Landscape and Complexity Catastrophe

Michael R. Douglas

[1]Simons Center / Stony Brook University

string_data, Munich, March 28, 2018

Applying the standard paradigm of eternal inflation and the mulitverse to the string landscape suggests that the preferred vacua are some of the most complicated possible vacua. We make the analogy to the ultraviolet catastrophe of pre-quantum statistical mechanics and discuss possible ways out.
I also report on developments in mathematical software which I learned about at AITP 2018 (concurrent in Aussois, France).

String Landscape and Vacuum Selection

Underlying “mathematical” (top-down) structures:

1. String compactifications: choice of manifold M, extra structure B (branes, bundles, fluxes)
2. Effective potential $\mathcal{V}_{M,B}$, choice of local minimum $\Rightarrow$ vacuum V_i.
3. Cosmological dynamics of the multiverse $\Rightarrow$ probability distribution over vacua P_i.

Physical consequences:

1. Determines number of matter fields and gauge groups $\Rightarrow$ complexity of each vacuum.
2. Determines KK scale, susy breaking scale of each vacuum.
3. While we need to know the set of vacua to define this at all, if it is true that there are enough vacua to realize all the options for (1) and (2), then understanding P_i is crucial – vacuum selection.

What simple hypotheses and pictures can we make about this complicated structure?

Roles of string vacua

Most of the discussion so far has tried to judge whether vacua are quasi-realistic – metastable with supesymmetry breaking and small c.c., and containing the Standard Model.

But string compactifications must also play a role in the dynamics of vacuum selection. Given that we have observational evidence for inflation, it is reasonble to think that this dynamics involves earlier unobservable inflationary phases. This leads to the picture advocated by Linde, Vilenkin, Guth and others, of a multiverse created by eternal inflation.

From this point of view, any configuration satisfying the conditions for eternal inflation, roughly $m^2 \lesssim \frac{\Lambda}{M_{Pl,4}^2}$, could be important.

String compactifications

All of the constructions (type II, het, M, F) lead to a qualitatively but not quantitatively similar picture:

- *M* comes from a finite list and thus the Betti numbers are bounded.
- In supersymmetric constructions with branes and fluxes, the numbers of these are also bounded by tadpole conditions, and thus the number of gauge groups and matter fields is bounded. This is much less clear for nonsupersymmetric constructions (*e.g.* branes and antibranes together) and it is important to know whether a large set of these can be metastable.
- F theory leads to by far the largest numbers: as we will discuss, the sum of the Betti numbers can be $\sim 2 \cdot 10^6$, while at maximally symmetric points the total gauge group can have rank $\sim 10^5$.
- We do not know whether the other constructions can reproduce such large numbers. One would think so by relating F theory to IIb with 7-branes and dualizing – *e.g.* see Halverson's talk here. If not, this would cast doubt on their existence.

The maximal fourfold

The CY_4 M_{max} with the largest Euler number (and Betti numbers) was identified in the late 90's. It can be realized as a hypersurface in the weighted projective space

$$\mathbb{WP}(1, 1806, 75894, 466206, 108714, 1631721), \tag{1}$$

and also as an elliptic fibration over a B_2 fibration over $\mathbb{P}^1$, where B_2 is a toric surface defined by a fan with 100 rays.

In Taylor and Wang `arXiv:1511.03209` this fourfold was identified as the one with the most flux vacua, of estimated number $10^{272,000}$. The generic gauge group (without flux) was computed from the geometry of non-Higgsable clusters to be

$$E_8^9 \times F_4^8 \times (G_2 \times SU(2))^{16} . \tag{2}$$

Flux vacua on manifolds of large Betti number

The number of supersymmetric flux vacua in IIb/F theory can be estimated along the lines of Ashok-Denef-Douglas. There are three important parameters: the Betti number $b_4 = 1819942$, the flux tadpole number $Q = \chi/24 = 75852$, and a certain Chern class of the moduli space

$$c(T^*\mathcal{M}_c \otimes \mathcal{L}) = \int_{\mathcal{M}_c} \det(R + \omega). \tag{3}$$

This last factor has never been estimated and Taylor and Wang set it to one, in which case the number of flux vacua is roughly the number of lattice points in a b_4-dimensional sphere of radius $\sqrt{2Q}$.

