Infrared Singularities and Soft Gluon Resummation with Massive Partons

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Outline

IR singularities in QCD

- In QCD we have:
  - soft divergences when gluon momenta go to zero;
  - collinear divergences when the momenta of two massless partons become parallel to each other.
- The soft divergences cancel between virtual and real contributions according to the KLN theorem.
- The remaining collinear divergences are absorbed into non-perturbative functions according to factorization theorems.
- The physical observables are free of IR singularities.

Why we care about IR singularities?

- Non-trivial property of non-abelian gauge theories.
  - Abelian case trivial: all information contained at one-loop [Yennie, Frautschi, Suura (1961)].
- Essential ingredient for factorization and resummation.
  - Important in proving the factorization theorems.
  - Predict logarithmic enhancements at higher orders.
  - Determine the evolution of various functions in the factorization formulas, which leads to the resummation of logarithmic enhancements.
- Consistency check on explicit loop calculations.

Soft gluon resummation

- Soft gluon resummation is based on the following kinds of factorization formula in certain kinematic limit:

$$\sigma \sim H(Q^2, \mu)\, S(\Lambda^2, \mu)\, J_1(Q\Lambda, \mu) \cdots J_n(Q\Lambda, \mu)$$

$$Q^2 \gg Q\Lambda \gg \Lambda^2 \longrightarrow \text{large logs!}$$

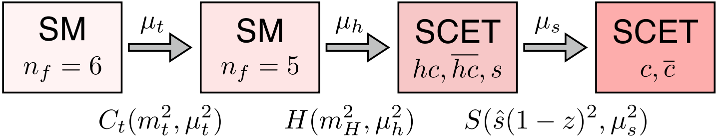
- Solution: evaluate the hard, soft and jet functions at their natural scales and use evolution equations to connect them

$$\sigma \sim U(\mu_h, \mu_s, \mu_j)\, H(Q^2, \mu_h)\, S(\Lambda^2, \mu_s)\, J_1(Q\Lambda, \mu_j) \cdots J_n(Q\Lambda, \mu_j)$$

- The evolution factor U resums the large logs between different scales.

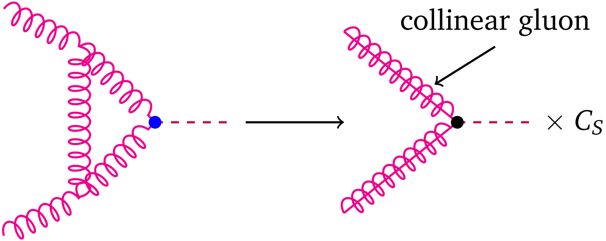
The effective theory comes into play

- Effective theories are useful to separate the different scales and treat them one by one. Example: Higgs production
  [Ahrens, Becher, Neubert, LLY (2008)]



- The relevant effective field theory here is soft-collinear effective theory (SCET).
  [Bauer, Fleming, Pirjol, Stewart (2000)]
  [Bauer, Pirjol, Stewart (2001)]
  [Beneke, Chapovsky, Diehl, Feldmann (2002)]

Demonstration of matching from QCD to SCET



$$\mathcal{M}^{\text{QCD}}(\epsilon_{\text{IR}}) = C_S(\mu)\,\langle O^{\text{ren}}(\epsilon_{\text{IR}},\mu)\rangle = C_S(\mu)\,Z(\epsilon_{\text{UV}},\mu)\,\langle O^{\text{bare}}(\epsilon_{\text{UV}},\epsilon_{\text{IR}})\rangle$$

- The IR divergences in QCD and SCET should agree by construction.
- All loop corrections to $\langle O^{\text{bare}}\rangle$ vanish in dimensional regularization for on-shell external partons.
- This implies: the UV poles in the bare operator matrix element are the negative of the IR poles in the QCD amplitude.
SCET relates UV and IR!

