Color Evolution and Resummation for Hadronic Event Shapes

Randall Kelley

Harvard University

(work with Matthew Schwartz)

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Introduction

At LHC, a vast majority of events will involve only quarks and gluons at the hard scale.

In e^+e^- colliders, event shapes characterize QCD events.

$$\text{thrust:} \quad T = \max_{\mathbf{n}} \frac{\sum |\mathbf{p}_i \cdot \mathbf{n}|}{\sum |\mathbf{p}_i|}$$

Natural to try to extend this work to Hadron colliders

Hadron Colliders

4 direction of large energy flow in $pp \to 2$ jets.

multiple channels

$$q\bar{q} \to q\bar{q} \qquad q\bar{q} \to gg \qquad gg \to gg$$

Multiple color structures

- 2 for $q\bar{q} \to q'\bar{q}'$, $qq \to qq$, $\cdots$
- 3 for $q\bar{q} \to gg$, $qg \to qg$, $\cdots$
- 8 for $gg \to gg$

Goal is to understand how to build a hadron observable?

- Build a simple simplest observable to NNLL
- Identify features that are Universal and observable specific.

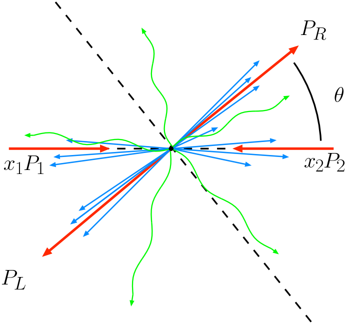
Application: Threshold Thrust

Define $P_i = E_i n_i$,

$$S = (P_1 + P_2)^2, \qquad T = (P_1 - P_R)^2, \qquad U = (P_2 - P_R)^2$$

$$S_4 = S + T + U = P_L^2 + P_R^2$$

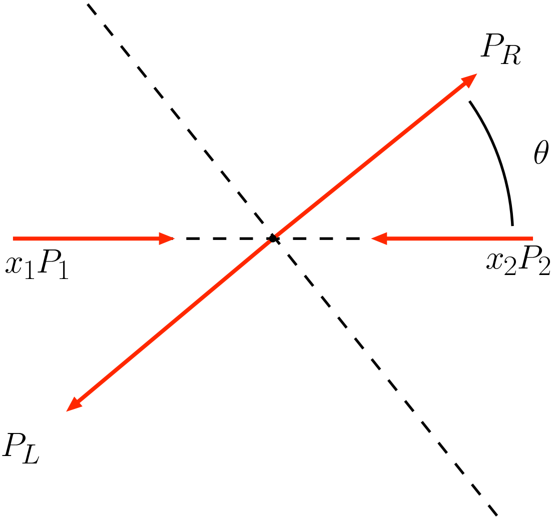
At threshold, $E_i = \sqrt{S}$, and $n_1 = \bar{n}_i$ and $n_L = \bar{n}_R$.



At threshold, $P_L^2 \ll E_R^2$, and so each hemisphere contains one jet and soft radiation