Counting flux vacua for $b \gg Q$ was discussed in Denef's Les Houches lectures `arXiv:0803.1194`. One has

$$N(b, Q) \sim \frac{1}{2\pi i} \int \frac{dt}{t} e^{-Qt} \sum_{\vec{n} \in \mathbb{Z}^b} e^{t\vec{n}^2/2} \tag{4}$$

which can be estimated by saddle point as $10^{272,000}$

Flux vacua on manifolds of large Betti number

The number of supersymmetric flux vacua in IIb/F theory can be estimated along the lines of Ashok-Denef-Douglas. There are three important parameters: the Betti number $b_4 = 1819942$, the flux tadpole number $Q = \chi/24 = 75852$, and a certain Chern class of the moduli space

$$\mathrm{c}(T^*\mathcal{M}_c \otimes \mathcal{L}) = \int_{\mathcal{M}_c} \det(R + \omega). \tag{3}$$

This last factor has never been estimated and Taylor and Wang set it to one, in which case the number of flux vacua is roughly the number of lattice points in a b_4-dimensional sphere of radius $\sqrt{2Q}$.
Counting flux vacua for $b \gg Q$ was discussed in Denef's Les Houches lectures `arXiv:0803.1194`. One has

$$N(b, Q) \sim \frac{1}{2\pi i} \int \frac{dt}{t}\, e^{-Qt} \sum_{\vec{n}\in\mathbb{Z}^b} e^{t\vec{n}^2/2} \tag{4}$$

which can be estimated by saddle point as $10^{272{,}000}$.

Effective potential

To lead to quasi-realistic vacua, a string compactification must lead to an effective potential with supersymmetry breaking metastable isolated vacua of near-zero cosmological constant.
For several reasons we restrict to positive cosmological constant:

- Empirically, it is easier to imagine getting the observed positive dark energy as a pure c.c..
- Theoretically, AdS and dS vacua behave rather differently under tunneling transitions. Tunneling to AdS results in a singular "crunch" solution, while tunneling to dS looks sensible.
- There is also a recent argument (Oogurl and Vafa `arXiv:1610.01533`) that nonsupersymmetric AdS is always unstable – as is the case for dS produced by KKLT or other compactifications with a large volume limit.

In a given flux sector, we can sketch an approximate effective potential as

$$\mathcal{V} = \mathcal{V}_{susy}(W = \int G \wedge \Omega(\phi)) + \mathcal{V}_{susybreaking}(\phi) \qquad (5)$$

where ϕ are the moduli. The GKPV-type superpotential W generically depends on all of the moduli, but we need quantitative results to estimate the signs and sizes of the moduli masses. Such estimates are important both for the metastability of quasi-realistic vacua, and also to determine the eternal inflation regime.

What do the flux vacua on $\mathcal{M}_{max}$ look like? Since $Q/b_4 \sim 1/24 \ll 1$, one might wonder whether the flux superpotential in fact has very weak dependence on most of the moduli.

In a given flux sector, we can sketch an approximate effective potential as

$$\mathcal{V} = \mathcal{V}_{susy}(W = \int G \wedge \Omega(\phi)) + \mathcal{V}_{susybreaking}(\phi) \quad (5)$$

where ϕ are the moduli. The GKPV-type superpotential W generically depends on all of the moduli, but we need quantitative results to estimate the signs and sizes of the moduli masses. Such estimates are important both for the metastability of quasi-realistic vacua, and also to determine the eternal inflation regime.

What do the flux vacua on $\mathcal{M}_{max}$ look like? Since $Q/b_4 \sim 1/24 \ll 1$, one might wonder whether the flux superpotential in fact has very weak dependence on most of the moduli.

Several arguments suggest that the measure (ignoring anthropic selection) will be dominated by vacua from M_{max}.

