NLO Parton Shower as Operator Replacement[†]

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April 7th, 2010

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Outline

Jets, Parton Showers, and Resummation

- Parton Showers allow us to bridge the order of magnitude gap between fixed order and final particle multiplicity.
- They achieve this by throwing out information we'd like to systematically put back.
- SCET, as an energy expansion, is well-suited to the kinematic ($1/Q$ power corrections, NLL resummation) part of this task.

Bauer-Schwartz*

- The first attempt to apply SCET to the parton shower met with many successes.
- The amplitude for gluon emission in SCET, $\bar{\chi}_n \rho^\alpha \Omega$ easily reproduces:
  - The usual $q \to qg$ splitting function, $|\rho|^2 \propto \frac{1+z^2}{1-z}$
  - The factorization of each emission from the rest of the process,

$$|A|^2 = \frac{\bar{p}_q}{2} \mathrm{Tr}[\not{n} \rho^\alpha \Omega \Omega^\dagger \rho_\alpha^\dagger] = \frac{\bar{p}_q}{2} |\rho|^2 \mathrm{Tr}[\not{n} \Omega \Omega^\dagger]$$

- SCET also allows a smooth interpolation between fixed-order QCD and shower monte carlo computations.

*hep-ph/0604065,hep-ph/0607296

Sudakov Factor as Operator Running

- Our justification of the no-branching, Sudakov factor came from treating the shower as a classical Markov process.
- In the Bauer-Schwartz picture, the parton shower comes from an operator $\mathcal{O}_n^{(2)}$ describing the radiation of $n-2$ partons from an initial $q\bar{q}$ pair.
- At LL level, Sudakovs ($\Delta_{q,\,g}$) are just the RG evolution kernel ($\Pi(Q,\mu)$) given by the n-parton operator's anomalous dimension:

$$\Pi_n(Q,\mu) \;=\; \Delta_q^{n_q}(Q,\mu)\Delta_g^{n_g}(Q,\mu).$$

Going Further

- Work by B&S has been to LO in the SCET power counting.
- We take as our observable of interest the fully differential cross-section: $\frac{d\sigma}{d^3k_1 \ldots d^3k_n}$.
- By going to NLO in λ, we hope to:
  - Describe partonic configurations away from the strongly ordered limit: $q_{0\perp} \gg q_{1\perp} \gg \ldots$
  - Include spin and color correlations as well as improving kinematic accuracy.
  - For simplicity we are only treating Abelian processes, though all results are extendable to non-Abelian case.
  - Determine how one can include NLL resummation.

$$\lambda^{-4} \sim \quad \frac{dq_{1\perp}^2}{q_{1\perp}^2}\frac{dq_{2\perp}^2}{q_{2\perp}^2} \rightarrow \quad \log^2$$

$$\lambda^{-2} \sim \quad \frac{dq_{\perp}^2}{q_{1\perp}^2} \rightarrow \quad \log$$

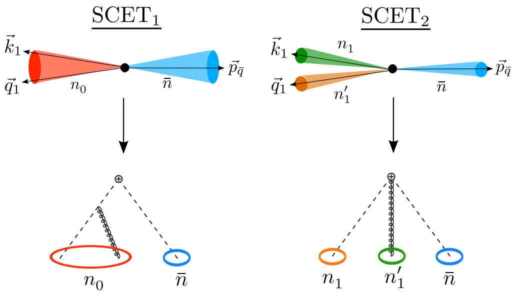
Points of Confusion I: Broken Symmetry

- Including such corrections is difficult with the Bauer-Schwartz setup because of their approach to LO.
- SCET fields (*e.g.* χ_{n_q}) are only allowed to create particles whose momentum is perfectly aligned with $n_q \equiv q/E_q$, *i.e.* $\chi_n|q\rangle = \delta_{n,\, n_q}$.
- This is partly *convention* and partly *approximation*, and we need to disentangle the two.
- This violates a symmetry of SCET (Reparametrization Invariance), which allows the creation of particles anywhere inside a cone with angle $\mathcal{O}(\lambda)$.

Points of Confusion II,III: Double Counting, Kinematic Restriction

- From the n-jet operator $\mathcal{O}_n$ we can project onto Fock states of any multiplicity $> n$.
- When we run $\mathcal{O}_n$ down to the scale of an emission, we avoid double counting by threshold matching onto $\mathcal{O}_n^{(n+1)}$.
- This intrinsic "fuzziness" to the operators makes it hard to implement corrections.
- The Threshold Matching is intimately tied to strong-ordering, $Q \gg p_{1\perp} \gg \ldots \gg p_{m\perp}$.
- It is not clear how to extend beyond this configuration.

