The Interplay of L_∞ Algebras and Field Theories

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The Interplay of L_∞ Algebras and Field Theories

Masters thesis at MPP/LMU, Supervisor: PD Dr. Ralph Blumenhagen

Starting Point:

- Field theories are L_∞ algebras [O. Hohm, B. Zwiebach, *Fortsch.Phys.* 65 (2017)]
- The L_∞ bootstrap for non-commutative gauge theories

  [R. Blumenhagen, I. Brunner, V. Kupriyanov, D. Lüst, *JHEP* 05 (2018)]

Questions:

- What about the uniqueness of the L_∞ bootstrap?
- What about supersymmetry?

Results:

- L_∞ quasi-isomorphisms are Seiberg-Witten maps.

  [R. Blumenhagen, M. Brinkmann,V. Kupriyanov, M. Traube, arXiv:1806.10314]

- SUSY symmetry-algebras are super-L_∞ algebras.

  [R. Blumenhagen, M. Brinkmann, arXiv:1809.10467]

Background: L_∞ Algebras

- Generalization of Lie algebras
- Symmetry of closed string field theory [Zwiebach 1992]
- Closely linked to field theories [Hohm, Zwiebach 2017]

From Lie to L_∞ Algebras

Lie algebras	generalize to	**L_∞ algebras**
vector space: $X = X_0$	$\longrightarrow$	$\mathbb{Z}$-graded vector space: $X = \bigoplus_{k\in\mathbb{Z}} X_k$
Lie bracket: $[x_1, x_2]$ of degree $\lvert[\cdot,\cdot]\rvert = 0$	$\longrightarrow$	L_∞ maps: $\ell_n(x_1, x_2, ..., x_n)$ of degree $\lvert\ell_n\rvert = n-2$
Jacobi Identity	$\longrightarrow$	L_∞ defining relations
Encodes symmetry	$\longrightarrow$	Encodes symmetry and dynamics

L_∞ Defining Relations

Defining relations: $\mathcal{J}_n = 0$

$$\mathcal{J}_1(x) = \ell_1(\ell_1(x))$$

$$\mathcal{J}_2(x_1, x_2) = \ell_1(\ell_2(x_1, x_2)) - \ell_2(\ell_1(x_1), x_2)) - (-1)^{x_1}\ell_2(x_1, \ell_1(x_2))$$

$$\begin{aligned}\mathcal{J}_3(x_1, x_2, x_3) &= \ell_2(\ell_2(x_1, x_2), x_3) + \text{ cyclic} \\ &\quad + \ell_1(\ell_3(x_1, x_2, x_3)) + \ell_3(\ell_1(x_1), x_2, x_3) + \dots\end{aligned}$$

$$\vdots$$

Field Theories are L_∞ Algebras

- Field theories and their dynamics are L_∞ algebras*
- The vector space is given by

$$X = \begin{array}{c|l} X_0 & \ni \lambda\text{: gauge parameters} \\ \bigoplus & \\ X_{-1} & \ni A\text{: physical fields} \\ \bigoplus & \\ X_{-2} & \ni \mathcal{F}\text{: field dynamics} \end{array}$$

*Hohm, Zwiebach (2017)

Dynamic Sector

The **action** of a field theory can be written as

$$S(A) = \frac{1}{2}\langle A, \ell_1(A)\rangle - \frac{1}{3!}\langle A, \ell_2(A^2)\rangle - \frac{1}{4!}\langle A, \ell_3(A^3)\rangle + \ldots$$

Example: Chern-Simons theory (with $\langle A, B\rangle = \int \mathrm{d}^3x\, \eta^{ab} A_a B_b$)

$$S(A) = \frac{1}{2}\int \mathrm{d}^3x\, \epsilon^{abc}\left(A_a\,\partial_b A_c + \frac{1}{3}A_a[A_b, A_c]\right)$$
$$= \frac{1}{2}\langle A, \epsilon_*\partial A\rangle + \frac{1}{3!}\langle A, \epsilon_*[A, A]\rangle$$

$$\Longrightarrow \quad \ell_1(A) = \epsilon_*\partial A\,, \quad \ell_2(A, B) = -\epsilon_*[A, B]$$

Gauge Sector

The fields transform under **gauge transformations** as

$$\delta_\lambda A = \ell_1(\lambda) + \ell_2(\lambda, A) - \frac{1}{2}\ell_3(\lambda, A^2) - \frac{1}{3!}\ell_4(\lambda, A^3) + \dots .$$

