Classifying fibration structures in Calabi-Yau constructions

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Based on work with:

Alexander Haupt and Andre Lukas:
arXiv:1303.1832, arXiv:1405.2073

Lara Anderson, Fabio Apruzzi, Xin Gao and Seung-Joo Lee:
arXiv:1507.03235

Lara Anderson, Xin Gao and Seung-Joo Lee
arXiv:1608.07554, 1608.07555 & 1708.07907

Lara Anderson and Brian Hammack
arXiv:1804.XXXXX

Complete Intersection Calabi-Yau (CICYs)

- A family of CICYs is described by a configuration matrix:

$$[\mathbf{n}|\mathbf{q}] \equiv \left[\begin{array}{c|ccc} n_1 & q_1^1 & \ldots & q_K^1 \\ \vdots & \vdots & \ddots & \vdots \\ n_m & q_1^m & \ldots & q_K^m \end{array}\right]$$

with m rows and K+1 columns.

- Ambient space is $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$
- Remaining columns give degree of defining relations:

Calabi-Yau condition:

$$\sum_{\alpha=1}^{K} q_\alpha^r = n_r + 1$$

D-fold condition:

$$\sum_r n_r - K \stackrel{!}{=} D$$

Example:

- An example of a configuration matrix (CICY four-fold 244):

$$\left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 1 & 2 \\ \mathbb{P}^3 & 0 & 4 \end{array}\right]$$

- The different choices of defining relation corresponds to a redundant description of part of complex structure moduli space:

$$p_1 = \sum_{i,a} c_{i,a} x^i y^a \quad p_2 = \sum_{i,\ldots,\delta} d_{iab\alpha\beta\gamma\delta} x^i y^a y^b z^\alpha z^\beta z^\gamma z^\delta$$

- This example is a Calabi-Yau four-fold.

CICY Data Sets:

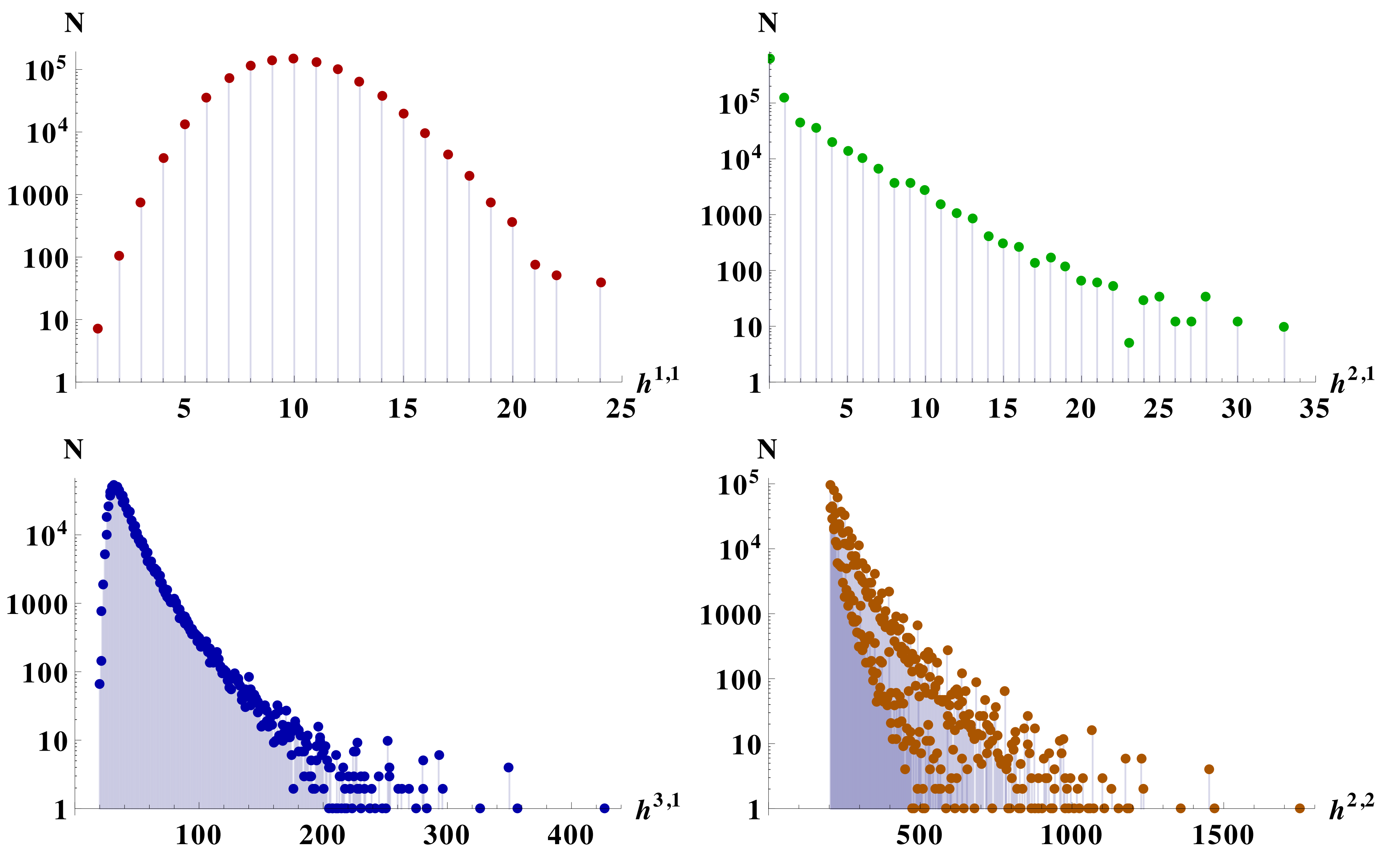
- Three-Folds:
    - Hübsch, Commun.Math.Phys. 108 (1987) 291
    - Green et al, Commun.Math.Phys. 109 (1987) 99
    - Candelas et al, Nucl.Phys. B 298 (1988) 493
    - Candelas et al, Nucl.Phys. B 306 (1988) 113
- Data Set classified: 7890 configuration matrices in the set.
- This is the data set that has been used extensively in heterotic compactifications leading to, for example, a classification of 1000s of heterotic standard models.