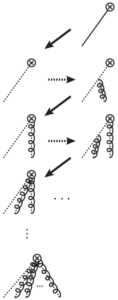
Our Resolution: Multiple SCETs



- A $\mathrm{SCET_i}$ is defined by a set of collinear directions $\{[n_i]\}$, plus a resolution parameter, λ_i.
- Same process seen in two different resolutions, related by standard running and matching.
- Will ultimately terminate in some $\mathrm{SCET_N}$ with $\mathcal{O}(\mathrm{GeV})$ resolution.

Building the Parton Shower with $\mathrm{SCET_i}$

$p^2 \sim Q^2$	QCD
$p^2 \sim Q^2\lambda^2$	SCET_1
$p^2 \sim Q^2\lambda^4$	SCET_2
$p^2 \sim Q^2\lambda^6$	SCET_3
$\vdots$	$\vdots$
$p^2 \sim Q^2\lambda^{2i}$	SCET_i



Sequence of emissions and matching to get one branch of a LO Shower.

- We can define sets Ω_i such that $\Omega_0 \supset \Omega_2 \supset \ldots \Omega_N$.
- These contain all momenta p such that $p^2 \lesssim Q^2\lambda^{2i}$.
- λ^i is resolution parameter for the theory, $\mathrm{SCET_i}$.
- We can remove an arbitrary number of particles going from Ω_i to Ω_{i+1}, allowing us to match any phase space configuration.

Advantages to $\mathrm{SCET_i}$ Picture

- Collinear fields with different label indices, n, do not overlap in Hilbert Space, which greatly simplifies squaring and integrating the amplitude.

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- Each $\mathrm{SCET_i}$ comes with its own symmetry group, $\mathrm{RPI_i}$, whose transformations disentangle convention-approximation issue in Bauer-Schwartz.
  - $\mathrm{RPI_{i+1}}$ transformation are convention chosen for convenience.
  - $\mathrm{RPI_i/RPI_{i+1}}$ define a set of corrections in $\mathrm{SCET_{i+1}}$.

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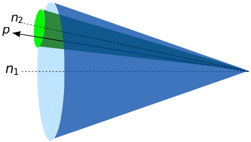
  - $\mathrm{RPI_i}/\mathrm{RPI_{i+1}}$ define a set of corrections in $\mathrm{SCET_{i+1}}$.
- By $\mathrm{SCET_i}$ symmetries and equations of motion, in matching to $\mathrm{SCET_{i+1}}$, collinear operator basis can be constructed from just three objects: $(\chi_n, \mathcal{B}_{\perp\, n}, \mathcal{P}_{\perp\, n})$

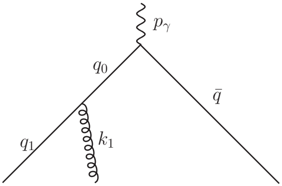
$\mathrm{RPI_i}$

- $\mathrm{SCET_i}$ possesses a symmetry called *Reparametrization Invariance* ($\mathrm{RPI_i}$).
- $\mathrm{RPI_i}$ transformation of interest sends $n \to n + \Delta_\perp$, where $\Delta_\perp \sim \lambda^i$.



Larger Cone is symmetric region for $\mathrm{SCET_i}$, smaller one for $\mathrm{SCET_{i+1}}$

Matching to Single Emission

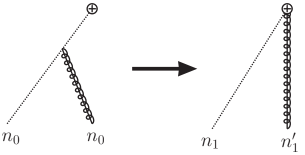


Kinematics for single emission

- For concreteness, we consider starting with a process in QCD, $J^\mu_{QCD} = \bar{q}\Gamma^\mu q$.
- Single gluon emission in SCET for the kinematics shown is $(q_{0\perp} = 0)$: $\bar{\xi}_{n_0}(q_1)\left(n_{0\alpha} + \frac{\not{q}_{1\perp}\gamma_{\perp\alpha}}{\bar{q}_1}\right)\frac{\bar{q}_0}{q_0^2} \times \Gamma^\mu \xi_{\bar{n}}(\bar{q})$.

Matching to Single Emission (cont'd)

Want to annihilate $|q(q_1)\,\bar{q}(p_{\bar{q}})\,g(k_1))\rangle_2$



L: Lagrangian emission in SCET_1

R: Matches to *qg* operator in SCET_2

- The matching to SCET_2 is straightforward.
- The 2 subscript on the state means that only fields whose *n*-index lies within a cone of size λ^2 about the particle can annihilate it.
- Need to use a finite Reparametrization Invariance transformation on the SCET_1 fields to guarantee this.

Using $\mathrm{RPI_i}$

- By performing an $\mathrm{RPI_1}$ transformation on the $\mathrm{SCET_1}$ objects, *e.g.*:

$$\begin{aligned} n_1 &= n_0 + \frac{2q_{1\perp}}{\bar{q}_1} - \frac{q_{1\perp}^2}{\bar{q}_1^2}\bar{n} \\ \xi_{n_0} &= \frac{\not{n}_0 \not{\bar{n}}}{4}\xi_{n_1} \end{aligned}$$

  we can move the $\mathrm{SCET_1}$ labels *inside* the $\mathrm{SCET_2}$ symmetry cones.
- In $\mathrm{SCET_2}$, the label can perfectly align with the field momentum, $n_1 = k_1/E_1$, for simplicity of matching.
- After getting operators in this way, we can get the results for other choices of n using an $\mathrm{RPI_2}$ transformation.