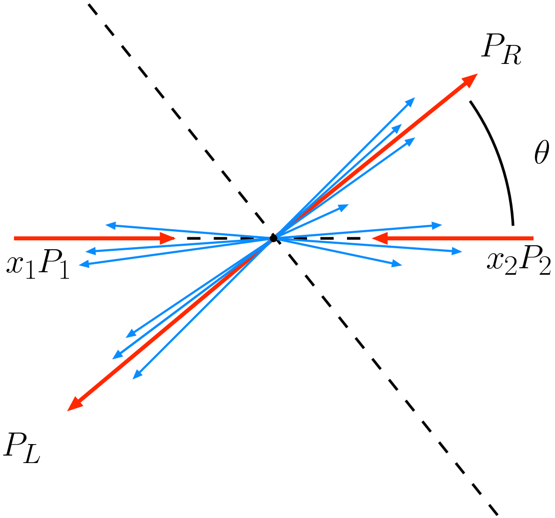
Hemisphere masses and Threshold Thrust

$$\tau \equiv 1 - T \approx \frac{P_L^2 + P_R^2}{S} + O(\tau^2)$$



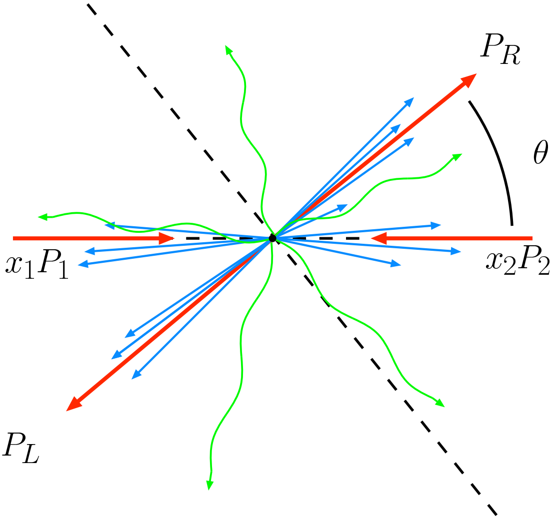
Hemisphere masses and Threshold Thrust

$$P_R^2 = \left[P_R^c + (1 - x_1) P_1 \right]^2$$



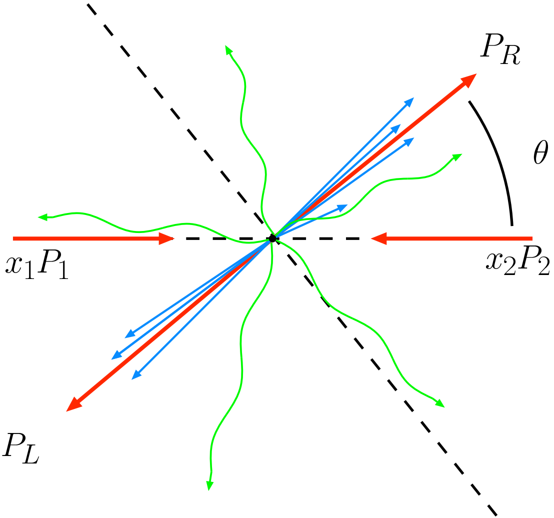
Hemisphere masses and Threshold Thrust

$$P_R^2 = \left[P_R^c + k_R + (1 - x_1) P_R \right]^2$$



Hemisphere masses and Threshold Thrust

$$P_R^2 = (P_R^c)^2 + 2E_R(n_R \cdot k_R) + t(1 - x_1) + O\Big(k_R, (1 - x_1)\Big)$$



Factorization Theorem: Threshold Thrust

$$\frac{d\sigma}{dP_L^2 dP_R^2} \sim \sum_{I,J,\Gamma} \hat{\sigma}^{\Gamma} C_I^{\Gamma}(s,t,u) C_J^{\Gamma *}(s,t,u) \times \int J(m_L^2) \otimes J(m_R^2) \otimes f_1(x_1) \otimes f_2(x_2) \otimes S_{IJ}\Big(k_R, k_L\Big)$$

$$\frac{d\sigma}{d\tau} = \sum_{\text{channels}} \int dP_L dP_R \left(\frac{d\sigma}{dP_L^2 dP_R^2}\right)^{\text{channel}} \delta\left(\tau - \frac{P_L^2 + P_R^2}{S}\right)$$

Factorization Theorem: Threshold Thrust

$$\frac{d\sigma}{dP_L^2 dP_R^2} = \sum_{I,J,\Gamma} \hat{\sigma}^{\Gamma} C_I^{\Gamma} C_J^{\Gamma} \int dx_1 dx_2 dm_L^2 dm_R^2 dk_R dk_L$$

$$\times J(m_L^2) J(m_R^2) f_1(x_1) f_2(x_2) S_{IJ}\Big(k_R, k_L\Big)$$

$$\times \delta\Big[(P_R^c)^2 + 2E_R k_R + t(1-x_1) - m_R^2\Big]$$

$$\times \delta\Big[(P_L^c)^2 + 2E_L k_L + t(1-x_2) - m_L^2\Big]$$

$$\frac{d\sigma}{d\tau} = \sum_{\text{channels}} \int dP_L dP_R \left(\frac{d\sigma}{dP_L^2 dP_R^2}\right)^{\text{channel}} \delta\left(\tau - \frac{P_L^2 + P_R^2}{S}\right)$$

Summary of Calculation

$$d\sigma^{\Gamma} \sim \sum_{IJ} \hat{\sigma}^{\Gamma} C_I^{\Gamma} S_{IJ} C_J^{\Gamma *} \otimes J \otimes J \otimes f \otimes f$$

Matching onto QCD is observable independent.

$$\mathcal{L}_{QCD} \to \sum_I C_I^{\Gamma} \mathcal{O}_I^{\Gamma}, \qquad I = \text{color}, \qquad \Gamma = \text{Helicity amplitude}$$

Soft function, S_{IJ}:

- cross section of emission from soft Wilson lines
- critical dependence on observable
- a function of $n_i \cdot n_j$ and any kinematic cuts

Use Inclusive Jet functions and threshold PDF's evolution.

