Conformal Symmetry and Feynman Integrals

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Conformal Symmetry and Feynman Integrals

Roadmap

1. Conformal symmetry breaking
2. A toy example: 6D 1-mass box
3. Bootstrap strategy
4. A non-trivial example: 6D penta-box
5. Outlook

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Conformal symmetry

- In massless theories the Poincaré group can be extended to the conformal group with the addition of
  - dilatations
  $$x^\mu \to \lambda x^\mu$$
  - conformal boosts
  $$x^\mu \to \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2 x^2}$$

  $\Rightarrow$ All local re-scalings of the metric

- Euclidean spacetime: transformations preserving angles

- Minkowski spacetime: transformations preserving causality

$$\begin{array}{ccc} \text{timelike points} & \longrightarrow & \text{timelike points} \\ \text{lightlike points} & \longrightarrow & \text{lightlike points} \\ \text{spacelike points} & \longrightarrow & \text{spacelike points} \end{array}$$

Conformal symmetry in momentum space

- Standard methods to compute correlation functions using conformal symmetry in position space date back to the '70s
- Goal: application to scattering amplitudes
  - momentum space
  - on-shell configuration $p_i^2 = 0$

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  - on-shell configuration $p_i^2 = 0$
- The generator of conformal boosts becomes 2nd order

$$K_{\mu;\Delta} = \sum_{i=1}^{n} \left[-p_{i\mu} \Box_{p_i} + 2 p_i^{\nu} \frac{\partial}{\partial p_i^{\nu}} \frac{\partial}{\partial p_i^{\mu}} + 2(D - \Delta_i) \frac{\partial}{\partial p_i^{\mu}} \right]$$

- We cannot Fourier-transform from position to momentum space

Collinear anomaly

- Consider *naïvely conformal* Feynman integrals:
  conformal symmetry may be broken in massless configurations

  [Chicherin, Sokatchev 2017]

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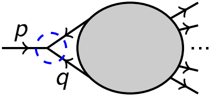
- E.g. $D = 6$ scalar Φ^3 theory
  - Contact anomaly

$$K_\mu \frac{1}{q^2(q+p)^2} = ?$$

on-shell corner
$p^2 = 0$



Collinear anomaly

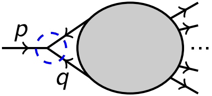
- Consider *naïvely conformal* Feynman integrals:
  conformal symmetry may be broken in massless configurations

  [Chicherin, Sokatchev 2017]

- E.g. $D = 6$ scalar Φ^3 theory
  - Contact anomaly

$$K_\mu \frac{1}{q^2(q+p)^2} = 4i\pi^3 p_\mu \int_0^1 d\xi \xi(1-\xi)\delta^{(6)}(q+\xi p)$$

on-shell corner
$p^2 = 0$



  - Localized on the collinear configuration $q = -\xi p$

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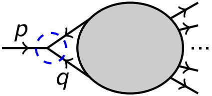


  - Localized on the collinear configuration $q = -\xi p$

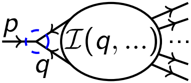
- The anomaly of tree amplitudes is localized on particular collinear configurations of particles

  [Cachazo, Svrcek, Witten 2004],[Beisert et al. 2009]

Powerful anomalous conformal Ward identities

- For Feynman integrals, the contact anomaly localizes a loop-integration

$$K_\mu \int d^6 q$$



$$\propto p_\mu \int_0^1 d\xi\,\xi(1-\xi)\,\mathcal{I}(q=-\xi p,\dots)$$

- System of linear non-homogeneous 2nd-order DEs

$$K_\mu \delta^{(6)}(P)\mathcal{I}^{(\ell)} = \delta^{(6)}(P)\mathcal{A}_\mu^{(\ell-1)}$$

  - $\mathcal{I}^{(\ell)}$ ℓ-loop Feynman integral
  - $\mathcal{A}_\mu^{(\ell-1)}$ anomaly, 1-fold integration of $\ell-1$-loop integrals