- Four-Folds:

    - Brunner et al, Nucl.Phys. B498 (1997) 156-174
    - JG et al, JHEP 1307 (2013) 070
    - JG et al, JHEP 1409 (2014) 093

- Data set classified: 921,497 configuration matrices in the set.
- We are currently building up the technology to use this data set for F-theory analysis and model building – as I will describe later.
- All Hodge data etc. are available for these manifolds:

Example: fourfold Hodge data



Generalized CICYs:

- One can consider CICY configuration matrices with negative signs in them and still obtain algebraic varieties!

Consider:

$$\left[\begin{array}{c|cccc} \mathbb{P}^1 & 1 & 1 & 1 & -1 \\ \mathbb{P}^1 & 1 & 1 & -1 & 1 \\ \mathbb{P}^5 & 3 & 1 & 1 & 1 \end{array}\right]$$

- In such a matrix we consider "applying" the hypersurfaces one at a time from left to right…

$$h^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(1,1,3)) = 224$$

- However

$$h^0(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^5, \mathcal{O}(1,-1,1)) = 0$$

there is no such polynomial in the ambient space

- Consider $\mathcal{O}(1,-1,1)$ restricted to one of the proceeding hypersurfaces, $\mathcal{M}$:

$$\mathcal{M} = \left[\begin{array}{c|c} \mathbb{P}^1 & 1 \\ \mathbb{P}^1 & 1 \\ \mathbb{P}^5 & 1 \end{array} \right] \qquad \mathcal{N}_{\mathcal{M}} = \mathcal{O}(1,1,1)$$

Using the Koszul sequence, one can show that:

$$h^0(\mathcal{M}, \mathcal{O}(1,-1,1)) = 1$$

- One might ask what does section look like?
  - If we could write it in coordinates on $\mathcal{M}$ it would be polynomial.
  - Thus the sections can't have singularities on $\mathcal{M}$ even though we might naively think of them as rational functions.

- To see a little more of the structure lets just consider the following piece of the configuration matrix:

$$\tilde{\mathcal{M}} = \left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^1 & 1 & -1 \\ \mathbb{P}^5 & 1 & 1 \end{array} \right]$$

- We have the defining relations:

$$p_1 = C_0(X,Z)Y^0 + C_1(X,Z)Y^1$$

$$p_2 = \frac{N}{D} = \frac{C_0(X,Z)}{Y^1} = -\frac{C_1(X,Z)}{Y^0}$$

- So on any patch in the normal open cover of the second $\mathbb{P}^1$ the defining relations are polynomial.
- But the co-dimension two manifold can not be written globally as a polynomial in ambient space homogeneous coordinates.
- Note the second defining relation above is unique.

