pySecDec: a toolbox for the numerical evaluation of multi-scale integrals

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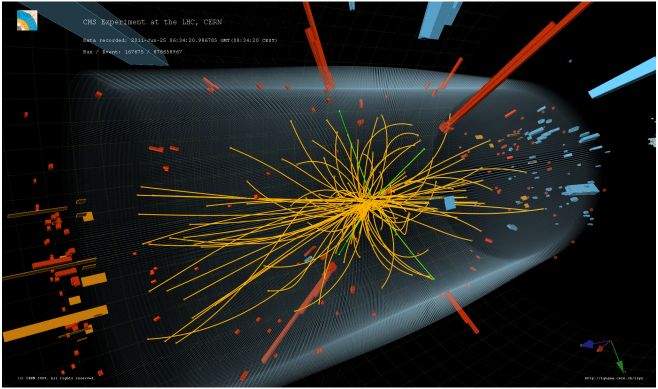
Introduction

precision tests of the Standard Model at the LHC
requires computation of multi-loop Feynman integrals



http://cds.cern.ch/journal/CERNBulletin/2011/41/News%20Articles/1387902

Introduction

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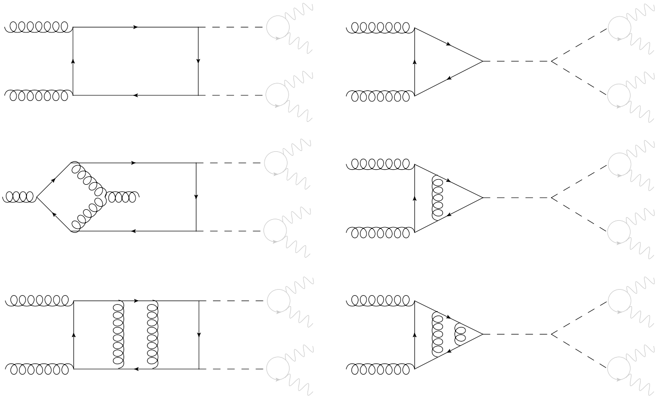
requires computation of multi-loop Feynman integrals



pySecDec

successor of SecDec-3

S. Borowka, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, T. Zirke [1502.06595]

Numerically computes regulated parameter integrals of the form

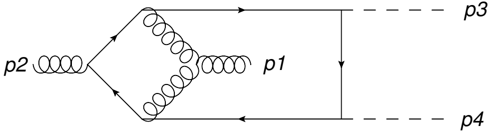
$$\mathcal{I} \equiv \int_0^1 \mathrm{d}x_1 \ldots \int_0^1 \mathrm{d}x_N \prod_{i=1}^{m} f_i\left(\vec{x}, \vec{a}\right)^{b_i + \sum_k c_{ik}\epsilon_k}$$

where the f_i are polynomials. Typically: $\mathcal{I}\big|_{\epsilon_k=0} = \infty$.

Example:
Loop integrals

after Feynman parametrization

in dimensional regularization



The SecDec collaboration

Sophia Borowka

Gudrun Heinrich

Stephan Jahn

Stephen Jones

Matthias Kerner

Johannes Schlenk

former members

Thomas Binoth

Jonathon Carter

Tom Zirke

Paper

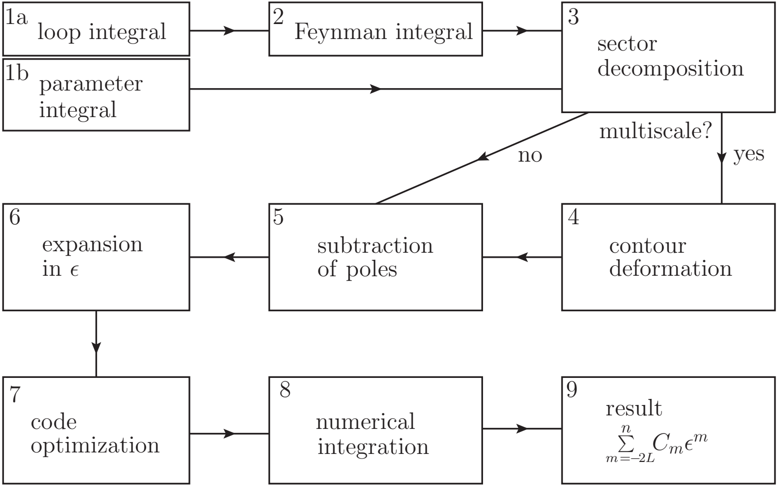
CPC 222 (2018) 313
[1703.09692]