- Vacuum counting, *i.e.* it will be true if the measure is in any sense uniform.
- The measure derived from eternal inflation at equilibrium (the long time limit) will be the dominant eigenvector of a transition matrix, and will be dominated by the longest lived metastable dS vacua. This will be the vacuum with the smallest supersymmetry breaking scale, which will come from the largest ratio of fluxes, and thus the largest Q (see `arXiv:1204.6626`).
- Many assumptions in the previous arguments, but they fit with the simple idea that equilibrium is a maximum entropy state.

Is this a sensible conclusion? The anthropic solution to the cosmological constant problem requires (depending on assumptions) 10^{60} to 10^{120} vacua. We might expect the number of string vacua to be comparable to this.

More pointedly, too many hidden sectors seem likely to contradict observational bounds. For example, axions in certain mass windows can be excluded by superradiance of rotating black holes.

How could we escape this claim?

1. Perhaps there aren't actually so many vacua because of dualities.
2. Perhaps almost all of the putative complex vacua are in fact unstable.
3. Perhaps the true dynamics does not respect a $4+6$-dimensional split of spacetime.
4. Perhaps quantum cosmology does not reach equilibrium.
5. The analogy to the "UV catastrophe" of classical stat mech suggests that there might be some sort of "quantum" nature to the problem which we are missing.

Is this a sensible conclusion? The anthropic solution to the cosmological constant problem requires (depending on assumptions) 10^{60} to 10^{120} vacua. We might expect the number of string vacua to be comparable to this.

More pointedly, too many hidden sectors seem likely to contradict observational bounds. For example, axions in certain mass windows can be excluded by superradiance of rotating black holes.

How could we escape this claim?

1. Perhaps there aren't actually so many vacua because of dualities.
2. Perhaps almost all of the putative complex vacua are in fact unstable.
3. Perhaps the true dynamics does not respect a $4+6$-dimensional split of spacetime.
4. Perhaps quantum cosmology does not reach equilibrium.
5. The analogy to the "UV catastrophe" of classical stat mech suggests that there might be some sort of "quantum" nature to the problem which we are missing.

Is this a sensible conclusion? The anthropic solution to the cosmological constant problem requires (depending on assumptions) 10^{60} to 10^{120} vacua. We might expect the number of string vacua to be comparable to this.

More pointedly, too many hidden sectors seem likely to contradict observational bounds. For example, axions in certain mass windows can be excluded by superradiance of rotating black holes.

How could we escape this claim?

1. Perhaps there aren't actually so many vacua because of dualities.
2. Perhaps almost all of the putative complex vacua are in fact unstable.
3. Perhaps the true dynamics does not respect a $4+6$-dimensional split of spacetime.
4. Perhaps quantum cosmology does not reach equilibrium.
5. The analogy to the "UV catastrophe" of classical stat mech suggests that there might be some sort of "quantum" nature to the problem which we are missing.

Point (1), duality relations between flux vacua, may not be widely appreciated. As a topological manifold, the only structure on the middle homology is the quadratic intersection form. Thus one can bring the flux to a canonical form using $SO(b_4/2, b_4/2; \mathbb{Z})$. This is not true geometrically and different cycles are distinguished by their periods. However the periods are of course only determined after solving for the moduli.

This is taken into account in the AD formula by the "volume" factor which is integrated over a fundamental region of the duality group. In known examples this factor is small, for example (Moore, `arXiv:1508.05612`)

$$\mathrm{vol}(\mathcal{M}_{K3}) \sim 1.66 \times 10^{-61}. \tag{6}$$

If the volume of $\mathcal{M}_{M_{max}}$ were the Q'th power of this, then the expected number of flux vacua would be very small. Nobody I have talked to knows how to estimate this number.

Point (1), duality relations between flux vacua, may not be widely appreciated. As a topological manifold, the only structure on the middle homology is the quadratic intersection form. Thus one can bring the flux to a canonical form using $SO(b_4/2, b_4/2; \mathbb{Z})$. This is not true geometrically and different cycles are distinguished by their periods. However the periods are of course only determined after solving for the moduli.