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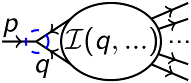


$$\propto p_\mu \int_0^1 d\xi \xi (1-\xi) \mathcal{I}(q=-\xi p,\dots)$$

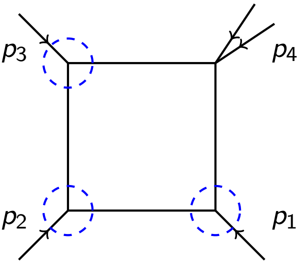
- System of linear non-homogeneous 2nd-order DEs

$$K_\mu \delta^{(6)}(P) \mathcal{I}^{(\ell)} = \delta^{(6)}(P) \mathcal{A}_\mu^{(\ell-1)}$$

  - $\mathcal{I}^{(\ell)}$ ℓ-loop Feynman integral
  - $\mathcal{A}_\mu^{(\ell-1)}$ anomaly, 1-fold integration of $\ell - 1$-loop integrals

- Assuming we know $\mathcal{A}_\mu^{(\ell-1)}$, can we solve for $\mathcal{I}^{(\ell)}$?

A toy example: the 6D 1-mass box



$p_i^2 = 0 \quad \forall i = 1, 2, 3$

$p_4^2 \neq 0$

$s_{ij} = 2 p_i \cdot p_j$

- anomalous conformal Ward identities
- symmetry under exchange $p_1 \Leftrightarrow p_3$
- absence of u-channel singularities

$$\mathcal{I}_{1m} = \frac{1}{s_{13}} \left[\mathrm{Li}_2 \left(1 - \frac{p_4^2}{s_{12}^2} \right) + \mathrm{Li}_2 \left(1 - \frac{p_4^2}{s_{23}^2} \right) + \frac{1}{2} \log^2 \left(\frac{s_{12}^2}{s_{23}^2} \right) + \frac{\pi^2}{6} \right]$$

A toy example: the 6D 1-mass box

$$\mathcal{I}_{1m}(x,z,s_{123}) = \frac{1}{s_{123}} g(x,z)$$

$$\begin{cases} s_{123} = s_{12} + s_{13} + s_{23} \\ x = s_{12}/s_{123} \\ z = s_{23}/s_{123} \end{cases}$$

Anomalous conformal Ward identities:

$$\left\{ z\left[x^2\frac{d^2}{dx^2} + (z-1)^2\frac{d^2}{dz^2} + 2x(z-1)\frac{d}{dx}\frac{d}{dz}\right] + 2x(2z-1)\frac{d}{dx} + 2(z-1)(2z-1)\frac{d}{dz} + 2(z-1)\right\} g(x,z) =$$
$$= -\frac{z(x-1) + (x+z-1)\log\left(\frac{1}{x}\right)}{(x-1)^2 z},$$

$$\left\{(1-x-z)\left[x^2\frac{d^2}{dx^2} + z^2\frac{d^2}{dz^2} + 2xz\frac{d}{dx}\frac{d}{dz}\right] + 2x(1-2x-2z)\frac{d}{dx} + 2z(1-2x-2z)\frac{d}{dz} - 2(x+z)\right\} g(x,z) =$$
$$= \frac{(z-1)^2\log\left(\frac{1}{x}\right) + (x-1)(z-1)(x+z-2) + (x-1)^2\log\left(\frac{1}{z}\right)}{(x-1)^2(z-1)^2},$$

$$\left\{ x\left[(x-1)^2\frac{d^2}{dx^2} + z^2\frac{d^2}{dz^2} + 2z(x-1)\frac{d}{dx}\frac{d}{dz}\right] + 2(2x-1)(x-1)\frac{d}{dx} + 2z(2x-1)\frac{d}{dz} + 2(x-1)\right\} g(x,z) =$$
$$= -\frac{x(z-1) + (x+z-1)\log\left(\frac{1}{z}\right)}{x(z-1)^2}$$

What do we learn?

