The Cauchy Problem for a dissipative scalar Wave Equation

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Introduction

We consider the Cauchy Problem for the dissipative scalar wave equation

$$\left(\partial_t^2 - \partial_u^2 - g(u)\partial_t + f(u)\right)\psi(t,u) = 0 \tag{1}$$

with initial data $\psi(0,u), i\partial_t\psi(0,u) \in C_0^\infty(\mathbb{R},\mathbb{C})$, $g \in C_0^\infty(\mathbb{R},\mathbb{C})$ and $f \in C^\infty(\mathbb{R})$ such that

$$\begin{cases} |f(u)| + u|f'(u)| + u^2|f''(u)| \leq \frac{c}{u^3} & \text{if } u > 0 \\ |f(u)| + |f'(u)| + |f''(u)| \leq ce^{\gamma u} & \text{if } u < 0 \end{cases}$$

$c, \gamma > 0$

Hamiltonian form of (1):

$$i\partial_t \Psi = H\Psi \tag{2}$$

with $H = \begin{pmatrix} 0 & 1 \\ A & g \end{pmatrix}$, $A = -\partial_u^2 + f(u)$ and domain of definition $\mathcal{D}(H) = (\mathcal{S}(\mathbb{R}, \mathbb{C}))^2$.

The initial data $\Psi(0,u) = \begin{pmatrix} \psi(0,u) \\ i\partial_t \psi(0,u) \end{pmatrix}$ is an element of $(C_0^\infty(\mathbb{R}, \mathbb{C}))^2$.

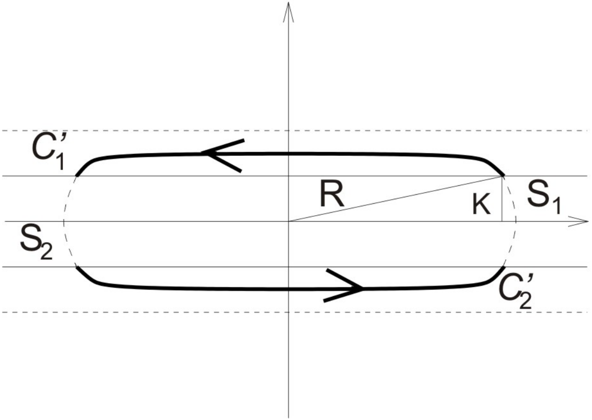
Remark: It is not possible to introduce a scalar product such that H is symmetric with respect to it.

The solution for the Cauchy problem

Theorem: The solution of (2) takes the form

$$\Psi(t,u) = -\frac{1}{2\pi i} \lim_{R\to\infty} \int_{C_1' \cup C_2'} e^{-i\omega t} (R_\omega \Psi_0(u)) \, \mathrm{d}\omega$$

Ψ_0 is the initial data and R_ω denotes the resolvent $(H-\omega)^{-1}$. It can be shown that the resolvent exists if $|\mathrm{Im}\ \omega|$ is large.


C'1
R
K
S1
S2
C'2

Separation of variables

We make the separation ansatz

$$\psi(t,u) = e^{-i\omega t}\phi(u)\,.$$

This leads us to the following radial equation of Schrödinger type:

$$\left(-\partial_u^2 + V(u)\right)\phi(u) = 0$$

with

$$V(u) := -\omega^2 + i\omega g(u) + f(u)$$

Now we want to construct solutions of this equation which decay exponentially at $+\infty$ or respectively $-\infty$.

Construction of the Jost solutions

In order to find a solution of the Schrödinger equation which decays at $+\infty$ we need the WKB wave function

$$\grave{\alpha}(u) := \grave{c}V(u)^{-\frac{1}{4}}\left(-\exp\int_{u_0}^{u}\sqrt{V}\right)$$

As we find that

$$\grave{\alpha} \sim \begin{cases} e^{i\omega u} & \text{if}\,\mathrm{Im}\ \omega > 0 \\ e^{-i\omega u} & \text{if}\,\mathrm{Im}\ \omega < 0 \end{cases}$$

we see that $\grave{\alpha}$ decays exponentially at $+\infty$.

Now we define

$$\begin{aligned} \grave{\phi}^{(0)} &:= \grave{\alpha} \\ \grave{\phi}^{(l)} &:= -\int_u^\infty S(u,v)W(v)\grave{\phi}^{(l-1)}\,\mathrm{d}v \end{aligned}$$

Here $S(u,v)$ is the Green's function for the differential operator $-\partial_u^2 + V_0(u)$ with

$$V_0 := V - \frac{V''}{4V} + \frac{5}{16}\left(\frac{V'}{V}\right)^2,$$

and $W := V - V_0$.

Then the series

$$\grave{\phi} := \sum_{l=0}^{\infty} \grave{\phi}^{(l)}$$

is a solution of the Schrödinger equation $(-\partial_u^2 + V(u))\phi = 0$ and has the same asymptotics as $\grave{\alpha}$.

In particular,

$$\lim_{u\to\infty} e^{-i\omega u}\grave{\phi} = 1, \quad \lim_{u\to\infty} (e^{-i\omega u}\grave{\phi})' = 0 \quad \text{if} \operatorname{Im}\,\omega > 0$$

and

$$\lim_{u\to\infty} e^{i\omega u}\grave{\phi} = 1, \quad \lim_{u\to\infty} (e^{i\omega u}\grave{\phi})' = 0 \quad \text{if} \operatorname{Im}\,\omega < 0$$

Similarly we can construct a solution $\acute{\phi}$ which decays at $-\infty$ by using the corresponding WKB function.

