Hidden symmetries in integrable models

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IMPRS Particle Physics Colloquium
MPP München, 14.12.2017

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- conceptual challenges in quantum field theory
  non-perturbative behaviour (spectrum, asymptotic freedom, solitons),...

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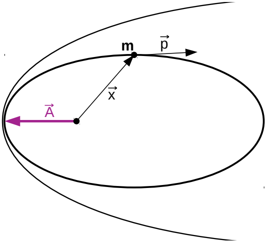
- here: **exactly solvable** (or **integrable**) toy models
  simple but non-trivial interacting theories
  - What is a 'complete' or 'exact' solution?
  - simplicity $\leftrightarrow$ (hidden) symmetries?
  - applications?

overview

1. classical integrability - symmetries and geometry
2. quantum integrability - symmetries and the S-matrix
3. applications

the (unexpected) beauty of the Kepler problem



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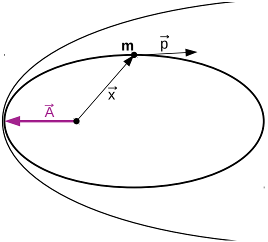

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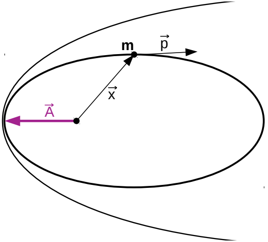

- effective one-body problem in central potential $V = -\frac{\alpha}{r}$

- conserved charges:
  - standard - energy E, angular momentum $\vec{L}$
  - accidental/'hidden' - perihel, Runge-Lenz vector $\vec{A} = \vec{p} \times \vec{L} - \alpha m \frac{\vec{r}}{r}$
  - $\vec{L}$ and $\vec{A}$: Noether charges of 'hidden' SO(4)
- algebraic solution via hidden symmetries

symmetries and geometry

Consider a system with n degrees of freedom

$\rightarrow$ $2n$-dimensional phase space $\mathcal{M}$ with $H(\mathbf{q}, \mathbf{p})$ and $\{\ ,\ \}$.

- When is this system called integrable?

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  Definition (Liouville integrability):

  - m functions, independent (on almost all $\mathcal{M}$),

    $$f_k(\mathbf{q},\mathbf{p}) \quad \text{with} \quad \{f_i, f_j\} = 0,\ \{f_k, H\} = 0$$

  - $n \leq m \leq 2n-1$: (super)**integrable**

    $\Rightarrow$ Kepler problem: 'maximally' integrable (n=3, m=5)

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- How do the 'solutions' look?
  Theorem (Arnold): Assume we have an integrable Hamiltonian system, if $\mathcal{M}_f = \{(\mathbf{q},\mathbf{p}) \in \mathcal{M} \mid f_k(\mathbf{q},\mathbf{p}) = c_k\}$ is compact and connected: $\mathcal{M}_f \sim T^n:\ S^1 \times S^1 \times ... \times S^1$.
  see e.g. harmonic oscillator, Kepler problem

conceptual lessons

- $\#$ conserved charges $\geq$ $\#$ d.o.f.
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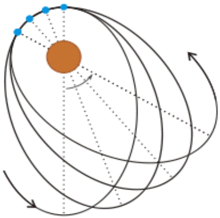
- disturbed integrable models:
  $\rightarrow$ violated conservation laws
  e.g. 'disturbed' Kepler problem: perihel rotation



Classical field theories

What about field theories?

so far only systems with finitly many degrees of freedom, integrability rather trivial

- field theories have an ∞-dimensional phase space

  degrees of freedom for every point in space

- $\infty - \infty = ?$

  how organise enough symmetries, what is a complete solution? - no universal definition of integrability

Lax integrability

description for a big class of integrable theories

- existence of a pair of (differential) operators $\mathbf{L}, \mathbf{M}$

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- $\Rightarrow$ eigenvalues of L are conserved!
  infinite tower of conserved charges,
  generating an ∞-dim. (hidden) symmetry group
- exact solution?

quantum integrability - naive

generalisation of classical integrability

symmetries:

functions on phase space $\mathcal{M}$ $\rightarrow$ operators on Hilbert space $\mathcal{H}$

$\exists n =$dim$(\mathcal{H})$ independent operators $\hat{I}_1, ..., \hat{I}_n$:

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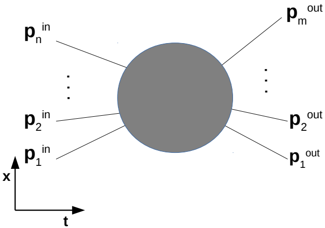
but: commuting operators are not independent on the whole Hilbert space

easy case: $\hat{A}, \hat{B}$ with non-degenerate spectrum and $[\hat{A}, \hat{B}] = 0$