- Observation
  - $\mathcal{I}_{1m}$ is weight-2 (Li_2)
  - $\mathcal{A}^\mu$ contains weight-1 (log) and weight-0 functions

  A great simplification is achieved by projecting the Ward identities along q^μ such that

  $$(q \cdot K)\mathcal{I}_{1m} = q \cdot A \equiv \text{weight-0}$$

  $\Rightarrow$ maximum weight drop

- The Ward identities are quite complicated already at 4-point $\Rightarrow$ different approach needed for more interesting applications

Bootstrap strategy

- Write down an ansatz ($s \equiv$ kinematic variables)

$$\mathcal{I}(s) = \sum_{i,j} c_{ij} r_i(s) f_j(s)$$

c_{ij} finite number of coefficients
$\Rightarrow$ to be fixed by imposing constraints
(anomalous conformal Ward identities)

$r_i(s)$ algebraic functions
$\Rightarrow$ leading singularities
[Cachazo 2008; Arkani-Hamed, Bourjaily, Cachazo, Trnka 2010]

$f_j(s)$ special functions (polylogarithms)
$\Rightarrow$ symbol alphabet

Iterated integrals and symbols

- MPLs can be written as Chen iterated integrals of logarithmic 1-forms

$$\int_\gamma d\log\alpha_1 \circ \ldots \circ d\log\alpha_k = \int_0^1 \left(\int_0^t d\log\alpha_1 \circ \ldots \circ d\log\alpha_{k-1} \right) d\log\alpha_k(t)$$

Weight: number of iterated integrations

- Letters α_i: algebraic functions of the kinematic variables
- Symbol map associates a “word” to each iterated integral

$$\mathcal{S}\left[\int_\gamma d\log\alpha_1 \circ \ldots \circ d\log\alpha_k\right] = \alpha_1 \otimes \ldots \otimes \alpha_k$$

- Alphabet: set of all independent letters

$$\Omega = \{\alpha_1, \ldots, \alpha_n\}$$

Massless 4-particle amplitudes @ 3-loop

- Alphabet $\Omega = \{x, 1+x\}$ $x = \frac{s}{t}$

[Remiddi, Gehrmann, Henn, Smirnov, Mistlberger...]

$\Rightarrow$ Harmonic polylogarithms

$$H(\vec{m}_w; x) = \int_0^x dx' f(a; x') H(\vec{m}_{w-1}; x') \qquad \vec{m}_w = (a, \vec{m}_{w-1})$$

$$f(0; x) = \frac{1}{x} \qquad f(1; x) = \frac{1}{1-x} \qquad f(-1; x) = \frac{1}{1+x}$$

- Symbol of the harmonic polylogarithms

$$H(0, -1; z) = \int_0^z \frac{dt}{t} H(-1, t) = \int_0^z d\log t \int_0^t d\log(1+t')$$

$$\mathcal{S}\Big[H(0, -1; z)\Big] = (1+z) \otimes z$$

Massless 5-particle amplitudes @ 2-loop

- Five independent kinematic variables $v_i = 2p_i \cdot p_{i+1}$

Planar, 26 letters
[Gehrmann, Henn, Lo Presti 2015]

$$\alpha_i = v_i$$
$$\alpha_{5+i} = v_{i+2} + v_{i+3}$$
$$\alpha_{10+i} = v_i - v_{i+3}$$
$$\alpha_{15+i} = v_i + v_{i+1} - v_{i+3}$$

Non-planar, 31 letters
[Chicherin, Henn, Mitev 2017]

$$\alpha_{20+i} = v_{2+i} + v_{3+i} - v_i - v_{i+1}$$
$$\alpha_{25+i} = \frac{a_i - \sqrt{\Delta}}{a_i + \sqrt{\Delta}}$$
$$\alpha_{31} = \sqrt{\Delta}$$

$$a_i = v_i v_{i+1} - v_{i+1} v_{i+2} + v_{i+2} v_{i+3} - v_{i+3} v_{i+4} - v_{i+4} v_i$$

$$\Delta = \det(2p_i \cdot p_j)$$

Massless 5-particle amplitudes @ 2-loop

Given the pentagon alphabet $\mathbb{A}$, is any word made of letters $\alpha_i \in \mathbb{A}$ allowed?

Massless 5-particle amplitudes @ 2-loop

Given the pentagon alphabet $\mathbb{A}$, is any word made of letters $\alpha_i \in \mathbb{A}$ allowed? No!

