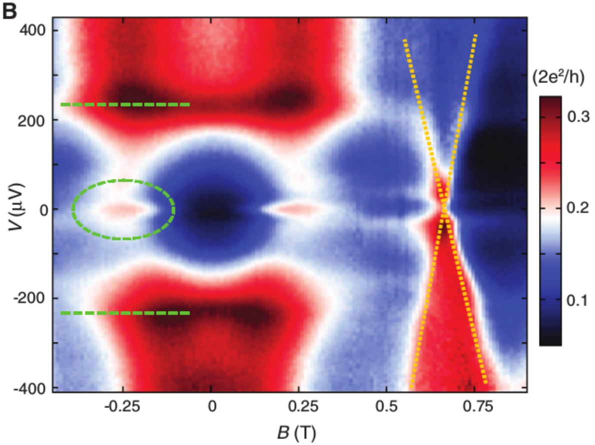




and *B*. The ZBP is highlighted by a dashed oval; green dashed lines indicate the gap edges. At ~0.6 T, a non-Majorana state is crossing zero bias with a slope equal to ~3 meV/T (indicated by sloped yellow dotted lines). Traces in

Thank you

Boundary Theory

The boundary of a 3+1 D topological superconductor admits a Majorana fermion

$$I_{\text{Euclidean}} = \int_Y \bar{\psi} i \mathrm{D} \psi d^3x,$$

where $\mathrm{D}_{\alpha\beta} = \epsilon_{\alpha\gamma}\mathcal{D}^{\gamma}_{\beta}$.
The partition function is given by the Pfaffian

$$Z_\psi = Pf(\mathrm{D}).$$

In the case of

$$\mathrm{D} = \begin{pmatrix} 0 & \mathcal{D} \\ -\mathcal{D}^T & 0 \end{pmatrix},$$

we have

$$Pf(\mathrm{D}) = \det\mathcal{D}.$$

Kramers rule once more

There is again a Kramers rule: If

$$\mathcal{D}\chi = \lambda\chi,$$

then $\tilde{\chi}^\alpha = (\mathcal{T}\chi)^\alpha = \epsilon^{\alpha\beta}\chi^*_\beta$ is also an eigenvector

$$\mathcal{D}\tilde{\chi} = \lambda\tilde{\chi}.$$

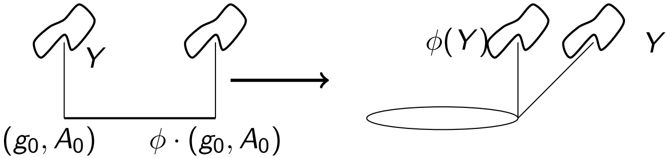
Using $[\mathcal{D}, \mathcal{T}] = 0$ this implies

$$\mathsf{D} = \oplus_i \left(\begin{array}{cc|cc} & -\lambda_i & & \\ \lambda_i & & & \\ \hline & & & -\lambda_i \\ & & \lambda_i & \end{array}\right) \Rightarrow Pf(\mathsf{D}) = \Pi'\lambda_i$$

An anomaly?

- Fermionic anomalies do not affect the absolute value of $Z_\psi = |Z_\psi| e^{i\phi}$
- Time reversal sends $Z_\psi \mapsto \bar{Z}_\psi$
- $Z_\psi \in \mathbb{R}_+$ implies no anomalies

We would like to check the sign of $Pf(\mathrm{D}) = \Pi' \lambda_i$.
Consider a family of these theories differing only by gauge transformations $\rightarrow$ apply index theory of the (lifted) Dirac operator (on the mapping torus)



It turns out that the index on a mapping torus of this form is always identically zero!!

But...

If we regularize the Pfaffian and then we obtain a complex partition function

$$Z_\psi = |Z_\psi| \exp(i\frac{\pi}{4}\eta).$$

But considering the interaction with some bulk theory we can restore $\mathcal{T}$ invariance. The result however depends on the four manifold X,

$$Z_\psi = (-1)^{\mathfrak{I}_X}.$$

$\mathfrak{I}$ is the index of a Dirac operator on a manifold with boundary.