Example: Lie algebra acting on a vector field

$$\delta_\lambda A_a = \partial_a \lambda + [A_a, \lambda]$$

$$\Longrightarrow \quad \ell_1(\lambda) = \partial\lambda \,, \quad \ell_2(\lambda, A) = [A, \lambda]$$

Gauge Sector

The **gauge algebra** is given by

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]\, A = \delta_{-\ell_2(\lambda_1,\lambda_2)-\ell_3(\lambda_1,\lambda_2,A)+\frac{1}{2}\ell_4(\lambda_1,\lambda_2,A^2)-\dots}\, A\,.$$

Example: Lie algebra

$$[\delta_{\lambda_1}\, \delta_{\lambda_2}] = \delta_{[\lambda_1,\lambda_2]}$$

$$\Longrightarrow \quad \ell_2(\lambda_1, \lambda_2) = -[\lambda_1, \lambda_2]$$

Example: Chern-Simons Theory

Traditional description

- Action
- Symmetry: Lie algebra
- Transformation properties

Equivalent L_∞ description

L_∞ algebra covers all aspects.

$$\ell_1(\lambda) = \partial\lambda$$
$$\ell_1(A) = \epsilon_*\partial A$$
$$\ell_2(\lambda_1, \lambda_2) = -[\lambda_1, \lambda_2]$$
$$\ell_2(\lambda, A) = [A, \lambda]$$
$$\ell_2(A, B) = -\epsilon_*[A, B]$$

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Questions:

- What about the uniqueness of the L_∞ bootstrap?
- What about supersymmetry?

Results:

- L_∞ quasi-isomorphisms are Seiberg-Witten maps.
- SUSY symmetry-algebras are super-L_∞ algebras.

The L_∞ Bootstrap

The L_∞ bootstrap was proposed for gauge theories where the action and equations of motion are unknown (e.g. noncommutative gauge theories).

The Idea:

Bootstrap an interacting gauge theory by imposing L_∞ algebra on the free theory.*

- Starting from known L_∞ maps, impose the defining relations.
- Where they fail to hold, define new maps so they are satisfied.
- Order by order deduce corrections to the theory.

*Blumenhagen, Brunner, Kupriyanov, Lüst (2018)

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Problem:

Solutions to the bootstrap are not unique!

*Blumenhagen, Brunner, Kupriyanov, Lüst (2018)

Seiberg-Witten Maps are Quasi-Isomorphisms

Seiberg-Witten maps

Seiberg-Witten maps are field redefinitions that ensure the physical properties stay the same by mapping gauge orbits to gauge orbits.

Quasi-isomorphisms

L_∞ algebras are categorially equivalent if they are quasi-isomorphic.

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Results:

- Gauge theories with quasi-isomorphic L_∞ algebras are physically equivalent via Seiberg-Witten maps.
- The L_∞ algebras of gauge theories related by Seiberg-Witten maps are (almost) quasi-isomorphic.

Seiberg-Witten Maps are Quasi-Isomorphisms

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All examples of different solutions to the bootstrap we have found are quasi-isomorphic.

*Blumenhagen, MB, Kupriyanov, Traube (2018)

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Symmetry Algebras

- For symmetry algebras like $\mathcal{W}$ algebras, the same dictionary holds as in the gauge sector of field theories*.
- No action: reduced vector space

$$X = X_{-1} \oplus X_0$$

- $\mathcal{W}$ algebras can be extended to super-$\mathcal{W}$ algebras
- Perfect setting to investigate SUSY extension of dictionary.

*Blumenhagen, Fuchs, Traube (2017)

Super-Virasoro Algebra

Fields / symmetry parameters

- Energy-momentum tensor $L \,/\, \epsilon^L$
- Superpartner $G \,/\, \epsilon^G$
- Algebra between them

Symmetry transformations & their algebra lead to L_∞ maps:

$$\ell_1^L(\epsilon^L) = \frac{c}{12}\partial^3\epsilon^L \qquad \ell_1^G(\epsilon^G) = \frac{c}{3}\partial^2\epsilon^G$$

$$\ell_2^{\epsilon^L}(\epsilon^L, \tilde{\epsilon}^L) = \epsilon^L\,\partial\tilde{\epsilon}^L - \partial\epsilon^L\,\tilde{\epsilon}^L \qquad \ldots$$

The superscript labels the vector space ℓ_n^X maps to.