IR renormalization

- The UV divergences in the matrix elements of the bare effective operators are removed by a multiplicative renormalization constant:

$$\langle O^{\text{ren}}(\epsilon_{\text{IR}}, \mu)\rangle = Z(\epsilon_{\text{UV}}, \mu)\,\langle O^{\text{bare}}(\epsilon_{\text{UV}}, \epsilon_{\text{IR}})\rangle = \mathcal{O}(\epsilon_{\text{UV}}^0)\,.$$

- This means that the IR divergences in QCD amplitudes can be absorbed into the same renormalization factor

$$Z^{-1}(\epsilon_{\text{IR}}, \mu)\,\mathcal{M}^{\text{QCD}}(\epsilon_{\text{IR}}) = \mathcal{O}(\epsilon_{\text{IR}}^0)\,.$$

- Extending this to arbitrary n-parton processes, the amplitudes and the renormalization factors become vectors and matrices in color space (more details later)

$$\boldsymbol{Z}^{-1}(\epsilon, \{\underline{p}\}, \{\underline{m}\}, \mu)\,|\mathcal{M}(\epsilon, \{\underline{p}\}, \{\underline{m}\})\rangle = \mathcal{O}(\epsilon^0)\,.$$

- This systematically generalizes a two-loop subtraction formula of [Catani (1998)] to all orders.

The anomalous dimension

- The renormalization factor satisfies a renormalization group equation

$$\boldsymbol{Z}^{-1}(\epsilon, \{\underline{p}\}, \{\underline{m}\}, \mu) \frac{d}{d \ln \mu} \boldsymbol{Z}(\epsilon, \{\underline{p}\}, \{\underline{m}\}, \mu) = -\boldsymbol{\Gamma}(\{\underline{p}\}, \{\underline{m}\}, \mu) \,.$$

- The same anomalous dimension $\boldsymbol{\Gamma}$ governs the evolution of the hard Wilson coefficient (and the effective operator)!

$$\frac{d}{d \ln \mu} \left|\mathcal{C}(\{\underline{p}\}, \{\underline{m}\}, \mu)\right\rangle = \boldsymbol{\Gamma}(\{\underline{p}\}, \{\underline{m}\}, \mu) \left|\mathcal{C}(\{\underline{p}\}, \{\underline{m}\}, \mu)\right\rangle \,.$$

- Now the two things — the structure of IR singularities and soft gluon resummation — both rely on the determination of this anomalous dimension.

All-order conjecture for massless case

- The anomalous dimensions for amplitudes involving only massless partons are conjectured to be extremely simple:
  [Becher, Neubert (2009)]
  [Gardi, Magnea (2009)]

$$\mathbf{\Gamma}(\{\underline{p}\},\mu) = \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\, \gamma_{\text{cusp}}(\alpha_s)\, \ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s)\,,$$

  where $s_{ij} = 2\sigma_{ij}\, p_i \cdot p_j$, $\sigma_{ij} = +1$ if both momenta are incoming or outgoing, and -1 otherwise.
- Minimal structure: two parton correlations only.
- Known at two-loop by explicit calculations.
  [Aybat, Dixon, Sterman (2006)]

All-order conjecture for massless case

- Supporting argument based on soft-collinear factorization, non-abelian exponentiation theorem and consistency with collinear limits.
- Implies Casimir scaling of the cusp anomalous dimensions:

$$\frac{\Gamma^q_{\rm cusp}}{C_F} = \frac{\Gamma^g_{\rm cusp}}{C_A} = \gamma_{\rm cusp} \, ,$$

which is known to hold up to three-loop by explicit calculations. [Moch, Vermaseren, Vogt (2004)]

When masses enter...

- For amplitudes involving massive partons, we need HQET in addition to SCET.
- Both the full and the effective theory know about the 4-velocities $v_I = p_I/m_I$ of the massive partons, which define the cusp angles

$$\cosh\beta_{IJ} = w_{IJ} = -\sigma_{IJ}\, v_I \cdot v_J\,.$$

- Much weaker constraints hold for the massive case:
  - no soft-collinear factorization
  - no constraint from (quasi-)collinear limits
- Non-abelian exponentiation theorem still apply.

Anomalous dimension to two loops

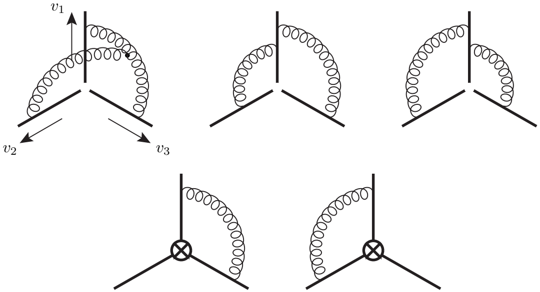
- General structure [Becher, Neubert (2009)]:

$$
\begin{aligned}
\boldsymbol{\Gamma}(\{\underline{p}\},\{\underline{m}\},\mu) = & \sum_{(i,j)} \frac{\boldsymbol{T}_i \cdot \boldsymbol{T}_j}{2}\, \gamma_{\text{cusp}}(\alpha_s)\, \ln\frac{\mu^2}{-s_{ij}} + \sum_i \gamma^i(\alpha_s) \\
& - \sum_{(I,J)} \frac{\boldsymbol{T}_I \cdot \boldsymbol{T}_J}{2}\, \gamma_{\text{cusp}}(\beta_{IJ},\alpha_s) + \sum_I \gamma^I(\alpha_s) \\
& + \sum_{I,j} \boldsymbol{T}_I \cdot \boldsymbol{T}_j\, \gamma_{\text{cusp}}(\alpha_s)\, \ln\frac{m_I \mu}{-s_{Ij}} \\
& + \sum_{(I,J,K)} i f^{abc}\, \boldsymbol{T}_I^a\, \boldsymbol{T}_J^b\, \boldsymbol{T}_K^c\, F_1(\beta_{IJ},\beta_{JK},\beta_{KI}) \\
& + \sum_{(I,J)} \sum_k i f^{abc}\, \boldsymbol{T}_I^a\, \boldsymbol{T}_J^b\, \boldsymbol{T}_k^c\, f_2\Big(\beta_{IJ}, \ln\frac{-\sigma_{Jk}\, v_J \cdot p_k}{-\sigma_{Ik}\, v_I \cdot p_k}\Big)\,.
\end{aligned}
$$

- New functions F_1 and f_2 appear! F_1 represents correlations among three massive partons, while f_2 among two massive and one massless partons. (Correlations among one massive and two massless partons vanish.)

Calculation of F_1

- Relevant two-loop Feynman diagrams:



- "Planar" and counter-term diagrams simple: evaluate using standard techniques.

$$F_1^{(2)\,\text{planar+CT}} = \frac{4}{3} \sum_{I,J,K} \epsilon_{IJK}\, \beta_{KI}\, \coth \beta_{KI}\, \coth \beta_{IJ}$$

$$\times \left[\beta_{IJ}^2 + 2\beta_{IJ} \ln(1 - e^{-2\beta_{IJ}}) - \text{Li}_2(e^{-2\beta_{IJ}}) + \frac{\pi^2}{6} \right],$$

Calculation of the triple-gluon diagram

- Mitov, Sterman and Sung calculated it numerically in the non-physical region.
  [Mitov, Sterman, Sung (2009)]
- We obtained the analytical result.
  [Ferroglia, Neubert, Pecjak, LLY (2009)]
- Our method is based on the following Mellin-Barnes representation:

$$I(w_{12}, w_{23}, w_{31}) = 2(w_{23}\, w_{31} + w_{12})\, \frac{1}{(2\pi i)^5} \int_{-i\infty}^{+i\infty} \left[\prod_{i=1}^{5} dz_i \right] (2w_{23})^{2z_1 - 1} (2w_{31})^{2z_2 - 1} (2w_{12})^{2z_3}$$
$$\times \frac{\Gamma(1-2z_1)\,\Gamma(1-2z_2)}{\Gamma(z_1+z_2+z_3+z_4+z_5)}\, \Gamma(-2z_3)\,\Gamma(-z_4)\,\Gamma(z_1+z_3)\,\Gamma(z_1+z_5)\,\Gamma(z_2-z_5)\,\Gamma(z_3+z_5)$$
$$\times\, \Gamma(z_1+z_2+z_4)\,\Gamma(z_2+z_3+z_4)\,\Gamma(z_2+z_4+z_5)\,\Gamma(1-z_2-z_4-z_5)\,,$$

from which the contribution to F_1 can be obtained:

$$F_1^{(2)\,\text{non-planar}} = \frac{4}{3} \sum_{I,J,K} \epsilon_{IJK}\, I(w_{IJ}, w_{JK}, w_{KI})\,.$$