Single Emission in SCET_2

- We get the operator: $C^{(1)}_{2,\,\mathrm{LO}}\mathcal{O}^{(1)}_2$, where

$$\begin{aligned}\mathcal{O}^{(1)}_2 &= \left(\bar\chi_j\right)_{n_1} \mathcal{B}^{\alpha}_{n_1'\perp}\Gamma^\mu \left(\chi_k\right)_{\bar n}\ , \\ C^{(1)}_{2,\,\mathrm{LO}} &= \left[\frac{\bar q_0}{q_0^2}\left(n_0^\alpha + \frac{(\not q_1)_{n_0\perp}\gamma^\alpha_{n_1'\perp}}{\bar q_1}\right)\frac{\not\bar n \not n_0}{4}\right]_{jk}\ .\end{aligned}$$

- This can be encoded as an operator replacement on $\bar\chi_{n_0}$ in $J^\mu = \bar\chi_{n_0}\Gamma^\mu\chi_{\bar n}$:

$$\bar\chi_{n_0} \to C^{(1)}_{2,\,\mathrm{LO}}\,\bar\chi_{n_1}\mathcal{B}_{n_1'\perp}$$

Replacement Rule

$$\bar{\chi}_{n_0} \to C^{(1)}_{2,\,\mathrm{LO}}\, \bar{\chi}_{n_1} \mathcal{B}_{n_1' \perp}$$

- Our EFT approach has taken the particle splitting, $q \to qg$ and turned it into an operator one.
- By iterating this over and, we can build up an entire parton shower.
- We can interpret the Wilson coefficient for this, $C^{(1)}_{2,\,\mathrm{LO}}$, as the square root of the splitting function.

$$|C^{(1)}_{2,\,\mathrm{LO}}|^2 \propto \frac{1+z^2}{1-z}$$

$\mathrm{SCET_i} \rightarrow \mathrm{SCET_{i+1}}$

- Subsequent strongly-ordered emissions factorize from the previous ones.
- We can generalize the procedure by going to even lower-scale theories.
- Doing $(\bar{\chi}_n \rightarrow c\,\bar{\chi}_{n'}\mathcal{B}_{\perp n''})$ i times gets $C^{(i)}_{i+1,\,\mathrm{LO}}\mathcal{O}^{(i)}_{i+1}$.
- This gives us the parton shower amplitude for i-gluon emission as a $\mathrm{SCET_{i+1}}$ operator.
- We never needed any information beyond that for single emission.

$C^{(i)}_{i+1,\,\mathrm{LO}}\mathcal{O}^{(i)}_{i+1}$

- Our operator for i-gluon emission is exactly what we would expect from our single emission calculation:

$$\mathcal{O}^{(i)}_{i+1} = \chi_{n_i}\left(\prod_{k=1}^{i}\mathcal{B}^{n'_k\perp}_{\alpha_k}\right)\Gamma_\mu\chi_{\bar{n}}\,,$$

$$C^{(i)}_{i+1,\,\mathrm{LO}} = \left(\prod_{k=1}^{i} c^{\alpha_k}_{\mathrm{LO}}\right),$$

$$c^{\alpha_k} = \frac{\overline{q}_{k-1}}{q^2_{k-1}}\left(n^{\alpha_k}_{k-1} + \frac{\left(\not{q}_{k\perp}\right)_{n_{k-1}}\gamma^{\alpha_k}_{n'_k\perp}}{\overline{q}_k}\right)\left(\frac{\not{\bar{n}}\not{n}_{k-1}}{4}\right)$$

Regaining the Parton Shower

- To LO, $|c^{\alpha_k}|^2 \propto \frac{1+z_k^2}{1-z_k}$. Thus, we reproduce the factorized product of $1 \to 2$ splittings.
- There are similar replacement formulas for gluon splittings as well:

$$\mathcal{B}_{\perp n_1} \to C_{q\to q\bar{q}}\ (\bar{\chi}_{n2}\gamma^{\mu}\chi_{n3}) + C_{q\to gg}\ (\mathcal{B}_{\perp n_2}\mathcal{B}_{\perp n_3})$$

Soft Emission

- Getting the parton shower also requires proper treatment of soft emissions.
- Redefining the quark field to decouple softs from the lagrangian introduces soft Wilson lines into our jet operators:

$$\mathcal{O}_i^{(i-1)} = \prod_{k=1}^{i-1} C_{\rm LO}^{\alpha_j} \bar{\chi}_{n_k}^{(0)} Y_{n_k}^\dagger Y_{n'_k} \mathcal{B}_{n'_k\perp}^{(0)\alpha_k} Y_{n'_k}^\dagger \Gamma_\mu Y_{\bar{n}} \chi_{\bar{n}}$$

- Note that in $\mathrm{SCET_j}$, for $j < i$, the n-index on some of the Y_n's would be the same, causing $Y_n Y_n^\dagger$ to cancel.
- More soft emissions become apparent as we run down in scale.