Matching

$q\bar{q} \to q'\bar{q}'$ (7 similar channels, 4 helicity structures)

$$\mathcal{O}_1^{\Gamma} = \left(\bar{q}_R T^a P_{\pm} q_L\right)\left(\bar{q}_2 T^a P_{\pm} q_1\right)$$
$$\mathcal{O}_2^{\Gamma} = \left(\bar{q}_R \mathbf{1} P_{\pm} q_L\right)\left(\bar{q}_2 \mathbf{1} P_{\pm} q_1\right)$$

where

$$q_i = Y_{n_i} W_{n_i}^{\dagger} \xi_{n_i,p_i}$$
$$Y_i = Y_{n_i} = P \exp\left(ig \int ds \; n \cdot A(s \cdot n)\right)$$
$$P_{\pm} = \gamma^{\mu} \frac{1 \pm \gamma_5}{2}, \qquad \Gamma = (+,+),(+,-),\cdots$$

The matching coefficients are helicity dependent.

$q\bar{q} \to q'\bar{q}'$: NLO matching

$$C_1^{(--)} = 1 + \frac{\alpha_s}{4\pi}\left[-2C_F \log^2\frac{s}{\mu^2} + \left(-\beta_0 + 6C_F + C_d \log\frac{u}{t} + C_A \log\frac{s^2}{tu}\right)\log\frac{s}{\mu^2} + X_0 - \frac{C_d - C_A}{4} Y_{stu} + 2\pi i(2C_F - C_A)\right]$$

$$C_1^{(--)} = \frac{\alpha_s}{4\pi}\left[4C_1 \log\frac{u}{t}\log\frac{s}{\mu^2} + X_0 + \frac{C_d - C_A}{4} Y_{sut}\right]$$

$$C_2^{(-+)} = 1 + \frac{\alpha_s}{4\pi}\left[-2C_F \log^2\frac{s}{\mu^2} + \left(-\beta_0 + 6C_F + C_d \log\frac{u}{t} + C_A \log\frac{s^2}{tu}\right)\log\frac{s}{\mu^2} - C_1 Y_{stu} + 2\pi i(2C_F - C_A)\right]$$

$$C_2^{(--)} = \frac{\alpha_s}{4\pi}\left[4C_1 \log\frac{u}{t}\log\frac{s}{\mu^2} - C_1 Y_{stu}\right]$$

(Chiu et al. 07, Kunszt et al. 93)

Matching

$q\bar{q} \to gg$ (4 similar channels)

$$\mathcal{O}_I^{\Gamma} = \left(\bar{\chi}_2^i \mathcal{A}_{R\perp}^{\mu a} \Gamma \mathcal{A}_{R\perp}^{\nu b} \chi_L^j\right)\left(Y_2^{\dagger} \mathcal{Y}_L^{aa'} \mathcal{G}_I^{a'b'} \mathcal{Y}_R^{bb'} Y_1\right)^{ij}$$

where

$$\mathcal{G}_1^{ab} = \delta^{ab}\mathbf{1}$$
$$\mathcal{G}_2^{ab} = d^{abc}\mathbf{T}^c$$
$$\mathcal{G}_3^{ab} = i f^{abc}\mathbf{1}^c$$

Matching

$gg \to gg$

(in the basis of Kidonakis et al.)

$$\mathcal{O}_I^{\Gamma} = \left(\mathcal{A}_{2\perp}^{\mu a}\mathcal{A}_{L\perp}^{\nu b}\mathcal{A}_{R\perp}^{\alpha c}\mathcal{A}_{1\perp}^{\beta d}\right)\left(\mathcal{G}_I^{a'b'c'd'}\mathcal{Y}_2^{aa'}\mathcal{Y}_L^{bb'}\mathcal{Y}_R^{cc'}\mathcal{Y}_1^{dd'}\right)$$

diagonal at Tree level for $N_c = 3$.