This is taken into account in the AD formula by the “volume” factor which is integrated over a fundamental region of the duality group. In known examples this factor is small, for example (Moore, `arXiv:1508.05612`)

$$\mathrm{vol}(\mathcal{M}_{K3}) \sim 1.66 \times 10^{-61}. \quad (6)$$

If the volume of $\mathcal{M}_{M_{max}}$ were the Q'th power of this, then the expected number of flux vacua would be very small. Nobody I have talked to knows how to estimate this number.

One argument for (2), instability, would follow from a (perhaps too literal) interpretation of work by Marsh, McAllister, Wrase `arXiv:1112.3034` and subsequent works. They worked out the counting of flux vacua under the assumption that the coefficients in a Taylor expansion of the superpotential are independent (Gaussian) random variables, and found that the fraction of stable vacua in a theory with N scalar fields goes as $\exp -cN^p$ with $p \gtrsim 1.3$.

If applicable to stringy flux vacua, this might kill them all, but the assumption that coefficients in the superpotential are independent random variables is suspect. One way it could fail is if we expand around a vacuum in which the different scalar fields have widely different mass scales. As the authors point out from the start, the N which controls the suppression is the number of scalar fields with masses below the susy breaking scale, while the number of vacua is controlled by all the scalar fields. So, given a small susy breaking scale one expects many metastable vacua.

Even without the low scale susy breaking assumption it seems unlikely (to me) that coefficients in an series expansion of a generic GKPV superpotential will have this form One can derive some results for the quintic threefold (in the large volume limit, and in expansions around the symmetric 1-parameter locus) and they do not look like this. Dine `arXiv:1512.08125` points out that such potentials would lead to quantum loop corrections growing as positive powers of N, which would drastically modify the Kähler potential.

It is also the case that analogous nonsupersymmetric problems do not behave this way. For a compact moduli space it is clear that each flux sector will have at least one vacuum (the global minimum). In the random potential language this comes from global constraints on the coefficients. This is the case even for problems such as critical points of a random Gaussian field, see for example Yamada and Vilenkin `arXiv:1712.01282`.

Point (3) about the need to justify using a $4+6$ split within a more general framework is pretty widely appreciated but there is no clear way to do this.

Perhaps the best argument for the split (?) is that it is hard to produce nonperturbative potentials in string/M theory that stabilize all the moduli, and so far as I know the KKLT type of model only exists in 4 and fewer space-time dimensions. This is related to the fact that supersymmetry in $D \geq 5$ is much more constraining than $D=4, N=1$. So maybe all the $D>4$ (equivalently very large volume) parts of the landscape are simple enough to analyze?

A (somewhat) related idea is that the complex vacua might necessarily be large volume and have a small KK energy scale (*e.g.* see McAllister's talk here). This would not in itself say why they do not dominate the measure. But perhaps this prevents stabilizing them.

Point (3) about the need to justify using a $4 + 6$ split within a more general framework is pretty widely appreciated but there is no clear way to do this.

Perhaps the best argument for the split (?) is that it is hard to produce nonperturbative potentials in string/M theory that stabilize all the moduli, and so far as I know the KKLT type of model only exists in 4 and fewer space-time dimensions. This is related to the fact that supersymmetry in $D \geq 5$ is much more constraining than $D = 4, N = 1$. So maybe all the $D > 4$ (equivalently very large volume) parts of the landscape are simple enough to analyze?

A (somewhat) related idea is that the complex vacua might necessarily be large volume and have a small KK energy scale (*e.g.* see McAllister's talk here). This would not in itself say why they do not dominate the measure. But perhaps this prevents stabilizing them.

$$ds^2_{4+6} = u^2(t,y)\left(-dt^2 + a(t,y)^2 d\vec{x}^2\right) + g_{IJ}(y)dy^I dy^J. \tag{7}$$

Warp factor variation is another aspect of compactification which I feel should be better understood, especially its dynamics during inflation. Above we give a general ansatz for a cosmological solution in 4 dimensions with a general internal metric, warp factor and scale factor. The form $u^2(y)$ for the warp factor was introduced in `arXiv:0911.3378` where it was shown that the constraint determining it can be written (in $D=4$) as a linear equation,

$$\left(-\nabla^2 + T_{int}(y)\right) u(y) = C \tag{8}$$

where C is determined by the normalization of the 4d Planck scale.