Angular Ordering

- The soft-collinear coupling contributes a multiplicative factor $\frac{n\cdot\varepsilon_s}{n\cdot k} = \frac{p\cdot\varepsilon_s}{p\cdot k}$, where k is the gluon momentum
- This term reproduces the well-known angular ordering result for soft gluons, which states that soft gluons are only emitted in the cone between the quark and a previously emitted gluon.
- Note that here we avoid soft 0-bin subtraction by working at the level of the differential cross-section $d\sigma$.
- Tempting in SCET to distinguish soft and collinear. [In parton shower, though only one kind of emission.]

Mapping to the Parton Shower

- Our operators $\bar{\chi}\,(\mathcal{B}_\perp)^n\,\chi$ have the same one-loop cusp anomalous dimension calculation as Bauer-Schwartz, and we can also incorporate Sudakov factors as one-loop running.
- We therefore get the following SCET $\rightarrow$ Parton Shower map:

$$\begin{aligned} \text{One} - \text{Loop Cusp AD} &\rightarrow \text{No} - \text{branching Factor} \\ \text{LO Replacement Rule} &\rightarrow \text{Splitting Probability} \\ \text{Softs} &\rightarrow \text{Choice of soft implementation}^\dagger \end{aligned}$$

- Note that the information in the first two lines above comes from single branching considerations alone.

† *cf.* C. Bauer's talk

Not Mapping to the Parton Shower

- As EFTs, the $\mathrm{SCET_i}$ contain objects which are not needed for the above mapping.
- For example, we could get the contribution to a three parton amplitude in $\mathrm{SCET_2}$ from the following:

$$A^{q\bar{q}gg} \; = \; \langle 0 | T\{\mathcal{L}_2 \mathcal{L}_2 \mathcal{O}_2^{(0)}\} | q\bar{q}gg\rangle$$

- This corresponds to a quark which had not branched until after the scale of matching $Q\lambda$.
- However, the RG kernels already give us the no-branching probability.
- We perform standing matching between EFTs first, *then* worry about shower mapping.

λ

- While we could set up a $\mathrm{SCET_i}$ picture for a concrete numerical $\lambda \ldots$
- For us it is a bookeeping device to define classes of strongly-ordered partices. [In order to perturb about LL shower.]
- As we will see, a set of our $\mathcal{O}(\lambda)$ corrections will lead to collinear NLL resummation.
- A strongly ordered process removes one particle in going from $\Omega_i \to \Omega_{i+1}$.
- However, at $\mathcal{O}(\lambda)$, we will have to remove more. No obstacle to this with standard matching.

Types of Corrections

- We will find two distinct types of $\mathcal{O}(\lambda)$ corrections:
  1. Hard-scattering corrections that come from matching QCD to SCET at higher orders. They involve information about the process that created the partons and other fields involved.
  2. Jet-structure corrections that arise in matching $\mathrm{SCET_i} \to \mathrm{SCET_{i+1}}$ at higher orders. They appear throughout the shower and are related to $\mathcal{O}(\alpha_s)$ corrections to the splitting kernel.
- We will proceed by discussing corrections to single gluon emission first (just hard-scattering), then double emission (hard-scattering and jet-structure).

Single Emission

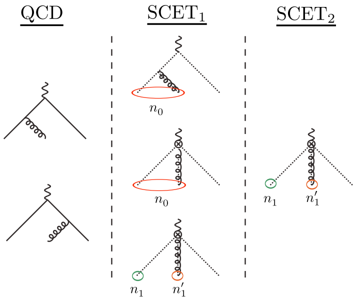
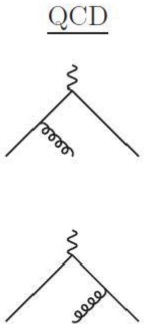


Figure: Diagrams for Matching QCD $\to$ SCET_1 $\to \mathrm{SCET}_2$.

$$
\begin{aligned}
\mathcal{O}_1^{(0)}(n_0) &= \bar{\chi}_{n_0}\chi_{\bar{n}}\,, \\
\mathcal{O}_1^{(1)}(n_0, n_0) &= \bar{\chi}_{n_0}\, g\mathcal{B}^{\alpha}_{n_0\perp}\chi_{\bar{n}}\,, \\
\mathcal{T}_1^{(1)}(n_0, n_0) &= \bar{\chi}_{n_0}\left[\mathcal{P}^{\beta}_{n_0\perp}\, g\mathcal{B}^{\alpha}_{n_0\perp}\right] \\
&\quad \times\chi_{\bar{n}}\,, \\
\mathcal{O}_1^{(1)}(n_1, n_1') &= \bar{\chi}_{n_1} g\mathcal{B}^{\alpha}_{n_1'\perp}\chi_{\bar{n}}
\end{aligned}
$$