$$G_1^{abcd} = \frac{i}{4}\left(d^{ade}f^{cbe} - d^{bce}f^{dae}\right) \qquad G_2^{abcd} = \frac{i}{4}\left(d^{ade}f^{cbe} + d^{bce}f^{dae}\right)$$

$$G_3^{abcd} = \frac{i}{4}\left(d^{cde}f^{bae} + d^{bae}f^{cde}\right) \qquad G_4^{abcd} = \frac{1}{8}\delta^{ab}\delta^{cd}$$

$$G_5^{abcd} = \frac{3}{5}d^{abe}d^{cde} \qquad G_6^{abcd} = \frac{1}{3}f^{bae}f^{cde}$$

$$G_7^{abcd} = \frac{1}{2}\left(\delta^{ad}\delta^{bc} - \delta^{ac}\delta^{bd}\right) - \frac{1}{3}f^{bae}f^{cde}$$

$$G_8^{abcd} = -\frac{3}{5}d^{abe}d^{cde} - \frac{1}{8}\delta^{ab}\delta^{cd} + \frac{1}{2}\left(\delta^{ad}\delta^{bc} + \delta^{ac}\delta^{bd}\right)$$

Evolution of the High scale matching coefficients

The matching coefficients, C_I, evolve according to

$$\frac{d}{d\mu} C_I^{\Gamma}(\mu) = \Gamma^H_{IJ} C_J^{\Gamma}(\mu)$$

with

$$\Gamma^H_{IJ} = \left[\left(\gamma_{\text{cusp}} c_H \log \frac{-s}{\mu^2} + \gamma^H \right) \delta_{IJ} + \gamma_{\text{cusp}} M_{IJ} \right],$$

(Neubert + Becher 09, Chiu et. al 09)

1. All of the μ dependence is contained in $c_H = \frac{1}{2} \sum C_R$.
2. Only need to compute γ^H, which is related to the non-cusp anomalous dimension of the Sudakov Form factor. (Chiu et al.)
3. The matrix M has universal form (to at least 3 loops).

$$M_{IJ} = -\sum_{\langle ij \rangle} \mathbf{T}_i \cdot \mathbf{T}_j \log \left(\frac{-n_i \cdot n_j - i0^+}{2} \right)$$

Evolution of the High scale matching coefficients

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$$M_{IJ} = \begin{pmatrix} \frac{C_d}{2} \log \frac{t}{u} + \frac{1}{2} C_A \log \frac{tu}{s^2} & 2 \log \frac{t}{u} \\ \frac{C_F}{C_A} \log \frac{t}{u} & 0 \end{pmatrix}$$

Diagonalization of High Scale Matching Coefficients

Universality of M_{IJ} allows it to be diagonalized to all orders (for which Casimir scaling holds) by $\hat{C}_K$ with eigenvalue λ_K

$$\frac{d}{d\mu}\hat{C}_K^{\Gamma}(\mu) = \left[\gamma_{\mathsf{cusp}} c_H \log\frac{-s}{\mu^2} + \gamma^H + \gamma_{\mathsf{cusp}}\lambda_K\right]\hat{C}_K^{\Gamma}(\mu)$$

Similar to the diagonalization of (Kidonakis et al.) at NLL.

"Natural" basis for color evolution is not the one convenient for matching.

Evolution is independent of helicity structure. Choose a basis of physical spins and helicity so there is no interference of spin states.

$$d\sigma \sim \sum_{K,K',\Gamma}\left(C_K^{\Gamma} S_{KK'} C_{K'}^{\Gamma *}\right)\hat{\sigma}^{\Gamma}$$

Evolution of the inclusive Jet function

Laplace transformed inclusive jet function:

$$\tilde{j}_i\left(Q^2,\mu\right) \equiv \tilde{j}_i\left(\log\frac{Q^2}{\mu^2},\mu\right) = \int_0^\infty dm^2\ \exp\left(-\frac{m^2}{Q^2 e^{\gamma_E}}\right) j(m^2,\mu)$$

$$\frac{d}{d\log\mu}\tilde{j}_i\left(\log\frac{Q^2}{\mu^2},\mu\right) = \left[-2C_{R_i}\gamma_{\text{cusp}}\log\frac{Q^2}{\mu^2} - 2\gamma^{J_i}\right]\tilde{j}_i\left(\log\frac{Q^2}{\mu^2},\mu\right)$$

where $i = q,\bar{q},g$, $C_{R_q} = C_{R_{\bar{q}}} = C_F$, $C_{R_g} = C_A$.