The Vector Space

Consistency requires

$$\ell_n^X(A, B, \ldots) \neq 0 \quad \Rightarrow \quad |X| = |A| + |B| + \ldots + n - 2\,.$$

This is sufficient information to deduce the gradings

$$|L| = |G| = -1\,, \quad |\epsilon^L| = |\epsilon^G| = 0\,.$$

The vector space can be written as

$$X = X_0 \oplus X_{-1}\,, \qquad X_{i\in\{0,-1\}} = X_i^{\text{bos}} \oplus X_i^{\text{ferm}}\,.$$

Super-L_∞ Algebra

A super-L_∞ algebra is an L_∞ algebra over a graded super vector space

$$X = \bigoplus_{n\in\mathbb{Z}} \left(X_n^{\text{bos}} \oplus X_n^{\text{ferm}} \right) .$$

This is exactly the structure we have found for the super-Virasoro algebra!

We must check if the maps defined above satisfy the L_∞ relations.

Super-Virasoro Algebra is a Super-L_∞ Algebra

Results:

- The super-Virasoro algebra is a super-L_∞ algebra.
- The dictionary works for supersymmetric symmetry algebras.*

*Blumenhagen, Brinkmann (2018)

Super-Virasoro Algebra is a Super-L_∞ Algebra

Results:

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Conjecture?

- Supersymmetric field theories are super-L_∞ algebras.
- The dictionary works for supersymmetric field theories.

*Blumenhagen, Brinkmann (2018)

Conclusions & Outlook

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- We have found an L_∞ dictionary for SUSY theories.
- A bootstrap method for unknown gauge theories has been proposed.
- Many redundancies in the bootstrap can be traced to Seiberg-Witten maps.

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- A bootstrap method for unknown gauge theories has been proposed.
- Many redundancies in the bootstrap can be traced to Seiberg-Witten maps.

Outlook

- Can we apply the bootstrap to non-commutative gravity?
- Can we apply the bootstrap to SUSY gauge theories?
- How are the L_∞ algebras of holographic theories related?

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Thank you for your attention!

L_∞ Algebra - Maps on a Graded Vector Space

Let X be a $\mathbb{Z}$-graded vector space.

$$X = \bigoplus_{n \in \mathbb{Z}} X_n$$

Let $\{\ell_n : X^{\otimes n} \to X\}_{n>0}$ be graded anti-commutative multilinear maps of degree $|\ell_n| = n - 2$ acting on X.

$$\ell_n(\dots, x_i, x_j, \dots) = (-1)^{1+x_i x_j} \ell_n(\dots, x_j, x_i, \dots)$$

$\{\ell_n : X^{\otimes n} \to X\}_{n>0}$ form an L_∞ algebra if they satisfy the L_∞ *defining relations.*

L_∞ Algebra - An Example

Lie algebras are the simplest L_∞ algebras:

A graded vector space:	$X = X_0$
A collection of maps:	$\ell_2(x_1, x_2) = \mathcal{L}_{x_1} x_2 = [x_1, x_2]$
A collection of conditions:	Only $\mathcal{J}_3$ nontrivial.

$$\mathcal{J}_3 = \ell_2(\ell_2(x_1, x_2), x_3) + \ell_2(\ell_2(x_2, x_3), x_1) + \ell_2(\ell_2(x_3, x_1), x_2 = 0$$

is just the Jacobi identity for the Lie algebra.

Equations of Motion

The **equations of motion** $\mathcal{F} \in X_{-2}$ are given by

$$\mathcal{F} = \sum_{n\geq 1} \frac{1}{n!}(-1)^{\frac{n(n-1)}{2}} \ell_n(A^n)$$
$$= \ell_1(A) - \frac{1}{2}\ell_2(A^2) - \frac{1}{3!}\ell_3(A^3) + \frac{1}{4!}\ell_4(A^4) + \dots .$$

Example: Chern-Simons theory

$$\mathcal{F} = \epsilon_* \left(\partial A + \frac{1}{2}[A, A] \right) = \ell_1(A) - \frac{1}{2}\ell_2(A^2)$$

The Dictionary

$$\delta_\lambda A = \ell_1(\lambda) + \ell_2(\lambda, A) - \frac{1}{2}\ell_3(\lambda, A^2) - \frac{1}{3!}\ell_4(\lambda, A^3) + \ldots$$

$$[\delta_{\lambda_1}, \delta_{\lambda_2}]\, A = \delta_{\ell_2(\lambda_1,\lambda_2)+\ell_3(\lambda_1,\lambda_2,A)-\frac{1}{2}\ell_4(\lambda_1,\lambda_2,A^2)-\ldots}\, A$$

$$S(A) = \frac{1}{2}\langle A, \ell_1(A)\rangle - \frac{1}{3!}\langle A, \ell_2(A^2)\rangle - \frac{1}{4!}\langle A, \ell_3(A^3)\rangle + \ldots$$

$$\mathcal{F}(A) = \ell_1(A) - \frac{1}{2}\ell_2(A^2) - \frac{1}{3!}\ell_3(A^3) + \frac{1}{4!}\ell_4(A^4) + \ldots$$