- This manifold can be shown to be smooth.

$$\left[\begin{array}{c|cccc} \mathbb{P}^1 & 1 & 1 & 1 & -1 \\ \mathbb{P}^1 & 1 & 1 & -1 & 1 \\ \mathbb{P}^5 & 3 & 1 & 1 & 1 \end{array}\right]$$

- Its Hodge data and Euler Characteristic are:

$$\chi = -156$$

$$h^{1,1}(X) = 3 \quad h^{2,1}(X) = 81$$

- This is not a manifold in the regular CICY list.
- This Hodge pair does appear in the Kreuzer-Skarke data set – but this construction does give rise to manifolds that are definitely new.
- It is important to check the cohomology of the trivial bundle on these manifolds to ensure connectedness.

Properties of CICYs: Torus Fibrations

- Consider configuration matrices which can be put in the form:

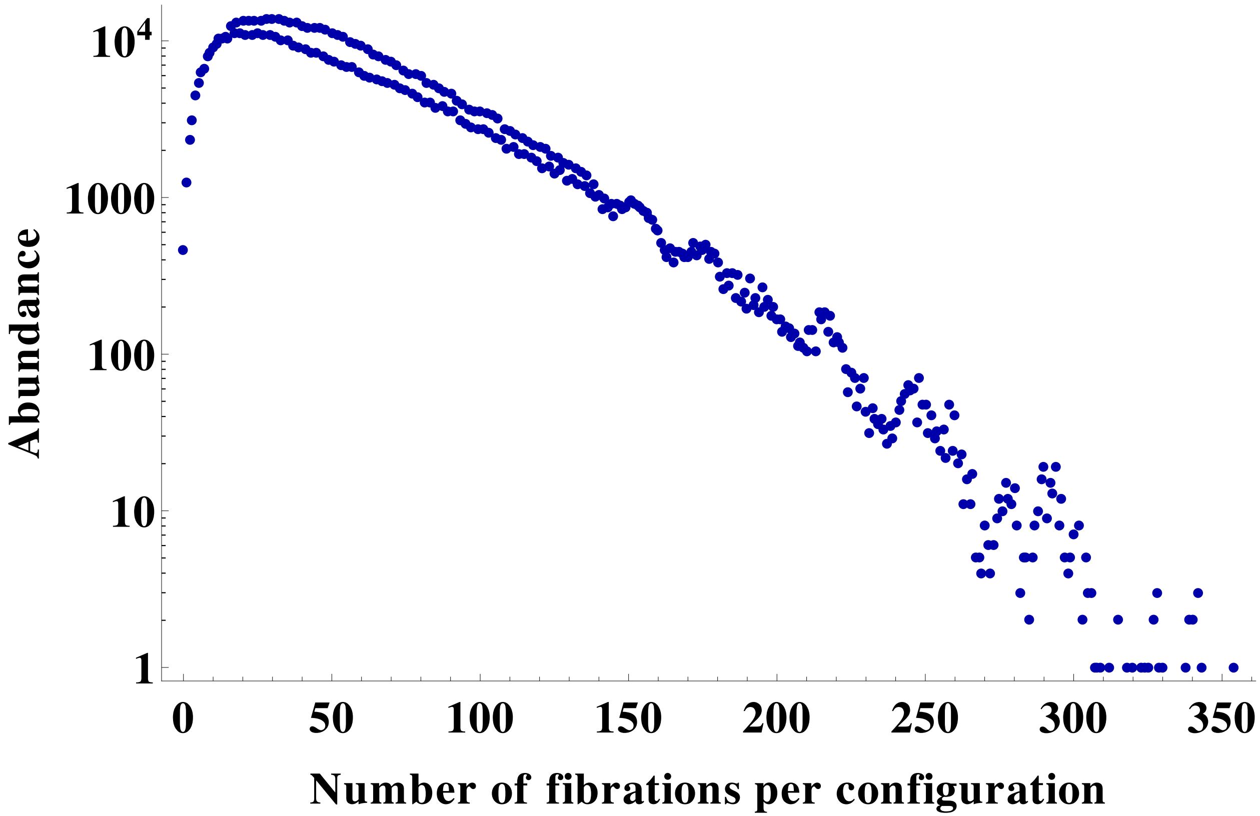
$$\left[\begin{array}{c|c} \mathcal{A}_1 & \mathcal{F} \end{array}\right] = T^2$$