Evolution of the PDF's

Laplace transform of the PDF's:

$$\tilde{f}_i\,(\tau,\mu) = \int_0^\infty dx\; e^{\frac{(1-x)}{\tau}} f_i(x,\mu)$$

Use the threshold PDF evolution,

$$\frac{d}{d\log\mu}\tilde{f}_q(\tau,\mu) = \left[2C_{R_i}\gamma_{\rm cusp}\log\tau + 2\gamma^{f_q}\right]\tilde{f}_i(\tau,\mu)$$

where $i = q, \bar{q}, g$, $C_{R_q} = C_{R_{\bar{q}}} = C_F$, $C_{R_g} = C_A$.

Soft function: General remarks

$$S_{IJ}(\{k\}, n_i) = \sum_{X_s} \langle 0 | \mathcal{W}_J^\dagger | X_s \rangle \langle X_s | \mathcal{W}_I | 0 \rangle F_s(\{k\})$$

$F_s(\{k\})$ denotes the various projections on soft momentum related to the specific observable. For $q\bar{q} \to q'\bar{q}'$,

$$\mathcal{W}_1 = \mathbf{T}\Big\{\Big(Y_R^\dagger T^a Y_L\Big)\Big(Y_2^\dagger T^a Y_1\Big)\Big\}$$
$$\mathcal{W}_2 = \mathbf{T}\Big\{\Big(Y_R^\dagger \mathbf{1} Y_L\Big)\Big(Y_2^\dagger \mathbf{1} Y_1\Big)\Big\}$$

In Laplace space $\{k\} \to \{Q\}$,

$$\frac{d}{d\log\mu}\widetilde{S}_{IJ}(\{Q\}, \mu) = -\Big[\widetilde{S}_{IK}(\{Q\}, \mu)\Gamma^S_{KJ} + \Gamma^{S*}_{IK}\widetilde{S}_{KJ}(\{Q\}, \mu)\Big]$$

Soft function: General remarks

$$\frac{d}{d\log\mu}\widetilde{S}_{IJ} = -\left[\widetilde{S}_{IK}\Gamma^S_{KJ} + \Gamma^{S*}_{IK}\widetilde{S}_{KJ}\right]$$

$$\Gamma^S_{IJ} = \left(\gamma_{\mathsf{cusp}} c_S \log\left(\frac{\{Q\}}{\mu}, \{n_i\}\right) + \gamma_S\right)\delta_{IJ} + \gamma_{\mathsf{cusp}} M_{IJ}(\{n_i\})$$

Similarities with Γ^H:

1. μ dependent terms proportional to γ_{cusp}, diagonal in color space
2. color mixing is due to a single matrix, M_{IJ}, proportional to γ_{cusp}
3. γ_S depends on the channel, not the color structure

For Γ^S:

1. M_{IJ} can be made the same as the hard function ($\frac{n_i \cdot n_j}{n_k \cdot n_l} = \frac{p_i \cdot p_j}{p_k \cdot p_l}$)
2. $\gamma_{\mathsf{cusp}}\delta_{IJ}$-term can depend on $\{n_i\}$ and $\{Q\}$
3. RG invariance allows us to solve γ_S in terms of $\gamma_H, \gamma^J, \gamma^f$

Soft function: General remarks

Γ^S can be diagonalized by the same matrix that diagonalizes Γ^H.

$$\hat{\Gamma}^S_{KK'} = \left(\gamma_{\text{cusp}} c_S \log\left(\frac{\{Q\}}{\mu}, \{n_i\}\right) + \gamma_S\right)\delta_{IJ} + \gamma_{\text{cusp}}\left(\lambda_K + \lambda_{K'}\right)$$

There is no mixing in this basis. At NNLL, one needs the non-diagonal parts as well.

$$d\sigma \sim \sum_{KK'} \hat{\sigma}^{\Gamma} \hat{C}_K S_{KK'} \hat{C}^*_{K'}$$

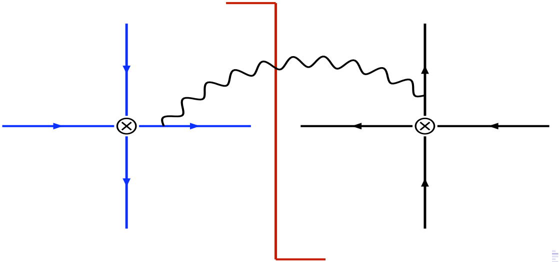
From the factorization theorem, one can calculate γ_S and the dependence on $\{Q\}$ and n_i

These results allow for resummation to NLL for observables for which you don't know finite parts of the soft function.