L_∞ Field Theory Example: Chern-Simons

There are 7 nontrivial L_∞ relations to check.

$$\begin{aligned}
\mathcal{J}_1(\lambda):&\quad 0=\ell_1(\ell_1(\lambda))=\epsilon_a^{\ bc}\partial_b\ell_1(\lambda)_c=\epsilon_a^{\ bc}\partial_b\partial_c\lambda \quad \checkmark\\
\mathcal{J}_2(\lambda,A):&\quad 0=\ell_1(\ell_2(\lambda,A))-\dots \quad \checkmark\\
&\vdots \quad \checkmark
\end{aligned}$$

The previously defined maps are not sufficient to satisfy all relations. This can be resolved by additionally defining $\ell_2(E,\lambda)$.

Note that the new map does not appear in any physical quantities. The physics stay the same after defining this new map.

Super-Virasoro Algebra

The super-Virasoro algebra is the simplest super-$\mathcal{W}$ algebra.

Fields

- Energy-momentum tensor L
- Superpartner G

OPE Algebra

$$L(z)\,L(w) = \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w} + \dots,$$

$$L(z)\,G(w) = \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial_w G(w)}{z-w} + \dots,$$

$$G(z)\,G(w) = \frac{\frac{2}{3}c}{(z-w)^3} + \frac{2L(w)}{z-w} + \dots$$

Symmetry Transformations

The fields transform as

$$\delta_{\epsilon^Y} X(w) = \frac{1}{2\pi i} \oint_{\mathcal{C}(w)} dz\, \epsilon^Y(z) \Big(Y(z) X(w) \Big)$$

with $X, Y \in \{L, G\}$.

Note that G and ϵ^G are Grassmann odd.

The L_∞ Maps

The transformations and their commutators lead to L_∞ maps:

$$\ell_1^L(\epsilon^L) = \frac{c}{12}\partial^3\epsilon^L \qquad \ell_1^G(\epsilon^G) = \frac{c}{3}\partial^2\epsilon^G$$

$$\ell_2^{\epsilon^L}(\epsilon^L, \tilde{\epsilon}^L) = \epsilon^L\,\partial\tilde{\epsilon}^L - \partial\epsilon^L\,\tilde{\epsilon}^L \qquad \ell_2^{\epsilon^L}(\epsilon^G, \tilde{\epsilon}^G) = -2\epsilon^G\,\tilde{\epsilon}^G$$

$$\ell_2^L(\epsilon^L, L) = 2\partial\epsilon^L L + \epsilon^L\partial L \qquad \ell_2^L(\epsilon^G, G) = -\frac{3}{2}\partial\epsilon^G G - \frac{1}{2}\epsilon^G\partial G$$

$$\ell_2^G(\epsilon^L, G) = \frac{3}{2}\partial\epsilon^L G + \epsilon^L\partial G \qquad \ell_2^G(\epsilon^G, L) = 2\epsilon^G L$$

$$\ell_2^{\epsilon^G}(\epsilon^L, \epsilon^G) = \epsilon^L\,\partial\epsilon^G - \frac{1}{2}\partial\epsilon^L\,\epsilon^G$$

The superscript labels the vector space ℓ_n^X maps to.

Uniqueness of Solutions to the L_∞ Bootstrap

Problem:

Solutions to the bootstrap are not unique!

Example: Abelian Chern-Simons

$$\ell_1(\lambda) = \partial\lambda\,, \qquad \ell_1(A) = \epsilon_*\partial A\,, \qquad \ell_2(\lambda_1,\lambda_2) = 0\,.$$

In the bootstrap, $\ell_2(\lambda, A) = 0$ is the obvious solution to

$$0 = \ell_2(\ell_1(\lambda_1),\lambda_2)) + \ell_2(\lambda_1,\ell_1(\lambda_2))\,.$$

There are many other solutions!

$$\ell_2(\lambda, A)_a = v^i\,(A_i\partial_a\lambda - A_a\partial_i\lambda)$$

and many more.