Matching QCD $\rightarrow$ $\mathrm{SCET_1}$



- Expand the QCD amplitude in the limit of the gluon being collinear to the quark.
- We match to the SCET lagrangian emission at LO. This process is at $\mathcal{O}(\lambda^{-1})$, since the virtual quark propagator goes like $(Q^2\lambda^2)^{-1}$ and The vertex, $\sim q_{1\perp}$ is $\mathcal{O}(Q\lambda)$.
- Quark emission graph contributes both Wilson-line gluons and $\mathcal{B}_\perp$ ones. Antiquark emission only gives Wilson-line.

Subleading SCET_1 Operators

For $\mathcal{O}_1^{(1)}(n_0, n_0) = \bar{\chi}_{n_0}\, g\mathcal{B}^{\alpha}_{n_0\perp}\chi_{\bar{n}}$, there are two contributions:

1. $\mathcal{B}_\perp$ emission from the antiquark. The virtual antiquark propagator $\sim \lambda^0$, as is the perp momentum of the gluon.
2. We can write the QCD spinor in terms of the collinear SCET spinor in direction n.

$$u_{\mathrm{QCD}}(p) = (1 + \frac{p_\perp \bar{\not{n}}}{2\bar{p}})u_n(p).$$

We can get a vertex off the "suppressed" propagator, corresponding to the second term. The vertex factor is $\mathcal{O}(\lambda^2)$, which multiplies the propagator $\sim \mathcal{O}(\lambda^{-2})$.

Subleading $\mathrm{SCET_1}$ Operators (cont'd)

- Running $\mathcal{O}_1^{(1)}(n_0, n_0)$ will its anomalous dimension. Fortunately, this is the same as $\mathcal{O}_1^{(0)}(n_0) = \bar{\chi}_{n_0}\chi_{\bar{n}}$ as the soft Wilson-line structure is the same. (Also true for $\mathcal{T}_1^{(1)}(n_0, n_0)$).
- The Ward identity forces these two contributions to join up with the same coefficient.
- If the $q\bar{q}$ pair comes from a vector current, $\bar{q}\gamma^\mu q$, then this subleading operator has the structure different from a single-field replacement rule:

$$\bar{\chi}_{n_{q_1}}\mathcal{B}^{\alpha}_{\perp n_{q_1}}(\bar{n}_\mu - n_\mu)\chi_{\bar{n}}.$$

Hard-scattering corrections

- These corrections do involve information about the parton's coupling to the rest of the process under consideration.
- They affect partonic production from the hard vertex, but do not appear in the subsequent shower.
- They represent a non-universal set of corrections that one can use to improve hard matrix elements.
- Fortunately, at the lowest orders they have a small basis, and one can compute them with just a few partons.
- Traditional shower has difficulty with double-counting between fixed order and shower contributions. In SCET, this distinction is straightforward as each type simply originates with different operators.

Hard-scattering corrections (cont'd)

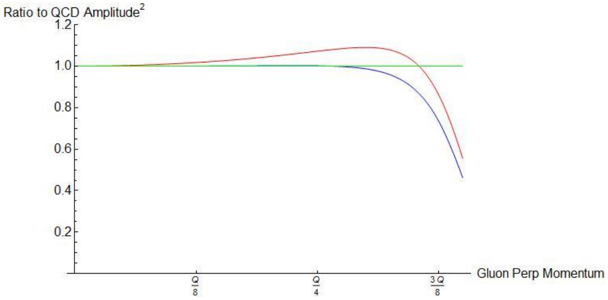


Figure: LO (red), LO+NLO (blue) for $q\bar{q}g$ emission.

- Including the $\mathcal{O}(\lambda^2)$ corrections extends the region where tree-level SCET and QCD agree.

NLO Two Gluon Emission: Hard-Scattering

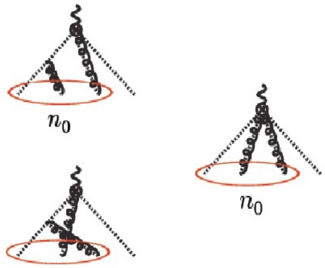


Figure: Two-gluon processes from **3,4-parton** hard-scattering operators in $\mathrm{SCET_1}$