Calculation of the Soft function

An explicit calculation a check to one loop.

$$S_{IJ}(\{k\}, n_i) = \sum_{X_s} \langle 0| \mathcal{W}_J^\dagger |X_s\rangle \langle X_s| \mathcal{W}_I |0\rangle \times \delta(k_L - n_L \cdot P_L^X)\delta(k_R - n_R \cdot P_R^X)$$

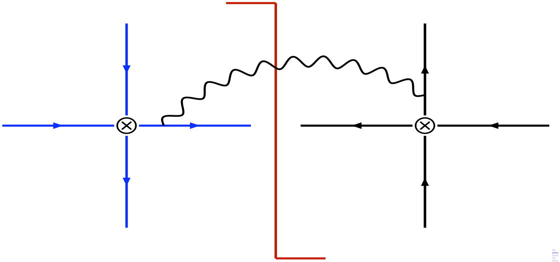


Calculation of the Soft function

An explicit calculation a check to one loop.

For $q\bar{q} \to q'\bar{q}'$: $\mathcal{W}_1 \sim T^a \otimes T^a$ and $\mathcal{W}_2 \sim \mathbf{1} \otimes \mathbf{1}$

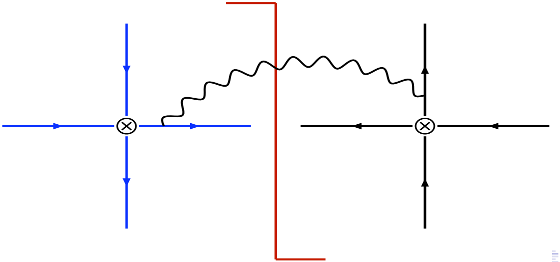
$$S_{21} = \sum_{X_s} \langle 0 | \bar{\mathbf{T}} \left\{ (Y_L^\dagger T^a Y_R)_{i_L i_R} (Y_1^\dagger T^a Y_2)_{i_1 i_2} \right\} | X_s \rangle$$
$$\times \langle X_s | \mathbf{T} \left\{ (Y_R^\dagger \mathbf{1} Y_L)_{i_R i_L} (Y_2^\dagger \mathbf{1} Y_1)_{i_2 i_1} \right\} | 0 \rangle$$
$$\times \delta(k_L - n_L \cdot P_L^X) \delta(k_R - n_R \cdot P_R^X)$$



Calculation of the Soft function

An explicit calculation a check to one loop.

$$S_{21} \sim \mathrm{Tr}\left[(Y_L^\dagger T^a Y_R)(Y_R^\dagger \mathbf{1} Y_L)\right] \mathrm{Tr}\left[(Y_1^\dagger T^a Y_2)(Y_2^\dagger \mathbf{1} Y_1)\right]$$



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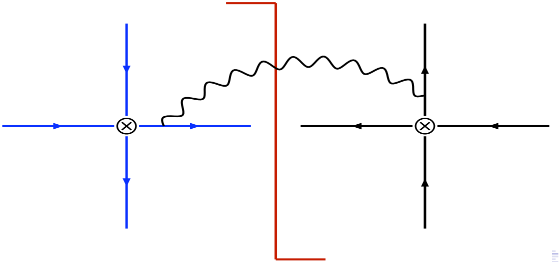

Calculation of the Soft function

$$S_{IJ}^{(\text{channel})}(k_L, k_R) = \sum_{n_a, n_b} D_{IJ}^{(\text{channel})}(n_a, n_b) I_S(n_a, n_b, k_L, k_R)$$