- Constraints on the symbols from mathematics
  - Integrability conditions

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Given the pentagon alphabet $\mathbb{A}$, is any word made of letters $\alpha_i \in \mathbb{A}$ allowed? No!

- Constraints on the symbols from mathematics
  - Integrability conditions
- and from physics
  - First entry condition
    - planar symbols $\to \{\alpha_i\}_{i=1}^{5} = \{s_{12}$ and cyclic$\}$
    - non-planar symbols $\to \{\alpha_i\}_{i=1}^{5} \cup \{\alpha_i\}_{i=16}^{20} = \{s_{ij}\}_{i<j=1}^{5}$
  - Conjectured second entry condition

[Chicherin, Henn, Mitev 2017]

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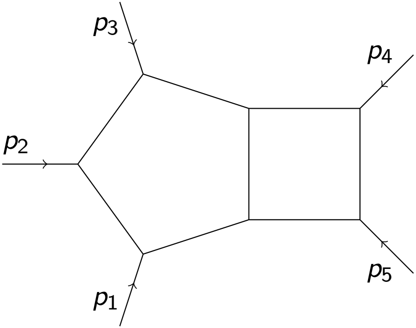
[Chicherin, Henn, Mitev 2017]

How many functions? E.g. for the planar alphabet (26 letters)

Weight	1	2	3	4	5	6
1^{st} entry cond.	5	25	126	651	3436	18426
2^{nd} entry cond.	5	20	81	346	1551	7201

[Chicherin, Henn, Mitev]

6D penta-box



Basic facts:

1. naïvely conformal $\rightarrow$ 6D scalar ϕ^3 theory, finite
2. planar alphabet $\mathbb{A}_{\mathrm{P}}$ (26 letters, 1st entry condition)
3. symmetric under exchange $\{1 \leftrightarrow 3, 4 \leftrightarrow 5\}$
4. parity even
5. leading singularity $1/\sqrt{\Delta}$, $\Delta = \det\left(2p_i \cdot p_j\right)$

Last entry condition from conformal symmetry

- Conformal symmetry constrains the ansatz even before knowing the explicit expression of the anomaly

$$(q \cdot K)\left[\frac{\alpha_1 \otimes \ldots \otimes \alpha_k}{\sqrt{\Delta}}\right] = (q \cdot K)\left[\frac{\log \alpha_k}{\sqrt{\Delta}}\right](\alpha_1 \otimes \ldots \otimes \alpha_{k-1}) + \text{weight-}(k-2)$$

- If there exists a vector q^μ such that

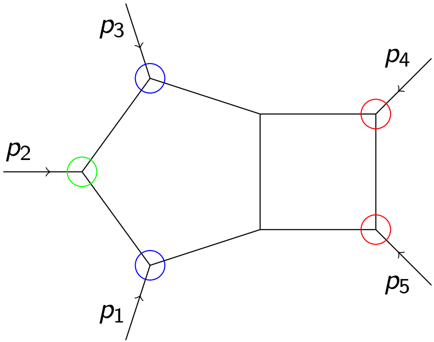
$$(q \cdot K)\mathcal{I}_k = (q \cdot A) \equiv \text{weight-}(k-2)$$

⇒ Constraint on the allowed last entries

$$(q \cdot K)\left[\sum_i c_i \frac{\log \alpha_i}{\sqrt{\Delta}}\right] = 0$$

6D penta-box: anomaly

$(q \cdot A) = q\cdot$



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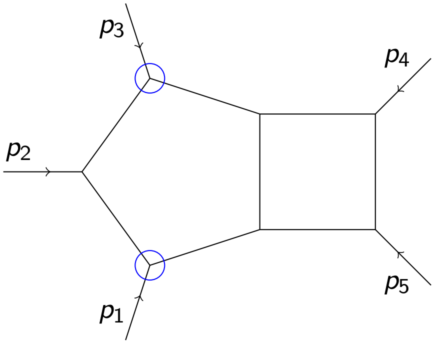

$$q \perp p_2, p_4, p_5$$

$$(q \cdot A) = \frac{\text{(weight-3)}}{s_{12} s_{14} (s_{23} - s_{45})} + (1 \leftrightarrow 3, 4 \leftrightarrow 5)$$