$$\text{Base: } \left[\begin{array}{c|c} \mathcal{A}_2 & \mathcal{B} \end{array}\right] \qquad \left[\begin{array}{c|cc} \mathcal{A}_1 & 0 & \mathcal{F} \\ \mathcal{A}_2 & \mathcal{B} & \mathcal{T} \end{array}\right]$$

- This is an torus fibred four-fold
- In our list of 921,497 matrices, 921,020 have such a fibration structure.

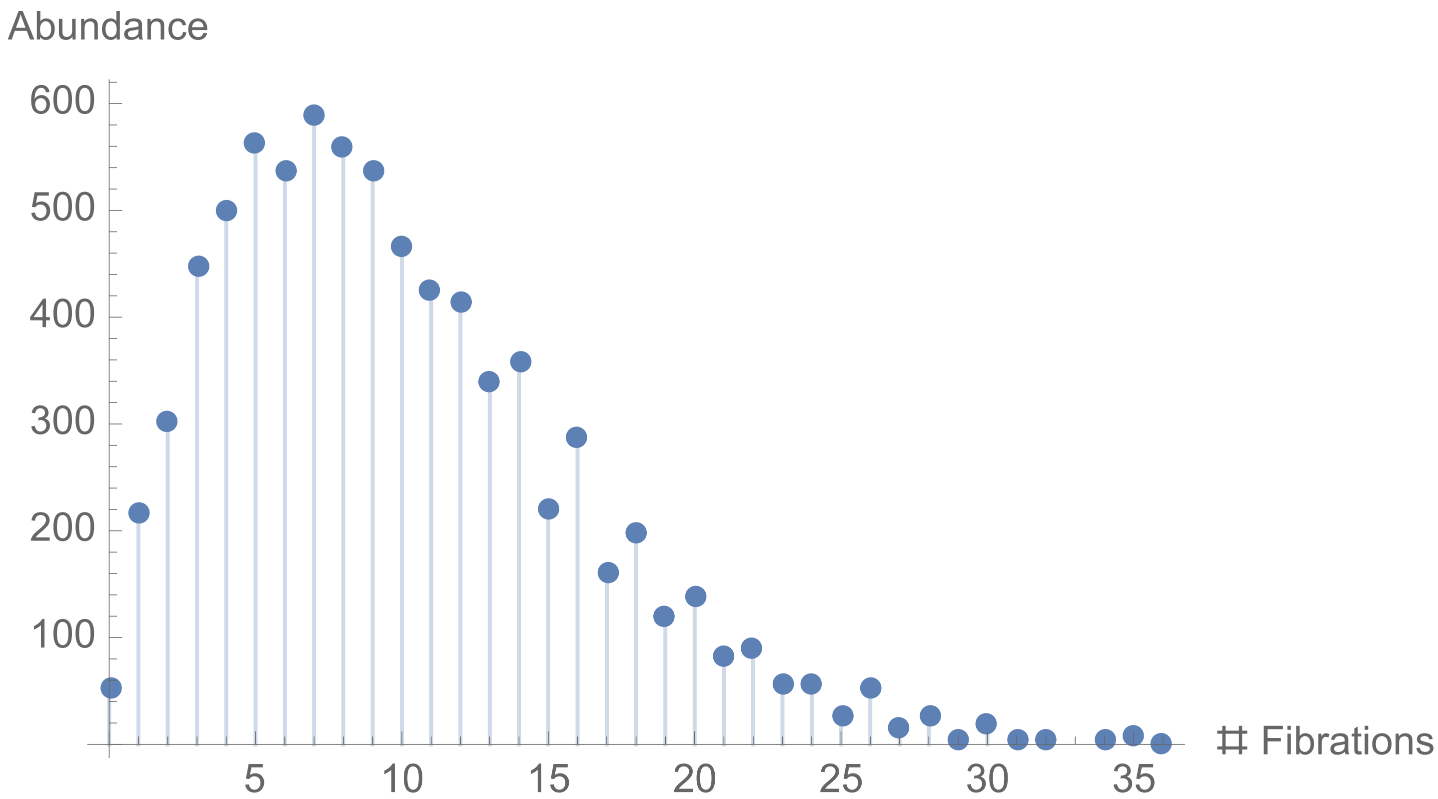
A given manifold/configuration matrix may admit many obvious torus fibrations…

See also S. Johnson and W. Taylor arXiv:1406.0514 and arXiv:1605.08052



- Total of 50,114,908 different torus fibrations.
- Average of 54.4 fibrations per manifold.