One can show that in a general time-dependent solution, both the warp factor and scale factor must depend on the extra dimensions, so inflation and warping are in fact linked. This hasn't been studied so far as I know.

$$ds^2_{4+6} = u^2(t,y)\left(-dt^2 + a(t,y)^2 d\vec{x}^2\right) + g_{IJ}(y)dy^I dy^J . \qquad (7)$$

Warp factor variation is another aspect of compactification which I feel should be better understood, especially its dynamics during inflation. Above we give a general ansatz for a cosmological solution in 4 dimensions with a general internal metric, warp factor and scale factor. The form $u^2(y)$ for the warp factor was introduced in `arXiv:0911.3378` where it was shown that the constraint determining it can be written (in $D = 4$) as a linear equation,

$$\left(-\nabla^2 + T_{int}(y)\right) u(y) = C \qquad (8)$$

where C is determined by the normalization of the 4d Planck scale.

One can show that in a general time-dependent solution, both the warp factor and scale factor must depend on the extra dimensions, so inflation and warping are in fact linked. This hasn't been studied so far as I know.

Cosmology,equilibrium and initial conditions

Conceptually, the most important of the questions we raised is (4), the assumption that multiverse cosmology reaches an equilibrium. It was first assumed since it is a natural way to remove the dependence on the initial conditions. This both maintains the analogy with observed inflation, and would seem attractive given that we don't know the initial conditions.

It is a very far reaching assumption and it leads to the dominance of entropy and the "catastrophe." One can ask if there are any well motivated axioms which lead to a non-equilibrium cosmology. In my recent `arXiv:1706.06430` with Denef, Greene and Zukowski, we postulate a definition of "simple" vacua in which simplicity is defined in terms of a hypothetical quantum computer which simulates the multiverse. This tends to lead to measure factors close to the initial conditions and, assuming that the initial conditions are simple, eliminates the catastrophe.

Cosmology,equilibrium and initial conditions

Conceptually, the most important of the questions we raised is (4), the assumption that multiverse cosmology reaches an equilibrium. It was first assumed since it is a natural way to remove the dependence on the initial conditions. This both maintains the analogy with observed inflation, and would seem attractive given that we don't know the initial conditions.

It is a very far reaching assumption and it leads to the dominance of entropy and the "catastrophe." One can ask if there are any well motivated axioms which lead to a non-equilibrium cosmology. In my recent `arXiv:1706.06430` with Denef, Greene and Zukowski, we postulate a definition of "simple" vacua in which simplicity is defined in terms of a hypothetical quantum computer which simulates the multiverse. This tends to lead to measure factors close to the initial conditions and, assuming that the initial conditions are simple, eliminates the catastrophe.

There may be other modifications to the dynamics of eternal inflation with similar effects, but to change the nature of the long time limit, they must create terminal vacua, break detailed balance, or both. Perhaps a proper understanding of tunneling to AdS will have such effects (see Garriga and Vilenkin `arXiv:1210.7540`).

New "quantum" effects on geometry and cosmology

The most provocative option is (5), that the analogy between the "complexity catastrophe" and the ultraviolet catastrophe suggests an analogous resolution, that there will be some change in our understanding analogous to quantization, which much reduces the weight of complex vacua in the measure. Can we suggest anything?

- Strong coupling effects – naively quantum corrections are controlled by g_s, but perhaps they are actually controlled by Ng_s where N is the complexity or number of fields.
- Quantum gravity effects – naively we establish the existence of a compactification by solving supergravity equations with stringy corrections in the extra dimensions, but maybe this is not right. Especially, it might be that in an inflating universe with rapidly varying warp factor in the extra dimensions, there might be event horizons in the 10-dimensional solution which are not visible from the 4-dimensional point of view, invalidating the $4 + 6$ split.
- Limits on quantum complexity as in the work with Denef *et.al.*

New "quantum" effects on geometry and cosmology

The most provocative option is (5), that the analogy between the "complexity catastrophe" and the ultraviolet catastrophe suggests an analogous resolution, that there will be some change in our understanding analogous to quantization, which much reduces the weight of complex vacua in the measure. Can we suggest anything?