Besides, it can be shown that the so called Jost solutions $\acute{\phi}, \grave{\phi}$ are holomorphic in ω.

We will use these functions to construct the resolvent of H. Then the resolvent is also holomorphic because $\acute{\phi}$ and $\grave{\phi}$ are holomorphic. We will use this later to deform our contour of integration.

Construction of the resolvent

For $\omega \in \mathbb{C}$ with $|\mathrm{Im}\ \omega| > K$ the resolvent $R_\omega = (H-\omega)^{-1}$ has the following representation:

$$(R_\omega \Psi)(u) = \int_{-\infty}^{\infty} R_\omega(u,v)\Psi(v)\,\mathrm{d}v$$

with

$$R_\omega(u,v) = \begin{pmatrix} 0 & 0 \\ \delta(u,v) & 0 \end{pmatrix} + G(u,v) \begin{pmatrix} \omega - g(v) & 1 \\ \omega(\omega - g(v) - 1) & \omega \end{pmatrix}.$$

Here $G(u,v)$ is the Green's function for $-\partial_u^2 + V(u)$ and is defined by

$$G(u,v) := \frac{1}{w(\acute{\phi}, \grave{\phi})} \cdot \begin{cases} \acute{\phi}(u)\grave{\phi}(v) & \text{if } v \geq u \\ \grave{\phi}(u)\acute{\phi}(v) & \text{if } v < u \end{cases}$$

Here $w(\acute{\phi}, \grave{\phi}) := \acute{\phi}' \grave{\phi} - \acute{\phi} \grave{\phi}'$ denotes the Wronskian of $\acute{\phi}$ and $\grave{\phi}$.
As domain of definition we choose $(C_0^\infty(\mathbb{R}, \mathbb{C}))^2$.

Resolvent estimates for large $|\omega|$

For $|\omega|$ large we can bound the resolvent in the following way:

$$|(R_\omega \Psi)(u)| \leq \frac{C}{|\omega|}$$

for any $\Psi \in (C_0^\infty(\mathbb{R}, \mathbb{C}))^2$ and any $u \in \mathbb{R}$.
We get this result by using that the Jost solutions are well approximated by the WKB functions:

$$\grave{\phi} = \grave{\alpha}(1 + \mathcal{O}(\frac{1}{\omega})), \quad \grave{\phi}' = \grave{\alpha}'(1 + \mathcal{O}(\frac{1}{\omega}))$$

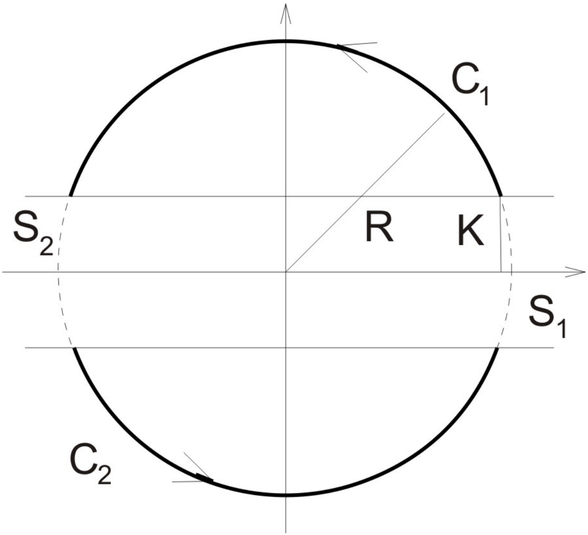
We can prove an analogue result for $\acute{\phi}$.

An integral representation for the solution of the Cauchy problem

To proof the formula for the solution of the Cauchy problem we show that $\forall\, \Psi \in (C_0^\infty(\mathbb{R}, \mathbb{C}))$ and $\forall\, u \in \mathbb{R}$:

$$\Psi(u) = -\frac{1}{2\pi i} \lim_{R\to\infty} \int_{C_1\cup C_2} (R_\omega \Psi)(u)\, \mathrm{d}\omega$$

After that we deform the contour $C_1 \cup C_2$ to $C_1' \cup C_2'$ and use Lebesgue's theorem to prove our main theorem.


C1
R
K
S2
S1
C2

Proof: We have

$$\left| \oint_{\partial B_R} \frac{\mathrm{d}\omega}{\omega} - \int_{C_1 \cup C_2} \frac{\mathrm{d}\omega}{\omega} \right| \leq \frac{1}{R} \int_{S_1 \cup S_2} |\mathrm{d}\omega| \xrightarrow{R \to \infty} 0$$

and so

$$\frac{1}{2\pi i} \lim_{R \to \infty} \int_{C_1 \cup C_2} \frac{\mathrm{d}\omega}{\omega} = 1 \tag{3}$$

We have $\forall \omega$ in $C_1 \cup C_2$: $\Psi = R_\omega(H-\omega)\Psi$

We divide by ω, integrate over $C_1 \cup C_2$ and apply (3).Then we obtain

$$\begin{aligned}\Psi(u) &= \frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_1\cup C_2}\frac{1}{\omega}(R_\omega(H-\omega))\Psi(u)\,\mathrm{d}\omega \\ &= -\frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_1\cup C_2}\left((R_\omega\Psi)(u)-\frac{1}{\omega}(R_\omega H\Psi)(u)\right)\mathrm{d}\omega\end{aligned}$$

As R_ω is bounded by $|(R_\omega H\Psi)(u)| \leq \frac{C}{|\omega|}$ the second term vanishes.