•Threefolds:



- Total of 77,744 different torus fibrations in data set.
- Average of 9.85 fibrations per manifold…

- As a simple example we can have:

$$\left(\begin{array}{c|cc:cccc} \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 0 & 1 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 1 & 0 & 1 & 0 & 0 \\ \mathbb{P}^2 & 1 & 2 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c|c:ccccc} \mathbb{P}^1 & 0 & 1 & 1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 0 & 0 & 1 & 0 & 0 \\ \mathbb{P}^2 & 2 & 1 & 0 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c|c:ccccc} \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{c|:cccccc} \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \left(\begin{array}{c|:cccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \end{array}\right) \left(\begin{array}{c|:cccccc} \mathbb{P}^1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \mathbb{P}^1 & 0 & 0 & 1 & 1 & 0 & 0 \\ \mathbb{P}^2 & 1 & 0 & 2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hdashline \mathbb{P}^2 & 0 & 0 & 0 & 0 & 2 & 1 \end{array}\right)$$

- Note that we have a variety of different bases here (Hirzebruchs, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ etc in this case).
- It doesn't just have to be *torus* fibration structures that exist in a CICY… lower dimensional Calabi-Yau of all kinds are ubiquitous (results coming soon).

Can we go beyond these obvious fibrations?

- Conjecture by Kollár (rough description):

  A Calabi-Yau threefold is genus one fibered if and only if there exists a divisor D *such that*

$$D \cdot C \geq 0 \quad \textit{for every algebraic curve} \quad C$$

$$D^3 = 0$$

$$D^2 \neq 0$$

(and similarly in higher dimensional cases)

- *Proven in threefold case by Oguiso, Wilson.*

- The question is, do we have good computational control over all of the elements of $h^{1,1}$?
- In favorable cases we do. For example in the case,

$$X = \left[\begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array} \right]$$

all divisor classes descend from divisor classes in the ambient space.

- In non-favorable cases we don't. For example

$$X' = \left[\begin{array}{c|cc} \mathbb{P}^1 & 1 & 1 \\ \mathbb{P}^2 & 3 & 0 \\ \mathbb{P}^2 & 0 & 3 \end{array} \right]$$

has $h^{1,1} = 19$ but $h^{1,1}$ of the ambient space is only 3 .

- Of 7890 CICY threefolds in the original list, only 4874 are favorable.

- We can obtain new configuration matrices describing the same manifolds by the process of contraction/splitting:

$$\left[\begin{array}{c|ccccc} n & 1 & 1 & \dots & 1 & 0 \\ \mathbf{n} & \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_{n+1} & \mathbf{q} \end{array}\right] \longleftrightarrow \left[\begin{array}{c|cc} \mathbf{n} & \sum_{a=1}^{n+1} \mathbf{u}_a & \mathbf{q} \end{array}\right]$$

Euler number doesn't change $\Leftrightarrow$ manifolds same

  - Use this to increase the size of the ambient space affording the configuration a better chance of being favorable

- By splitting we have obtained favorable descriptions of all but 7842 of the 7890 CICYS.

- We can then compute data such as intersection numbers, line bundle cohomology etc completely in these cases.

What about the remaining 48?

- It turns out that these can all be written as hypersurfaces in direct products of del Pezzo surfaces.
- For example:

$$X_3 = \left[\begin{array}{c|cccc} \mathbb{P}^1 & 1 & 0 & 0 & 1 \\ \mathbb{P}^2 & 2 & 0 & 0 & 1 \\ \mathbb{P}^4 & 0 & 2 & 2 & 1 \end{array} \right]$$

  can be written as the anti-canonical hypersurface inside

$$dP_4 = \left[\begin{array}{c|c} \mathbb{P}^1 & 1 \\ \mathbb{P}^2 & 2 \end{array} \right] \quad \text{times} \quad dP_5 = \left[\begin{array}{c|cc} \mathbb{P}^4 & 2 & 2 \end{array} \right]$$

- Enough is known about the divisors of del Pezzo's that we can then find a favorable description of these spaces too.