Other public Implementations

- *C. Bogner, S. Weinzierl*
  Sector decomposition
  [0709.4092]

  supplemented by:
  J. Gluza, K. Kajda, T. Riemann, V. Yundin
  CSectors
  [1010.1667]
- *A.V. Smirnov*
  FIESTA 4
  [1511.03614]

Flowchart


1a loop integral
2 Feynman integral
3 sector decomposition
1b parameter integral
multiscale?
no
yes
6 expansion in ϵ
5 subtraction of poles
4 contour deformation
7 code optimization
8 numerical integration
9 result $\sum_{m=-2L}^{n} C_m \epsilon^m$

Loop Integral - Momentum Representation

$$\mathcal{I} = \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \cdot \ldots \cdot \frac{d^D k_L}{i\pi^{\frac{D}{2}}} \frac{1}{P_1^{\nu_1} \cdot \ldots \cdot P_N^{\nu_N}}$$

D: dimensionality

L: number of loops

N: number of propagators

P_i: propagators $(< momentum >^2 \left[- < mass >^2\right] + i\delta)$

ν_i: propagator powers

Loop Integral - Feynman Representation

$$\mathcal{I} = (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^1 \prod_{j=1}^{N} dx_j x_j^{\nu_i - 1} \delta\left(1 - \sum_{n=1}^{N} x_n\right) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$

D: dimensionality

L: number of loops

N: number of propagators

ν_i: propagator powers

$N_\nu = \sum_{i=1}^{N} \nu_i$

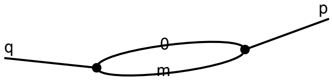
$\mathcal{U} = \mathcal{U}(\vec{x})$: 1st Symanzik polynomial

$\mathcal{F} = \mathcal{F}(\vec{x}, p_i \cdot p_j, m_i^2)$: 2nd Symanzik polynomial

Feynman Parametrization with pySecDec

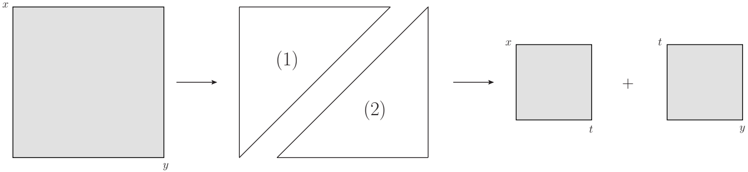
```
>>> from pySecDec.loop_integral import LoopIntegralFromPropagators

>>> one_loop_bubble = LoopIntegralFromPropagators(
...   propagators=['k^2-m^2', '(k-p)^2'],
...   loop_momenta=['k']
...)

>>> one_loop_bubble.exponentiated_U
( + (1)*x0 + (1)*x1)**(2*eps - 2)

>>> one_loop_bubble.exponentiated_F
( + (m**2 - p**2)*x0*x1 + (m**2)*x0**2)**(-eps)

>>> one_loop_bubble.Gamma_factor
gamma(eps)
```

```
>>> from pySecDec.loop_integral import LoopIntegralFromGraph, plot_diagram

>>> one_loop_bubble = LoopIntegralFromGraph(
...   internal_lines=(['m',(1,2)], [0,(1,2)]),
...   external_lines=[('p',1), ('q',2)],
...   replacement_rules=[('q','p')]
...)

>>> one_loop_bubble.exponentiated_U
( + (1)*x0 + (1)*x1)**(2*eps - 2)

>>> one_loop_bubble.exponentiated_F
( + (m**2 - p**2)*x0*x1 + (m**2)*x0**2)**(-eps)

>>> one_loop_bubble.Gamma_factor
gamma(eps)

>>> plot_diagram(
...    one_loop_bubble.internal_lines,
...    one_loop_bubble.external_lines,
...    filename='bubble1L',
...    Gstart=986089
... )
```



Sector Decomposition

or: Resolution of Overlapping Singularities



$$\int_0^1 dx \int_0^1 dy \; (x+y)^{a+b\epsilon} \, f(x,y)$$

$$= \int_0^1 dx \int_0^1 dy \; (x+y)^{a+b\epsilon} \, f(x,y) [\underbrace{\Theta(x-y)}_{(1)} + \underbrace{\Theta(y-x)}_{(2)}]$$