Calculation of the triple-gluon diagram

- The above representation is not reducible with Barnes' Lemmas, and is also difficult to evaluate by residue method.
- The key observation here is that it is much more natural to work with cusp angles β_{IJ} instead of scalar products w_{IJ}.
- Decomposing w_{IJ} as $w_{IJ} = \cosh\beta_{IJ} = (\alpha_{IJ} + \alpha_{IJ}^{-1})/2$ with $\alpha_{IJ} \equiv e^{\beta_{IJ}}$, and introducing three more Mellin-Barnes parameters, the resulting representation can be reduced using Barnes' Lemmas to a three-fold one:

$$I(w_{12}, w_{23}, w_{31}) = 2(w_{23}\, w_{31} + w_{12})\, \frac{1}{(2\pi i)^3} \int\limits_{-i\infty}^{+i\infty} dz_1\, dz_2\, dz_3\, \alpha_{12}^{-2z_3} \alpha_{23}^{-1-2z_1} \alpha_{31}^{-1-2z2}$$
$$\times\, \Gamma(-z_1 - z_3)\, \Gamma(1 + z_1 - z_3)\, \Gamma(-z_1 + z_3)\, \Gamma(1 + z_1 + z_3)$$
$$\times\, \Gamma^2(-z_2 - z_3)\, \Gamma^2(1 + z_2 - z_3)\, \Gamma^2(-z_2 + z_3)\, \Gamma^2(1 + z_2 + z_3)\,.$$

Final result for F_1

- The remaining integrals can be performed by closing the contours and summing up the residues. The result turns out to be amazingly simple after anti-symmetrized sum:

$$F_1^{(2)\,\text{non-planar}} = -\frac{4}{3} \sum_{I,J,K} \epsilon_{IJK}\, \beta_{IJ}^2\, \beta_{KI}\, \coth \beta_{KI}\,.$$

- Together with the planar and counter-term diagrams, the final result for F_1 is

$$F_1^{(2)}(\beta_{12}, \beta_{23}, \beta_{31}) = \frac{4}{3} \sum_{I,J,K} \epsilon_{IJK}\, r(\beta_{KI})\, g(\beta_{IJ})\,,$$

where

$$r(\beta) = \beta \coth \beta$$
$$g(\beta) = \coth \beta \left[\beta^2 + 2\beta\, \ln(1 - e^{-2\beta}) - \mathrm{Li}_2(e^{-2\beta}) + \frac{\pi^2}{6}\right] - \beta^2 - \frac{\pi^2}{6}\,.$$

Derivation of f_2

- The derivation of f_2 is straightforward by observing that f_2 is the limit of F_1 when one of the partons becomes massless:

$$f_2^{(2)}\left(\beta_{12}, \ln\frac{-\sigma_{23}\, v_2\cdot p_3}{-\sigma_{31}\, v_1\cdot p_3}\right) = 3\lim_{m_3\to 0} F_1^{(2)}(\beta_{12},\beta_{23},\beta_{31})$$
$$= -4\, g(\beta_{12})\,\ln\frac{-\sigma_{23}\, v_2\cdot p_3}{-\sigma_{13}\, v_1\cdot p_3}\,,$$

Properties of F_1 and f_2

- F_1 and f_2 do not vanish for $v_1 \to v_2$! In contrast to naive expectations based on anti-symmetry.

$$\lim_{\beta_{12}\to i\pi} F_1^{(2)}(\beta_{12},\beta_{23},\beta_{31}) = -\sigma_{13}\,\frac{4}{3}\,\left\{\left[\pi^2 + 2i\pi\ln(2|\vec{v}_{12}|)\right] r'(\beta_{31}) - i\pi\, g'(\beta_{31})\right\} \times \left(\frac{\vec{v}_{12}}{|\vec{v}_{12}|}\cdot\frac{\vec{v}_3}{|\vec{v}_3|}\right),$$

$$\lim_{\beta_{12}\to i\pi} f_2^{(2)}\left(\beta_{12}, \ln\frac{-\sigma_{23}\,v_2\cdot p_3}{-\sigma_{13}\,v_1\cdot p_3}\right) = 4\left[\pi^2 + 2i\pi\ln(2|\vec{v}_{12}|)\right]\frac{\vec{v}_{12}}{|\vec{v}_{12}|}\cdot\frac{\vec{p}_3}{|\vec{p}_3|}\,.$$

- In the massless limit $m_I \to 0$ or $|w_{IJ}| \to \infty$, F_1 and f_2 vanish like $(m_I m_J/s_{IJ})^2$, in accordance with a mass factorization theorem.
  [Mitov, Moch (2007)], [Becher, Melnikov (2007)]

First application: top quark pair production

- Predict all IR poles in two-loop amplitudes in analytic form.
  - Results for $q\bar{q}$ channel verified with numeric results of [Czakon (2008)] and analytic results of [Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)], [Bonciani, Ferroglia, Gehrmann, Studerus (2009)].
  - Results for gg channel are new, and were later confirmed by [Czakon: RADCOR 2009].
- Predict logarithmic terms at next-to-next-to-leading order.
- Soft gluon resummation at next-to-next-to-leading-log.