- We match $\mathrm{SCET_1} \to \mathrm{SCET_2}$ for the hard-scattering contributions to double gluon emission.
- Three-parton operators in $\mathrm{SCET_2}$ have same structure as we found in $\mathrm{SCET_1}$,
- We also get the four-parton contributions:

$$\begin{aligned}
\mathcal{O}_2^{(2)}(n_2, n_2, n_1') &= \bar{\chi}_{n_1} \mathcal{B}^{\alpha}_{n_1\perp} \mathcal{B}^{\beta}_{n_1'\perp} \chi_{\bar{n}} \,, \\
\mathcal{O}_2^{(2)}(n_2, n_1', n_1') &= \bar{\chi}_{n_1} \mathcal{B}^{\alpha}_{n_1'\perp} \mathcal{B}^{\beta}_{n_1'\perp} \chi_{\bar{n}} \,, \\
\mathcal{O}_2^{(2)}(n_2, n_1', n_2') &= \bar{\chi}_{n_2} \mathcal{B}^{\alpha}_{n_1'\perp} \mathcal{B}^{\beta}_{n_2'\perp} \chi_{\bar{n}}
\end{aligned}$$

NLO Two Gluon Emission: Hard-Scattering (cont'd)

$$
\begin{aligned}
&[C^{(1)}_{1,\,\mathrm{NLO}}(n_0,\ n_0) + C^{(1)}_{1,\,\mathrm{NLO}}(n_1,\ n_1')]\langle 0|\,T\{\mathcal{L}_{\mathrm{SCET}_1}\mathcal{O}^{(1)}_{1,\,\mathrm{NLO}}\}|q\bar{q}gg\rangle_1 \\
&+C^{(1)}_{1,\,\mathrm{NNLO,T}}(n_0, n_0)\langle 0|\,T\{\mathcal{L}_{\mathrm{SCET}_1}\mathcal{T}^{(1)}_{1,\,\mathrm{NNLO}}\}|q\bar{q}gg\rangle_1 \\
&+C^{(2)}_{1,\,\mathrm{NNLO}}(n_0,\ n_0,\ n_0)\langle 0|\mathcal{O}^{(2)}_{1,\,\mathrm{NNLO}}|q\bar{q}gg\rangle_1 \\
&= [C^{(1)}_{2,\,\mathrm{NLO}}(n_0,\ n_0) + C^{(1)}_{2,\,\mathrm{NLO}}(n_1,\ n_1')]\langle 0|\,T\{\mathcal{L}_{\mathrm{SCET}_2}\mathcal{O}^{(1)}_{2,\,\mathrm{NLO}}\}|q\bar{q}gg\rangle_2 \\
&+C^{(1)}_{2,\,\mathrm{NNLO,T}}(n_0, n_0)\langle 0|\,T\{\mathcal{L}_{\mathrm{SCET}_2}\mathcal{T}^{(1)}_{2,\,\mathrm{NNLO}}\}|q\bar{q}gg\rangle_2 \\
&+[C^{(2)}_{2,\,\mathrm{NNLO}}(n_2, n_2, n_1') + C^{(2)}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_1') + C^{(2)}_{2,\,\mathrm{NNLO}}(n_2, n_1', n_2')] \\
&\langle 0|\mathcal{O}^{(2)}_{2,\,\mathrm{NNLO}}|q\bar{q}gg\rangle_2
\end{aligned}
$$

- $\mathcal{O}^{(j)} \sim \bar{\chi}(\mathcal{B}_\perp)^j\chi$ and $\mathcal{T}^{(j)} \sim (\mathcal{P}_\perp)^\ell\bar{\chi}(\mathcal{B}_\perp)^j\chi$
- $C^{(2)}_{1,\,\mathrm{NNLO}}(n_0,\ n_0,\ n_0)\mathcal{O}^{(2)}_{1,\,\mathrm{NNLO}}$ only contributes at $\mathrm{N^3LO}$.

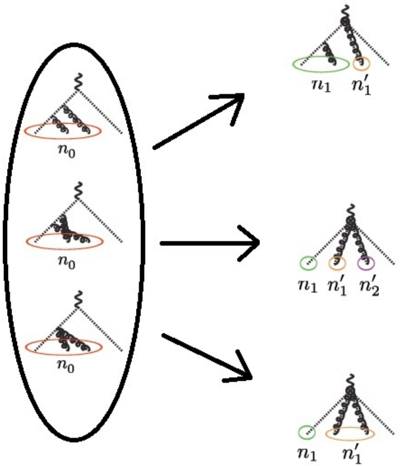
NLO Two Gluon Emission: Jet-Structure

Subleading terms can come from $\mathrm{SCET_1}$ lagrangian emission

- Previously, we calculated the corrections from hard-scattering operators.
- However, the $\mathrm{SCET_1}$ lagrangian contains more than the LO replacement rule.
- To get our LO replacement rule, we threw out subleading contributions.
- Now we need to put this back.