- Strong coupling effects – naively quantum corrections are controlled by g_s, but perhaps they are actually controlled by Ng_s where N is the complexity or number of fields.
- Quantum gravity effects – naively we establish the existence of a compactification by solving supergravity equations with stringy corrections in the extra dimensions, but maybe this is not right. Especially, it might be that in an inflating universe with rapidly varying warp factor in the extra dimensions, there might be event horizons in the 10-dimensional solution which are not visible from the 4-dimensional point of view, invalidating the $4+6$ split.
- Limits on quantum complexity as in the work with Denef *et al.*

Developments in computational mathematics

- AITP (Artificial Intelligence and Theorem Proving) 2018, `http://aitp-conference.org/2018/`

Formal methods – branch of computer science which develops tools for proving software correct (*i.e.*, implements some specification). Automatic proof of mathematical theorems has long been considered a benchmark problem. Not much success at autonomous theorem provers, but verification works and has been used for large theorems such as the four color theorem and the Feit-Thompson theorem. Theorem verification does not work as well as one would like. One must spell out many more steps, generally a verifiable theorem is 5 times as long as a human readable one. There is much hope that recent progress in AI will improve this, for example the success of Alpha Go has suggested reformulating theorem proving as a game. Why is this relevant for us? A way to organize and integrate the many mathematical definitions, algorithms, theorems, datasets, etc. we use.

Developments in computational mathematics

- AITP (Artificial Intelligence and Theorem Proving) 2018, `http://aitp-conference.org/2018/`

Formal methods – branch of computer science which develops tools for proving software correct (*i.e.*, implements some specification). Automatic proof of mathematical theorems has long been considered a benchmark problem. Not much success at autonomous theorem provers, but verification works and has been used for large theorems such as the four color theorem and the Feit-Thompson theorem. Theorem verification does not work as well as one would like. One must spell out many more steps, generally a verifiable theorem is 5 times as long as a human readable one. There is much hope that recent progress in AI will improve this, for example the success of Alpha Go has suggested reformulating theorem proving as a game.

Why is this relevant for us? A way to organize and integrate the many mathematical definitions, algorithms, theorems, datasets, etc. we use.

Developments in computational mathematics

- AITP (Artificial Intelligence and Theorem Proving) 2018, `http://aitp-conference.org/2018/`

Formal methods – branch of computer science which develops tools for proving software correct (*i.e.*, implements some specification). Automatic proof of mathematical theorems has long been considered a benchmark problem. Not much success at autonomous theorem provers, but verification works and has been used for large theorems such as the four color theorem and the Feit-Thompson theorem. Theorem verification does not work as well as one would like. One must spell out many more steps, generally a verifiable theorem is 5 times as long as a human readable one. There is much hope that recent progress in AI will improve this, for example the success of Alpha Go has suggested reformulating theorem proving as a game. Why is this relevant for us? A way to organize and integrate the many mathematical definitions, algorithms, theorems, datasets, etc. we use.

- OpenDreamKit http://opendreamkit.org/ This is a large European collaboration (50 people, 7.6 MEUR over 4 years, ending mid-2019) to develop what they call Virtual Research Environments incorporating many open source mathematical platforms (Sage, Singular, Pari/GP, GAP) and other UI tools such as Jupyter.
- Formal Abstracts. This is a description of a mathematical result or paper in a precise (formal) version which could be understood by a computer and serve as a basis for (for example) a mathematical search engine. Thomas Hales, a pure mathematician who proved the Kepler conjecture and then organized a group collaboration Flyspeck which verified the proof by computer, has been developing this idea.
- International Conference on Mathematical Software, July 24-28, Notre Dame.