6D penta-box: ansatz

- Info from conformal symmetry:

  6. at most weight 5

$$\underbrace{(q \cdot K)}_{-2 \text{ weight drop}} \quad \mathcal{I}_5 = \underbrace{q \cdot A}_{\text{weight } 3}$$

6D penta-box: ansatz

- Info from conformal symmetry:

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  7. last entry condition (11 allowed letters out of 26)

6D penta-box: ansatz

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  7. last entry condition (11 allowed letters out of 26)

Constraints	weight-5 integrable symbols ($\mathbb{A}_{\mathrm{P}}$)
1st entry condition	3436
parity	161
last entry condition	59
graph symmetry	33

- Ansatz

$$\mathcal{S}\left[\mathcal{I}_5\right] = \frac{1}{\sqrt{\Delta}}\sum_{i=1}^{33} c_i \left(\alpha_{i_1}\otimes\ldots\otimes\alpha_{i_5}\right), \qquad \alpha_i \in \mathbb{A}_{\mathrm{P}}\ \forall i$$

6D penta-box: symbol

- Plug the ansatz

$$\mathcal{S}\left[\mathcal{I}_5\right] = \frac{1}{\sqrt{\Delta}} \sum_{i=1}^{33} c_i \left(\alpha_{i_1} \otimes \ldots \otimes \alpha_{i_5}\right), \qquad \alpha_i \in \mathbb{A}_{\mathrm{P}} \; \forall i$$

into the anomalous conformal Ward identity

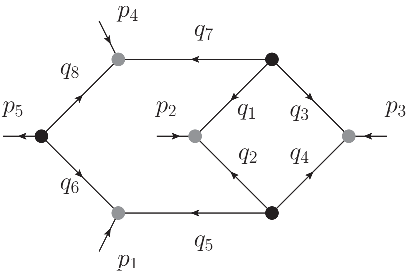
$$(q \cdot K)\, \mathcal{S}\left[\mathcal{I}_5\right] = q \cdot \mathcal{S}\left[A\right]$$

▷ **All coefficients are fixed**!
▷ Only **one projection** of the Ward identities is needed

$$\mathcal{S}\left[\mathcal{I}_5\right] \sim 10000 \text{ terms}$$

Outlook

- Solve the anomalous conformal Ward identities
- Upgrade bootstrap to function level (planar pentagon functions are known [Gehrmann, Henn, Lo Presti 2018])
- Study interplay with the β function
- Super-conformal symmetry [Chicherin, Henn, Sokatchev 2018]
  - Wess-Zumino model of $\mathcal{N} = 1$ massless supersymmetric matter
  - 1st order Ward identities $\Rightarrow$ more powerful!



Backup slides

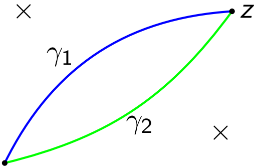
Integrable symbols

- Given an alphabet $\Omega = \{\alpha_1, ..., \alpha_k\}$, is any word made of the letters $\alpha_i \in \Omega$ allowed? No!
- Local path independence

$$\mathcal{S}(f) = \alpha_1 \otimes ... \otimes \alpha_k$$

$$\omega = d\log\alpha_1 \circ ... \circ d\log\alpha_k$$

$$f(z) = \int_{\gamma_1}\omega \overset{!}{=} \int_{\gamma_2}\omega$$



$\gamma_1 \sim \gamma_2$ same homotopy class

- Integrability conditions for $\mathcal{S} = \sum_{i_1,\ldots,i_n} c_{i_1\ldots i_n}(\alpha_{i_1} \otimes ... \otimes \alpha_{i_n})$

$$\sum_{i_1,\ldots,i_n} c_{i_1\ldots i_n}(d\log\alpha_{i_a} \wedge d\log\alpha_{i_{a+1}})\,\alpha_{i_1} \otimes ... \otimes \hat{\alpha}_{i_a} \otimes \hat{\alpha}_{i_{a+1}} \otimes ... \otimes \alpha_{i_n} = 0$$

$$\forall a = 1, ..., n-1$$

Symbols vs functions

Integrand $\longrightarrow$ Symbol $\longrightarrow$ Function (polylogs...)