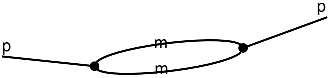
$$= \int_0^1 dx \int_0^1 dt \, x \, x^{a+b\epsilon} \, (1+t)^{a+b\epsilon} \, f(x,xt) + \int_0^1 dt \int_0^1 dy \, y \, y^{a+b\epsilon} \, (t+1)^{a+b\epsilon} \, f(yt,y)$$

Subtraction of Poles

$$
\begin{aligned}
&\int_0^1 dt\, t^{-1+b\epsilon} g(t) \\
&= \int_0^1 dt\, t^{-1+b\epsilon} \left(g(0) + g(t) - g(0)\right) \\
&= \underbrace{\int_0^1 dt\, t^{-1+b\epsilon} g(0)}_{=\frac{1}{b\epsilon} g(0)} + \underbrace{\int_0^1 dt\, t^{-1+b\epsilon} \left(g(t) - g(0)\right)}_{\text{finite for } \epsilon \to 0 \text{, expand integrand in } \epsilon}
\end{aligned}
$$

Contour Deformation - A Simple Example

consider the massive one loop bubble



$$\int d^D k \frac{1}{\left(k^2 - m^2 + i\delta\right)\left((k-p)^2 - m^2 + i\delta\right)}$$

taking $\delta\left(1 - x_1 - x_2\right)$ into account:

$$\begin{aligned}\mathcal{F}\left(\vec{x}, p_i \cdot p_j, m_i^2\right) &= \left[m^2\left(x_1 + x_2\right)^2 - p^2 x_1\, x_2\right]_{x_2 = 1 - x_1} \\ &= m^2 - p^2 x_1 (1 - x_1)\end{aligned}$$

- physical threshold $p^2 \geq 4m^2$
- can have $\mathcal{F} = 0$ although $x_1 \neq 0$

Contour Deformation

or: Satisfying the Feynman Prescription

D. E. Soper [hep-ph/9910292], T. Binoth et al. [hep-ph/0504267], C. Anastasiou et al. [hep-ph/0703282], S. Borowka et al. [1204.4152]

$$\mathcal{I} = \int \frac{d^D k_1}{i\pi^{\frac{D}{2}}} \cdot \ldots \cdot \frac{d^D k_L}{i\pi^{\frac{D}{2}}} \frac{1}{P_1^{\nu_1} \cdot \ldots \cdot P_N^{\nu_N}}$$

$$= (-1)^{N_\nu} \frac{\Gamma(N_\nu - LD/2)}{\prod_{j=1}^{N} \Gamma(\nu_j)} \int_0^1 \prod_{j=1}^{N} dx_j x_j^{\nu_i - 1} \delta\left(1 - \sum_{n=1}^{N} x_n\right) \frac{\mathcal{U}^{N_\nu - (L+1)D/2}}{\mathcal{F}^{N_\nu - LD/2}}$$

P_i: propagators $(< \mathit{momentum} >^2 \; [- < \mathit{mass} >^2] + \boxed{i\delta})$

$\mathcal{F} = \mathcal{F}(\vec{x}, p_i \cdot p_j, m_i^2) - \boxed{i\delta}$: 2^{nd} Symanzik polynomial

Contour Deformation

or: Satisfying the Feynman Prescription

D. E. Soper [hep-ph/9910292], T. Binoth et al. [hep-ph/0504267], C. Anastasiou et al. [hep-ph/0703282], S. Borowka et al. [1204.4152]

$$\mathcal{F}(\vec{x}, p_i \cdot p_j, m_i^2) - \boxed{i\delta}$$

move integration contour to the complex plane

$$\begin{aligned} \vec{z}: \quad [0,1]^n &\longrightarrow \mathbb{C}^n \\ x_k &\to z_k(\vec{x}) \\ &\equiv x_k - i\lambda_k\, x_k\,(1-x_k)\frac{\partial\,\mathrm{Re}\left[\mathcal{F}(\vec{x})\right]}{\partial x_k} \end{aligned}$$

such that

$$\mathrm{Im}\left[\mathcal{F}(\vec{z}(\vec{x}))\right] \leq \mathrm{Im}\left[\mathcal{F}(\vec{x})\right] \quad \forall\, \vec{x} \in [0,1]^n$$