Thus we find a favorable description of all CICYs.

- With these descriptions we can compute almost all of the information we need to investigation the fibrations of all CICYs. There are, however, some subtleties associated to the Kahler cone structure.
- For the 4874 “Kahler favorable” cases which are favorable in products of projective spaces, and for which the Kahler cone is the naive one induced from the ambient space, obvious fibrations and Kollár fibrations coincide.

  However, in general there can be many more Kollár fibrations than obvious ones.

- A good example is the Schoen manifold – which admits an infinite number of genus one fibrations!

(See also Grassi,Morrison; Aspinwall, Gross; Oguiso; Piateckii-Shapiro, Shafarevich).

Fibrations and quotients

- One can create a new (non-simply connected) Calabi-Yau by quotienting a CICY by a freely acting symmetry.
- Example: Take the bicubic:

$$X = \left[\begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array} \right]$$

- With homogeneous coordinates:

$$x_{a,i} \qquad\qquad a = 1, 2 \quad i = 0, 1, 2$$

- And quotient by the following $\mathbb{Z}_3$ group action:

$$g : x_{a,j} \to \omega^j x_{a,j}$$

- Clear in this case, the quotienting preserves the fibration.

- More generally what can we say about fibrations in quotients of CICYs?
- Classification of symmetries:
  - Braun, JHEP 1104 (2011) 005

(The equivalent classifications for the four-folds has not yet been carried out – although may not be as interesting.)

- A lot of work has already been done classifying the properties of the associated quotients:
  - Candelas et al, arXiv:1602.06303
  - Braun et al, arXiv:1512.08367
  - Candelas et al, arXiv:1511.01103
  - Constantin et al, arXiv:1607.01830

Upcoming work with Lara Anderson and Brian Hammack:

- Of the 1632 symmetry-CICY pairs (for manifolds with fibration), 1552 of them preserve *some* fibration (95%).
- Of 20700 fibration/symmetry pairs, 17161 preserved.

Symmetry	Fibs preserved	Fibs not preserved	%preserved
$\mathbb{Z}_2$	8812	464	95%
$\mathbb{Z}_3$	175	201	46.5%
$\mathbb{Z}_4$	120	244	33.0%
$\mathbb{Z}_5$	0	30	0.0%
$\mathbb{Z}_6$	62	438	12.4%
$\mathbb{Z}_2 \times \mathbb{Z}_2$	7711	1488	83.8%
$\mathbb{Z}_2 \times \mathbb{Z}_4$	105	200	34.4%
$\mathbb{Z}_3 \times \mathbb{Z}_3$	176	0	100%

- There are several larger symmetries that appear (including non-Abelian symmetries), none of which preserve any fibrations:

$$\mathbb{Z}_8\ ,\ \mathbb{Z}_{10}\ ,\ \mathbb{Z}_{12}\ ,\ Q_8\ ,\ \mathbb{Z}_2 \times Q_8\ ,\ \mathbb{Z}_3 \rtimes \mathbb{Z}_4\ ,$$
$$\mathbb{Z}_8 \times \mathbb{Z}_2\ ,\ \mathbb{Z}_4 \rtimes \mathbb{Z}_4\ ,\ \mathbb{Z}_8 \rtimes \mathbb{Z}_2\ ,\ \mathbb{Z}_4 \times \mathbb{Z}_4\ ,$$
$$\mathbb{Z}_{10} \times \mathbb{Z}_2$$

- In any case where the fibration is preserved, the base of the quotiented fibration is divided by same group as total space.
- Classifications of the bases that appear will be provided in the paper.

Summary:

- We reviewed some recent progress in studying the geometry of Calabi-Yau manifolds described as complete intersections in products of projective spaces.
- Essentially all known Calabi-Yau are genus one fibered
- In every case where we have looked, essentially all known Calabi-Yau are multiply fibered
- For the CICYs, on small enough data sets integer computation can be performed using existing techniques…
- … but polynomial based computations can be prohibitively expensive to perform exhaustively, even over these small data sets, and we could benefit from new tools.