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→ See the talk by Ben Pecjak

IR singularities to two-loops

- The IR singularities in the amplitudes are determined to two-loops via

$$\left|\mathcal{M}^{(1),\,\text{sing}}\right\rangle = \boldsymbol{Z}^{(1)}\left|\mathcal{M}^{(0)}\right\rangle \,,$$

$$\left|\mathcal{M}^{(2),\,\text{sing}}\right\rangle = \left[\boldsymbol{Z}^{(2)} - \left(\boldsymbol{Z}^{(1)}\right)^2\right]\left|\mathcal{M}^{(0)}\right\rangle + \left(\boldsymbol{Z}^{(1)}\left|\mathcal{M}^{(1)}\right\rangle\right)_{\text{poles}} \,,$$

- The renormalization factor is given by

$$\begin{aligned}\boldsymbol{Z} = 1 &+ \frac{\alpha_s^{\text{QCD}}}{4\pi}\left(\frac{\Gamma_0'}{4\epsilon^2} + \frac{\boldsymbol{\Gamma}_0}{2\epsilon}\right)\\ &+ \left(\frac{\alpha_s^{\text{QCD}}}{4\pi}\right)^2\left\{\frac{(\Gamma_0')^2}{32\epsilon^4} + \frac{\Gamma_0'}{8\epsilon^3}\left(\boldsymbol{\Gamma}_0 - \frac{3}{2}\beta_0\right) + \frac{\boldsymbol{\Gamma}_0}{8\epsilon^2}\left(\boldsymbol{\Gamma}_0 - 2\beta_0\right) + \frac{\Gamma_1'}{16\epsilon^2} + \frac{\boldsymbol{\Gamma}_1}{4\epsilon}\right.\\ &\left. - \frac{2T_F}{3}\left[\Gamma_0'\left(\frac{1}{2\epsilon^2}\ln\frac{\mu^2}{m_t^2} + \frac{1}{4\epsilon}\left[\ln^2\frac{\mu^2}{m_t^2} + \frac{\pi^2}{6}\right]\right) + \frac{\boldsymbol{\Gamma}_0}{\epsilon}\ln\frac{\mu^2}{m_t^2}\right]\right\} + \mathcal{O}(\alpha_s^3)\,.\end{aligned}$$

- The anomalous dimension $\boldsymbol{\Gamma}$ is expressed in terms of color generators $\boldsymbol{T}_i$ and the functions γ_{cusp}, γ^q, γ^g, γ^Q and f_2.

IR singularities to two-loops

- The IR singularities in the amplitudes are determined to two-loops via

$$\left|\mathcal{M}^{(1),\,\text{sing}}\right\rangle = \boldsymbol{Z}^{(1)} \left|\mathcal{M}^{(0)}\right\rangle ,$$

Still need to explain

$$\left|\mathcal{M}^{(2),\,\text{sing}}\right\rangle = \left[\boldsymbol{Z}^{(2)} - \left(\boldsymbol{Z}^{(1)}\right)^2\right] \left|\mathcal{M}^{(0)}\right\rangle + \left(\boldsymbol{Z}^{(1)} \left|\mathcal{M}^{(1)}\right\rangle\right)_{\text{poles}} ,$$

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- The anomalous dimension $\boldsymbol{\Gamma}$ is expressed in terms of color generators $\boldsymbol{T}_i$ and the functions γ_{cusp}, γ^q, γ^g, γ^Q and f_2.