Contour Deformation

or: Satisfying the Feynman Prescription

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$$
\begin{aligned}
\mathcal{F}\left(\vec{z}(\vec{x})\right) = &+ \mathrm{Re}\,\mathcal{F}(\vec{x}) + i\,\mathrm{Im}\,\mathcal{F}(\vec{x}) \\
&+ \sum_k \lambda_k \left[\boxed{-i\left(\frac{\partial \mathrm{Re}\,\mathcal{F}(\vec{x})}{\partial x_k}\right)^2} + \frac{\partial\,\mathrm{Re}\,\mathcal{F}(\vec{x})}{\partial x_k}\frac{\partial\,\mathrm{Im}\,\mathcal{F}(\vec{x})}{\partial x_k} \right] x_k(1-x_k) \\
&+ \sum_{k,l} \frac{\lambda_k \lambda_l}{2} \left[-\frac{\partial^2\,\mathrm{Re}\,\mathcal{F}(\vec{x})}{\partial x_k \partial x_l} \boxed{-i\frac{\partial^2\,\mathrm{Im}\,\mathcal{F}(\vec{x})}{\partial x_k \partial x_l}} \right] \prod_{i=k,l} \frac{\partial \mathrm{Re}\,\mathcal{F}(\vec{x})}{\partial x_i} x_i(1-x_i) \\
&+ \mathcal{O}(\lambda^3)
\end{aligned}
$$

- correct sign at $\mathcal{O}(\lambda)$
- indeterminate sign at $\mathcal{O}(\lambda^2)$

correct sign if λ_k "small enough"

Expansion in the Regulator(s) ϵ

- Taylor expansion and series multiplication

  poles in ϵ explicitly factorize by construction

- cumbersome bookkeeping

  individual factors must be expanded to different orders

Code Optimization

- faster numerics due to **Code Optimization in FORM**

  J. Kuipers, T. Ueda, J.A.M. Vermaseren [1310.7007]

Numerical Integration

- CUBA and `gsl_integration_cquad` (GNU scientific library)

  T. Hahn [hep-ph/0404043], *M. Galassi et al.* [GSL Reference Manual]

- can link any numerical integrator using the C++ interface

A Simple Example - Analytic Calculation

$$\int_0^1 dx \int_0^1 dy \; (x+y)^{-2+\epsilon}$$

$$= 2 \int_0^1 dx \, x^{-1+\epsilon} \int_0^1 dt \; (1+t)^{-2+\epsilon}$$

$$= \frac{2}{\epsilon} \left[\int_0^1 dt \; (1+t)^{-2} + \epsilon \int_0^1 dt \; (1+t)^{-2} \log(1+t) + O(\epsilon^2) \right]$$

$$= \frac{2}{\epsilon} \left[\frac{1}{2} + \epsilon \, \frac{1}{2} (1 - \log(2)) + O(\epsilon^2) \right] = \frac{1}{\epsilon} + (1 - \log(2)) + O(\epsilon)$$

A Simple Example - Basic pySecDec Usage

$$\int_0^1 dx \int_0^1 dy (x+y)^{-2+\epsilon} = \frac{1}{\epsilon} + (1 - \log(2)) + O(\epsilon) \approx \frac{1}{\epsilon} + 0.306853 + O(\epsilon)$$

Step 1: write input files

generate_easy.py

```python
from pySecDec import make_package

make_package(

name = 'easy',
integration_variables = ['x','y'],
regulators = ['eps'],

requested_orders = [0],
polynomials_to_decompose = ['(x+y)^(-2+eps)'],

)
```