The integrals, I_S, are independent of color and channel,

$$I_S(n_a, n_b, k_L, k_R) = \left(\frac{\bar{\mu}^2 e^{\gamma_E}}{4\pi}\right)^{\epsilon} \int \frac{d^d q}{(2\pi)^{d-1}} \Theta(q_0)\delta(q^2) \frac{n_a \cdot n_a}{(n_a \cdot q)(n_b \cdot q)}$$
$$\times \Big\{ \Theta(n_R \cdot q - n_L \cdot q)\delta(k_R - n_R \cdot q)\delta(k_L)$$
$$+ \Theta(n_R \cdot q - n_L \cdot q)\delta(k_L - n_L \cdot q)\delta(k_R) \Big\}$$

I_S is invariant under rescaling of n_1 and n_2, and so it can only depend on

$$\frac{n_1 \cdot n_L}{n_1 \cdot n_R} = \frac{n_2 \cdot n_R}{n_2 \cdot n_L} = \frac{-u}{-t} \equiv \nu$$

Calculation of the Soft function

The color factors, D_{IJ} are the same same for any observable since they are independent of phase restrictions.

For $q\bar{q} \to q'\bar{q}'$,

$$D(0,0) = \begin{pmatrix} \frac{1}{2}C_F C_A & 0 \\ & C_A^2 \end{pmatrix}, \qquad \text{tree result}$$

$$D(n_1, n_2) = D(n_L, n_R) = \begin{pmatrix} -\frac{1}{4}C_F & 0 \\ 0 & C_A^2 C_F \end{pmatrix}$$

$$D(n_1, n_L) = D(n_2, n_R) = \begin{pmatrix} \frac{1}{2}C_F & -\frac{1}{2}C_A C_F \\ -\frac{1}{2}C_A C_F & 0 \end{pmatrix}$$

$$D(n_1, n_R) = D(n_2, n_L) = \begin{pmatrix} \frac{C_F}{4}(C_A^2 - 2) & \frac{1}{2}C_A C_F \\ \frac{1}{2}C_A C_F & 0 \end{pmatrix}$$

Not just $\mathbf{T}_i \cdot \mathbf{T}_j$, although the two may be related.

Calculation of the Soft function

The color factors, D_{IJ} are the same same for any observable since they are independent of phase restrictions.

For $q\bar{q} \to gg$,

$$D(0,0) = \mathrm{diag}\Big(2C_A^2 C_F, C_F(C_A^2-4), C_A^2 C_F\Big)$$

$$D(n_1,n_2) = D(n_L,n_R) = \begin{pmatrix} 2C_A^2 C_F^2 & 0 & 0 \\ 0 & -\frac{1}{2C_A}(C_A^2-4)C_F & 0 \\ 0 & 0 & -\frac{1}{2}C_A C_F \end{pmatrix}$$

$$D(n_1,n_2) = D(n_L,n_R) = \begin{pmatrix} 0 & 0 & C_A^2 C_F \\ 0 & \frac{1}{4}C_A(C_A^2-4)C_F & \frac{1}{4}C_A\left(C_A^2-4\right)C_F \\ C_A^2 C_F & \frac{1}{4}C_A(C_A^2-4)C_F & \frac{1}{4}C_A^3 C_F \end{pmatrix}$$

$$D(n_1,n_2) = D(n_L,n_R) = \begin{pmatrix} 0 & 0 & -C_A^2 C_F \\ 0 & \frac{1}{4}C_A(C_A^2-4)C_F & -\frac{1}{4}C_A(C_A^2-4)C_F \\ -C_A^2 C_F & -\frac{1}{4}C_A(C_A^2-4)C_F & \frac{1}{4}C_A^3 C_F \end{pmatrix}.$$

Not just $\mathbf{T}_i \cdot \mathbf{T}_j$, although the two may be related.

Calculation of the Soft function

$$S_{IJ}(k) = \int dk_L dk_R \; S_{IJ}(k_L, k_R)\delta(k - k_L - k_R)$$

For $q\bar{q} \to q'\bar{q}'$,

$$S_{IJ}(k,\mu) = \delta(k)\begin{pmatrix} \frac{1}{2}C_F C_A & 0 \\ 0 & C_A^2 \end{pmatrix} + \frac{\alpha}{4\pi}\left\{ f_S\left(\frac{u}{t}\right)\delta(k) + \begin{pmatrix} 4C_F \log\frac{-st}{u^2} & 8C_A C_F \log\frac{t}{u} \\ 8C_A C_F \log\frac{t}{u} & -16C_A^2 C_F \log\frac{-t}{s} \end{pmatrix} \left[\frac{1}{k}\right]_*^{[k,\mu]} \right\}$$

No $\log Q\mu$ dependence. Accident of this channel.