Matching diagrams for two-gluon emission

$\mathrm{SCET_2}$ operators obtained from taking kinematic limits of double $\mathrm{SCET_1}$ lagrangian emission



Matching Calculation

- Now we want the coefficients for $\mathcal{O}_2^{(2)} = \bar{\chi}_{n_1}\mathcal{B}^{\alpha}_{n_1'\perp}\mathcal{B}^{\beta}_{n_1'\perp}\chi_{\bar{n}}$, $\mathcal{O}_2^{(2)} = \bar{\chi}_{n_2}\mathcal{B}^{\alpha}_{n_1'\perp}\mathcal{B}^{\beta}_{n_2'\perp}\chi_{\bar{n}}$.
- Matching equation for previous diagrams is as follows:

$$\begin{aligned} C^{(0)}_{1,\,\mathrm{LO}}\langle 0|T\{\mathcal{L}_{\mathrm{SCET}_1}\mathcal{L}_{\mathrm{SCET}_1}\mathcal{O}^{(0)}_1\}|q\bar{q}gg\rangle_1 \quad &= \\ C^{(1)}_{2,\,\mathrm{LO}}\langle 0|T\{\mathcal{L}_{\mathrm{SCET}_2}\mathcal{O}^{(1)}_2\}|q\bar{q}gg\rangle_2 \\ +[C^{(2)J}_{2,\,\mathrm{NLO}}(n_1,\,n_1',\,n_1') + C^{(2)J}_{2,\,\mathrm{NLO}}(n_1,\,n_1',\,n_1')]\langle 0|\mathcal{O}^{(2)}_2|q\bar{q}gg\rangle_2. \end{aligned}$$

- $C^{(1)}_{2,\,\mathrm{LO}}\mathcal{O}^{(1)}_2$ is the operator we already have from LO replacement rule.

Limit Matching

- Despite writing the matching as a sum, we get each term individually, by taking a particular limit of the $\mathrm{SCET_1}$ two-gluon amplitude, $A^{q\bar{q}gg}$.

$$\lim_{n_q\cdot n_{g2}\sim\lambda^4} A^{q\bar{q}gg} =$$
$$C^{(1)}_{2,\,\mathrm{LO}}\langle 0|T\{\mathcal{L}_{\mathrm{SCET}_2}\mathcal{O}^{(1)}_2\}|q\bar{q}gg\rangle$$

$$\lim_{n_q\cdot n_{g2}\sim\lambda^2} A^{q\bar{q}gg} =$$
$$C^{(2)J}_{2,\,\mathrm{NLO}}(n_1,\, n_1',\, n_2')\langle 0|\mathcal{O}^{(2)}_2|q\bar{q}gg\rangle$$

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Collinear Bin

- At the operator and $\mathrm{amplitude}^2$ level, the *n*-structure keeps the contributions distinct.
- In matching $\mathrm{SCET_i} \to \mathrm{SCET_{i+1}}$, formerly different operators, $\{\mathcal{O}_i(n_j, n'_j), \mathcal{O}_i(n_j, n_j)\}$ collapse to a single one $\mathcal{O}_{i+1}(n_j, n'_j)$.
- Distinction maintained using Wilsonian cutoff functions whose arguments areb dot products of *n*'s, *e.g.* $\theta(\lambda^{2i} - n_j \cdot n'_j)$. In practice we use smoothed *theta*'s to minimize cutoff dependence.
- Scaleless, dimreg type 0-bin procedures don't obviously work because our overlap involves angle not energy scale.
- Integrating also leads to double-counting issues resolved by θ's.

Operator Merging

- We can illustrate the θ-function procedure for the case of $q\bar{q}g$ configurations with 2 and 3 different collinear directions.

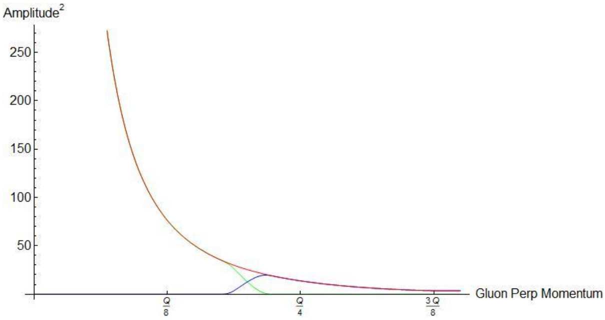


Figure: 2 and 3 "jet" merged $\mathrm{amplitude}^2$ in $q\bar{q}g$ process with two-jet (green), three-jet (blue), and total (red).

$C_{3,\,\mathrm{NLO}}^{(2)J}\mathcal{O}_3^{(2)}$

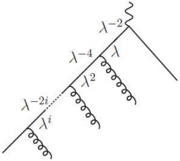
- The different jet-structure correction operators in SCET_2 collapse to the same form in SCET_3.
- Our suppressed operator has the following structure:

$$C_{3,\,\mathrm{NLO}}^{(2)J}\mathcal{O}_3^{(2)} \;=\; h_{\mathrm{I}}^{\alpha\beta}\,\bar{\chi}_{n_2}\mathcal{B}^{\alpha}_{n_1'\perp}\mathcal{B}^{\beta}_{n_2'\perp}\Gamma^{\mu}\chi_{\bar{n}}$$

- Just like our LO replacement rule, it doesn't depend on the rest of process.
- Thus, we can write it as an NLO replacement rule: $\bar{\chi} \;\to\; h_{\mathrm{I},\,\alpha\beta}\bar{\chi}B_{\perp}^{\alpha}B_{\perp}^{\beta}$.
- The I subscript refers to the different operators that contribute from SCET_2. Need to keep contributions distinct as the different terms have different anomalous dimensions.