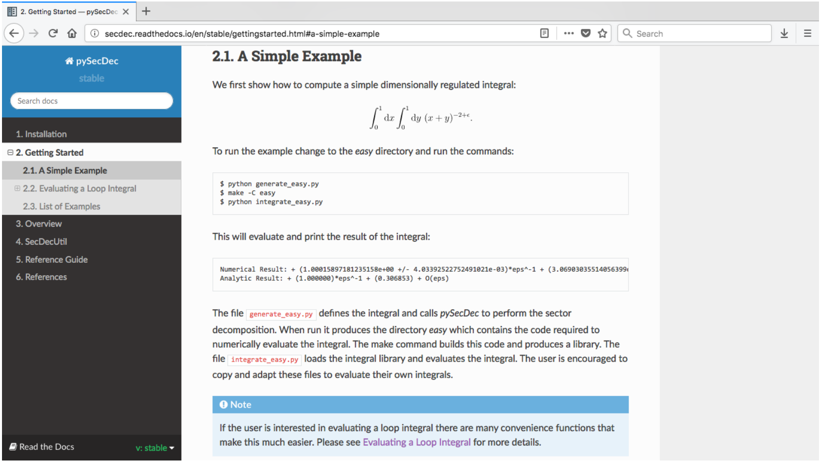
integrate_easy.py

```python
from pySecDec.integral_interface \
    import IntegralLibrary

# load c++ library
easy_integral = \
    IntegralLibrary('easy/easy_pylink.so')

# integrate
_, _, result = easy_integral()

# print result
print('Numerical Result:')
print(result)
```

Step 2: run pySecDec

```
$ python generate_easy.py && make -C easy && python integrate_easy.py
<skipped some output>
Numerical Result:
 + (1.00015897181235158e+00 +/- 4.03392522752491021e-03)*eps^-1 + (3.06903035514056399e-01 +/-
↪  2.82319349818329918e-03) + O(eps)
```

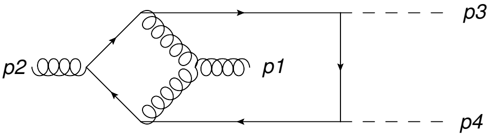
A Simple Example - pySecDec Documentation



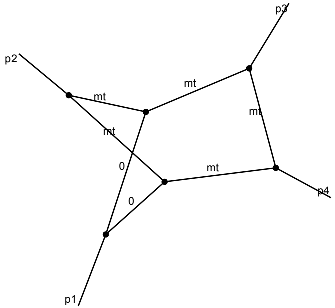
http://secdec.readthedocs.io/

Higgs Boson Pair Production

S. Borowka, N. Greiner, G. Heinrich, S. P. Jones, M. Kerner, J. Schlenk, T. Zirke [1608.04798]



$\mu^{-6-4\epsilon}\,\mathcal{I} \equiv \text{finite}\Bigg[$



$\Bigg],\ \mu = 1\text{GeV}$

Higgs Boson Pair Production

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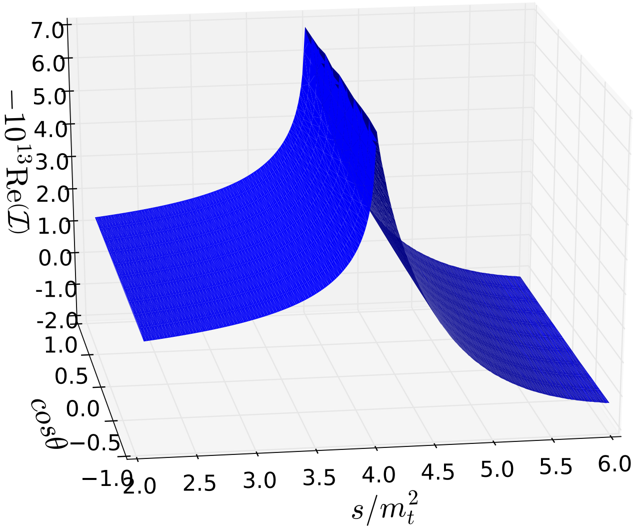

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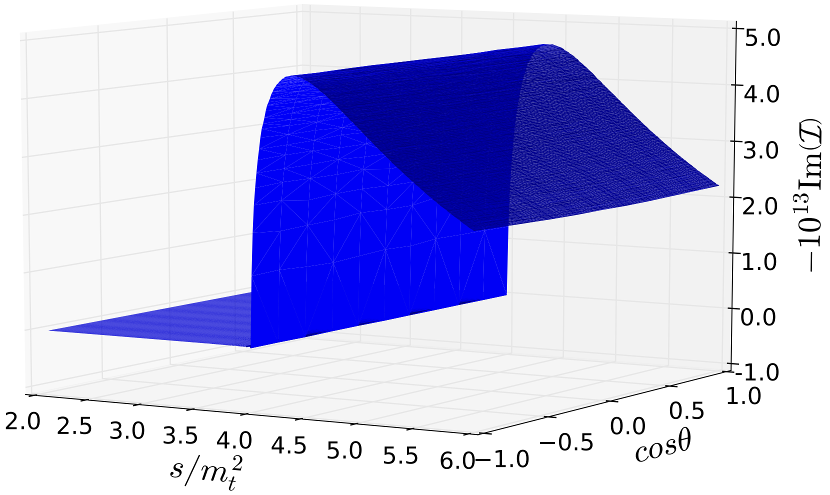