- Symbols fix the leading functional transcendentality piece

$$\mathrm{Li}_2(1-x) + \log(x)\log(1-x) = -\mathrm{Li}_2(x) + \cancel{\frac{\pi^2}{6}}$$

$$-x \otimes (1-x) + x \otimes (1-x) + (1-x) \otimes x + (1-x) \otimes x = (1-x) \otimes x$$

$\Rightarrow$ they capture the most complicated part of the function

- The lower functional transcendentality pieces can be fixed as well $\Rightarrow$ Bootstrap strategy: accommodate them in the ansatz
- Additional work required to fully upgrade a symbol to function (Goncharov polylogarithms)

Iterated integrals and symbols

- Differential equations in the canonical form

$$d\vec{f}(s;\epsilon) = \epsilon \; d\left(\sum_k A_k d\log \alpha_k(s)\right)\vec{f}(s;\epsilon) \qquad \text{[Henn]}$$

$\alpha_k(s)$ rational functions of the kinematic variables $s \equiv$ letters
A_k constant matrices

- Alphabet $\Omega = \{\alpha_1, ..., \alpha_n\}$ set of all possible letters
$\rightarrow$ specifies which class of functions is required

- General solution

$$\vec{f}(s;\epsilon) = \mathcal{P}\exp\left[\epsilon \int_\gamma d\left(\sum_k A_k d\log \alpha_k(s)\right)\right]\vec{f}_0(\epsilon)$$

Iterated integrals and symbols: a few examples

- Classical polylogarithms

$$\mathrm{Li}_s(z) = \int_0^z d\log(t)\mathrm{Li}_{s-1}(z), \qquad \mathrm{Li}_1(z) = -\log(1-z)$$

$$\mathrm{Li}_s(z) = -\int_0^z d\log(t_1)\int_0^{t_1} d\log(t_2)...\int_0^{t_{s-1}} d\log(1-t_s)$$

$$\mathcal{S}\Big[\mathrm{Li}_s(z)\Big] = -(1-z)\otimes\overbrace{z\otimes...\otimes z}^{s-1\ \text{times}} \quad \Rightarrow \Omega = \{z, 1-z\}$$

From symbols to functions

- Differentiation acts on the last entry of the symbol

$$d\left(\alpha_1 \otimes \ldots \otimes \alpha_n\right) = d \ln \alpha_n \left(\alpha_1 \otimes \ldots \otimes \alpha_{n-1}\right)$$

- The differential of a weight-n symbol is written in terms of weight-$(n-1)$ symbols

$$d\mathcal{S}\left[f^{(n)}\right] = \sum_i c_i d \ln \alpha_i \; \mathcal{S}\left[f_i^{(n-1)}\right]$$

- Differential equation with a natural grading

$$d \begin{pmatrix} f^{(n)} \\ \vec{f}^{(n-1)} \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & A & & & \\ & 0 & B & & \mathbf{0} \\ & & 0 & \ddots & \\ & \mathbf{0} & & \ddots & C \\ & & & & 0 \end{pmatrix} \begin{pmatrix} f^{(n)} \\ \vec{f}^{(n-1)} \\ \vdots \\ 1 \end{pmatrix}$$

[Caron-Huot, Henn]

From symbols to functions

- Algorithm: given a symbol $\mathcal{S}^n$ of weight-n
  - by differentiating from weight-n down to weight-0 we determine the block triangular matrix M in

$$d\vec{f} = dM\,\vec{f} \qquad \vec{f} = \left(f^{(n)}, \vec{f}^{(n-1)}, ..., 1\right)^T$$

  - we integrate iteratively from weight-0 up to weight-n

✓ Function whose symbol matches $\mathcal{S}^n$

X The lower functional transcendentality pieces are still missing!

$\Rightarrow$ Bootstrap strategy: accommodate them into the ansatz