NLO Replacement Rule

- The replacement rule suggests that we can generalize it to $\mathrm{SCET_i} \rightarrow \mathrm{SCET_{i+1}}$ matching.
- Consider the strongly-ordered emission of $i+1$ gluons.



- If instead the last gluon is as collinear as the previous one, we have a process suppressed by a single power of λ for all i.

Using the NLO Replacement Rule

- LO emissions factorize from everything, so we can just apply the replacement rule $\bar{\chi} \rightarrow h_{\alpha\beta}\bar{\chi}B_\perp^\alpha B_\perp^\beta$ to generate a subleading operator.
- We get a set of NLO operators for i-gluon emission by taking $(i-2)$ uses of the LO replacement rule, and using the NLO replacement rule once.

Interference: An Interesting Result for $\mathrm{SCET_i}$

- The $\mathrm{SCET_i}$ picture greatly simplifies the interference structure when we square amplitudes.
- Fields in SCET with a particular label index, n_i, can only interfere with fields carrying the same index.
- This is enforced in integration by the inclusion of smoothed θ-functions in the Wilson coefficients.

Interference: An Interesting Result for $\mathrm{SCET_i}$ (cont'd)

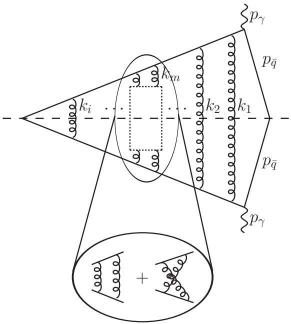
- Once again our formula for NLO jet-structure corrections is:

$$\mathcal{O}^{\mathrm{NLO}} = \bar{\chi}_{n_q}\left(h_{\beta_{j-1}\gamma_{j'}}\mathcal{B}^{\beta_{j-1}}_{n_{j-1}\perp}\mathcal{B}^{\gamma_{j'}}_{n_{j'}\perp}\right) \times\left(\prod_{k=1,\ k\neq j-1,\ j}^{i} C^{\mathrm{LO}}_{\alpha_k}(\mathcal{B}^{\alpha_k}_{n_k\perp})\right)\Omega,$$

- Different values of j lead to different stuctures in the indices.
- Thus, each term only interferes with itself:

$$|A_{q+(ig)}|^2_{\text{to NNLO}} = \langle|\mathcal{O}^{q(ig)}_{\mathrm{LO}}|^2\rangle + \sum_{j=1}^{i-1}\langle|\mathcal{O}^{q(ig)}_{\mathrm{NLO},j}|^2\rangle.$$

Interference Structure in $\mathrm{SCET_i}$



Deviation from strong-ordering in one location causes the above interference pattern.

It is just the usual parton shower with a two-parton phase space defect.

Subleading splitting function

- The coefficient of our LO replacement rule had a nice interpretation as the "square-root" of the usual splitting function.
- We want to test the idea that our subleading replacement rule captures the subleading behavior of the parton splitting.
- We checked if it could reproduce the $\mathcal{O}(\alpha_s)$ correction to the splitting function (Curci, Furmanski, and Petronzio).

Subleading splitting function

- We matched for the gauge invariant piece we checked:

$$
\begin{aligned}
P_{qq}^{(1)} &= C_F^2 \frac{\alpha_s^2}{2\pi} \left[(1-x)\ln(x) - \frac{3}{2}\frac{1+x^2}{1-x}\ln(x) \right. \\
&- 2\frac{1+x^2}{1-x}\ln(x)\ \ln(1-x) - \frac{1}{2}(1+x)\ln^2(x) \\
&- \left. 5(1-x) - \frac{5}{2}(1+x)\ln(x) \right] \\
&+ \{\text{other color structures}\}.
\end{aligned}
$$

- Including such contributions is a non-trivial step toward including NLL resummation.

Subleading splitting function (cont'd)

- Resumming NLL requires improving the Sudakov factor to include the subleading splitting:

$$\Delta(t_j) = \exp\left[-\int_{t_i}^{t_j} \frac{dt'}{t'} \int dz \, \frac{\alpha_s}{2\pi} \left[P_{jk}^{(0)}(z) + \frac{\alpha_s}{2\pi} P_{jk}^{(1)}(z)\right]\right].$$

- Doing this alone will violate conserved probability of the parton shower.
- We also need to include the subleading splitting information for real emissions discussed above.
- It is still an unsolved problem at the algorithmic level to have $1 \to 2$ and $1 \to 3$ splittings simultaneously.

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  7. Rederivation of subleading splitting function within SCET, allowing for inclusion of NLL resummation.

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