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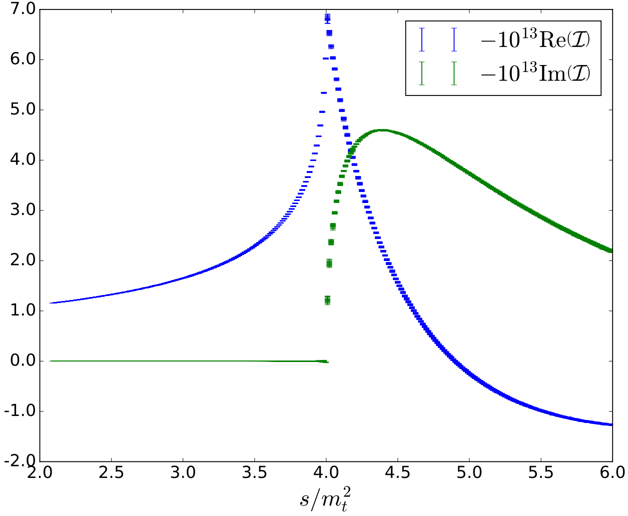

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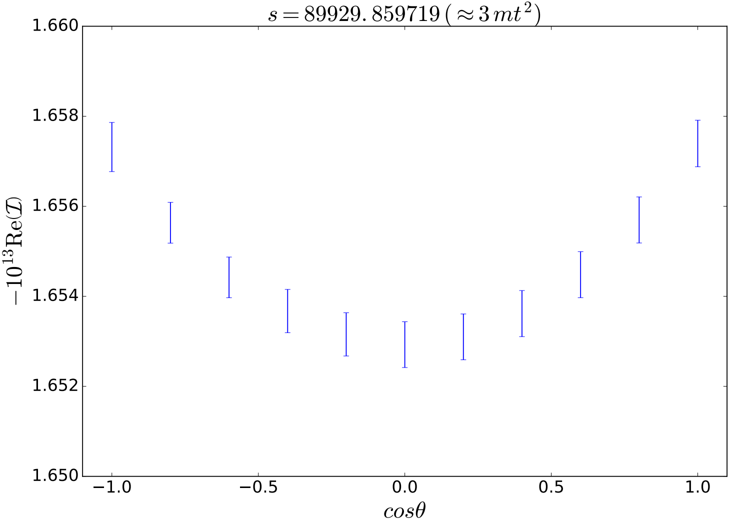

Conclusion

- pySecDec: automated numerical evaluation of loop integrals
- can be used to compute amplitudes and physical cross-sections
- further reading:
  - homepage: `http://secdec.hepforge.org/`
  - source code: `https://github.com/mppmu/secdec/`
  - documentation: `http://secdec.readthedocs.io/`

BACKUP

Timings

Table 5
Comparison of timings (algebraic, numerical) using pySecDec, SecDec 3 and Fiesta 4.1.

	pySecDec time (s)	SecDec 3 time (s)	Fiesta 4.1 time (s)
triangle2L	(40.5, 9.6)	(56.9, 28.5)	(211.4, 10.8)
triangle3L	(110.1, 0.5)	(131.6, 1.5)	(48.9, 2.5)
elliptic2L_euclidean	(8.2, 0.2)	(4.2, 0.1)	(4.9, 0.04)
elliptic2L_physical	(21.5, 1.8)	(26.9, 4.5)	(115.3, 4.4)
box2L_invprop	(345.7, 2.8)	(150.4, 6.3)	(21.5, 8.8)

GoSam-Xloop

The GoSam-Xloop collaboration

- Long Chen
- Nicolas Greiner
- Gudrun Heinrich
- Stephan Jahn
- Stephen Jones
- Matthias Kerner
- et al.