Color space formalism

[Catani, Seymour (1996)]

- Consider the on-shell amplitude ($\alpha\beta = q\bar{q}, gg$)

$$\mathcal{M}_{\{a\}} = \left\langle t^{a_3}(p_3)\,\bar{t}^{a_4}(p_4) \,\middle|\, \mathcal{H} \,\middle|\, \alpha^{a_1}(p_1)\,\beta^{a_2}(p_2) \right\rangle \,.$$

- Introduce an orthonormal basis of vectors $\{|a_1, a_2, a_3, a_4\rangle\}$ and a vector $|M\rangle$, so that

$$\mathcal{M}_{\{a\}} = \langle a_1, a_2, a_3, a_4 \,|\, \mathcal{M}\rangle \,.$$

- Color generators $\boldsymbol{T}_i$ are defined by

$$\boldsymbol{T}_i^c \, |\ldots, a_i, \ldots\rangle = (\boldsymbol{T}_i^c)_{b_i a_i} \, |\ldots, b_i, \ldots\rangle \,,$$

  where $(\boldsymbol{T}_i^c)_{ba}$ is
  - t^c_{ba} for a final-state quark or an initial-state anti-quark;
  - $-t^c_{ab}$ for a final-state anti-quark or an initial-state quark;
  - if^{abc} for a gluon.

Color space formalism

- Introduce the orthogonal basis

$$\left(c_1^{q\bar{q}}\right)_{\{a\}} = \delta_{a_1 a_2}\delta_{a_3 a_4}\,, \quad \left(c_2^{q\bar{q}}\right)_{\{a\}} = t^c_{a_2 a_1} t^c_{a_3 a_4}\,,$$

$$\left(c_1^{gg}\right)_{\{a\}} = \delta^{a_1 a_2}\delta_{a_3 a_4}\,, \quad \left(c_2^{gg}\right)_{\{a\}} = if^{a_1 a_2 c}\, t^c_{a_3 a_4}\,, \quad \left(c_3^{gg}\right)_{\{a\}} = d^{a_1 a_2 c}\, t^c_{a_3 a_4}\,,$$

 and the vectors

$$|c_I\rangle \equiv \sum_{\{a\}} \left(c_I\right)_{\{a\}} |\{a\}\rangle \ .$$

- Define the "vector components" of $|\mathcal{M}\rangle$ and the "matrix elements" of $\mathbf{\Gamma}$ as

$$\mathcal{M}_I = \frac{1}{\langle c_I \,|\, c_I \rangle} \langle c_I \,|\, \mathcal{M} \rangle \ , \quad \Gamma_{IJ} = \frac{1}{\langle c_I \,|\, c_I \rangle} \langle c_I \,|\, \mathbf{\Gamma} \,|\, c_J \rangle \ ,$$

 so that

$$\left(\mathbf{\Gamma}\,|\mathcal{M}\rangle\,\right)_I = \Gamma_{IJ}\,\mathcal{M}_J\,.$$

Two-loop anomalous dimension matrices

- Now we are ready to present

$$\begin{aligned}\mathbf{\Gamma}_{q\bar{q}} &= \left[C_F\,\gamma_{\rm cusp}(\alpha_s)\,\ln\frac{-s}{\mu^2} + C_F\,\gamma_{\rm cusp}(\beta_{34},\alpha_s) + 2\gamma^q(\alpha_s) + 2\gamma^Q(\alpha_s) \right] \mathbf{1} \\ &+ \frac{N}{2}\left[\gamma_{\rm cusp}(\alpha_s)\,\ln\frac{(-s_{13})(-s_{24})}{(-s)\,m_t^2} - \gamma_{\rm cusp}(\beta_{34},\alpha_s) \right] \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \gamma_{\rm cusp}(\alpha_s)\,\ln\frac{(-s_{13})(-s_{24})}{(-s_{14})(-s_{23})} \left[\begin{pmatrix} 0 & \frac{C_F}{2N} \\ 1 & -\frac{1}{N} \end{pmatrix} + \frac{\alpha_s}{4\pi}\,g(\beta_{34}) \begin{pmatrix} 0 & \frac{C_F}{2} \\ -N & 0 \end{pmatrix} \right].\end{aligned}$$

- Similar for gg, but a 3×3 matrix.

Conclusions

- Infrared singularities play an important role in QCD and can be determined systematically from anomalous dimensions.
- We compute the anomalous dimensions to two-loop order for scattering amplitudes involving arbitrary numbers of massless and massive partons.
- The infrared structure of any two-loop amplitude in non-abelian gauge theories is therefore well understood.
- Our results also enable the soft gluon resummation for such processes at next-to-next-to-leading-log level.
- First application: top quark pair production.