$\Rightarrow$ common eigenvectors $\Rightarrow$ $\hat{A}$ is a polynomial in $\hat{B}$.

quantum integrability

instead: properties of S-matrix under symmetries



- $n \to m$ scattering in $1{+}1d$:

$$\mathcal{A} \sim \langle \mathbf{p}_1^{out}; ...; \mathbf{p}_m^{out} | S | \mathbf{p}_1^{in}; ...; \mathbf{p}_n^{in} \rangle$$

- special in $1d$: spatial ordering of wavepackages

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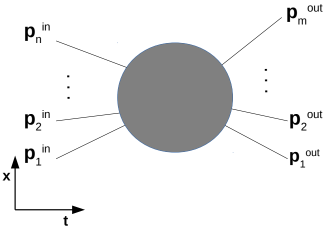

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candidates for (higher, hidden) symmetries:

- higher spin symmetries
- non-local symmetries

example: higher spin

- 'higher spin' symmetries
  - 'higher spin' generator $\hat{Q}_s$ (Lorentz tensor) schematical action on wavepackages $\phi(x, p)$:

    $$\hat{Q}_s \propto \hat{\mathbf{p}}^s, \qquad e^{-i\alpha \hat{Q}_s} |\phi(x,p)\rangle \propto |\phi(x + s\alpha p^{s-1}, p)\rangle$$

    momentum-dependent shifts for $s > 1$

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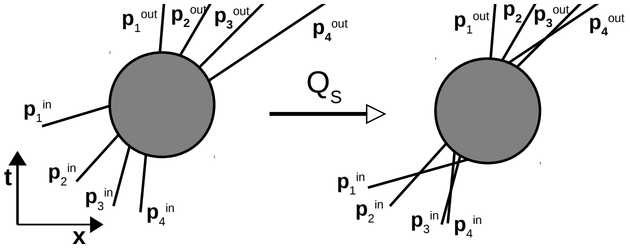

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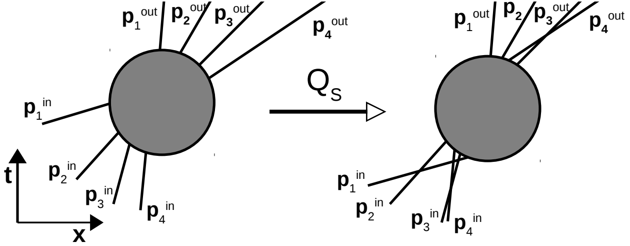

Comment: Coleman-Mandula theorem - S-matrix trivial in 3+1d, if Poincare symmetry is extended, but: loophole in 1+1d

exact S-matrices

'definition' of an integrable quantum theory

- elasticity - no particle production,
  initial set of momenta = final set of momenta
- factorisability
  $n \to n$ S-matrix is a product of $2 \to 2$ S-matrices

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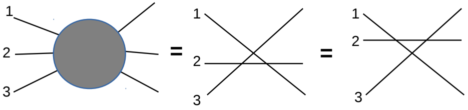
- Yang-Baxter equation: $S_{\underline{23}}S_{\underline{12}}S_{\underline{23}} = S_{\underline{12}}S_{\underline{23}}S_{\underline{12}}$



- additionally: standard (physical) constraints
  unitarity, crossing
  optional: Lorentz invariance, C, P, T

applications

Why should we care about integrability in $1{+}1d$?

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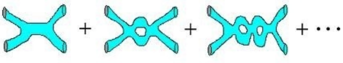
- string theory 

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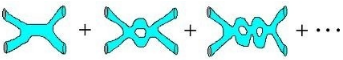
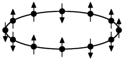
- string theory 

- spin chains 
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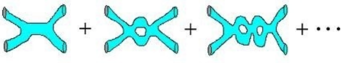
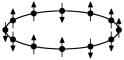
- string theory 
- spin chains 
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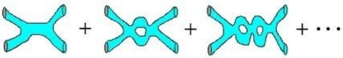
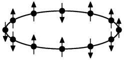
- string theory 
- spin chains 
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- toy models for otherwise inaccessible systems
  1d Bose condensates, 1+1d quantum gravity, ...

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- integrability and the role of (hidden) symmetries
  - classical (field) theory:
    symmetries organise the phase space,
    enough symmetry $\rightarrow$ purely algebraic solution
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  - **but**: non-trivial toy models for conceptual problems of more complicated theories
    non-perturbative QFT, quantum gravity, ...

Thank you for your attention!