$f_S(u/t)$, needed for NNLL, is the μ independent and can depends

$$\frac{t-u}{s} = \frac{s}{-t}\cos\theta_{1R}$$

$q\bar{q} \to q\bar{q}'$: Soft function

$$\nu = \frac{-u}{-t}, \qquad 1-\nu = \frac{s}{-t}\cos\theta_{1R}$$

$$S_{IJ}^{q\bar{q}q\bar{q}}(k) = \delta(k)\begin{pmatrix} \frac{1}{2}C_F C_A & 0 \\ 0 & C_A^2 \end{pmatrix}$$

$$+\left(\frac{\alpha_s}{4\pi}\right)\left(\begin{array}{c|c} \begin{array}{c} C_F\left(-\mathrm{Li}_2(1-\nu) - \frac{1}{2}\mathrm{Li}_2\frac{1}{\nu} + \frac{1}{2}\ln\nu\ln\frac{\nu^8(1+\nu)^2}{(1-\nu)^5}\right) \\ -\frac{\pi^2}{4}C_F + C_A^2 C_F\left(\ln\nu\ln\frac{(\nu-1)^2}{2\nu} + \frac{1}{2}\mathrm{Li}_2\frac{1}{\nu}\right) \end{array} & C_A C_F\left(\ln\frac{(\nu-1)^2}{\nu^3}\ln\nu - \mathrm{Li}_2\frac{1}{\nu}\right) \\ \hline C_A C_F\left(\ln\frac{(\nu-1)^2}{\nu^3}\ln\nu - \mathrm{Li}_2\frac{1}{\nu}\right) & \begin{array}{c} C_A^2 C_F(\pi^2 + 4\mathrm{Li}_2(1-\nu) + 6\mathrm{Li}_2(\frac{1}{\nu}) \\ +C_A^2 C_F\ln\nu\ln\frac{(\nu-1)^2}{(\nu+1)^4\nu^4} \end{array} \end{array}\right)\delta(k)$$

$$+\left(\frac{\alpha_s}{4\pi}\right)\begin{pmatrix} C_F\ln\frac{1+\nu}{\nu^2} & 2C_A C_F\ln\nu \\ 2C_A C_F\ln\nu & -4C_A^2 C_F\ln(1+\nu) \end{pmatrix}\left[\frac{1}{k}\right]_\star^{[k,\mu]} \qquad (214)$$

Evolution of the soft functions

$$\Gamma^S_{IJ} = \left(\gamma_{\mathsf{cusp}} c_S \log\left(\frac{\{Q\}}{\mu}, \{n_i\}\right) + \gamma_S\right)\delta_{IJ} + \gamma_{\mathsf{cusp}} M_{IJ}(\{n_i\})$$

For $q\bar{q} \to q'\bar{q}'$,

$$c_S \log\left(\frac{\{Q\}}{\mu}, \{n_i\}\right) = -2C_F \log\frac{-t}{s}$$

$$\gamma_S = 0$$

$$M = \begin{pmatrix} \frac{C_d}{2}\log\frac{t}{u} + \frac{C_A}{2}\log\frac{tu}{s^2} & 2\log\frac{t}{u} \\ \frac{C_F}{C_A}\log\frac{t}{u} & 0 \end{pmatrix}$$

Evolution of the soft functions

$$\Gamma^S_{IJ} = \left(\gamma_{\mathsf{cusp}} c_S \log\left(\frac{\{Q\}}{\mu}, \{n_i\}\right) + \gamma_S\right)\delta_{IJ} + \gamma_{\mathsf{cusp}} M_{IJ}(\{n_i\})$$

For $q\bar{q} \to gg$,

$$c_S \log\left(\frac{\{Q\}}{\mu}, \{n_i\}\right) = (2C_F - 2C_A)\log\frac{Q}{\mu} - 2C_F \log\frac{-t}{s}$$

$$\gamma_S = 0$$

$$M = \begin{pmatrix} 0 & 0 & \log\frac{u}{t} \\ 0 & \frac{C_A}{2}\log\frac{tu}{s^2} & \frac{C_A}{2}\log\frac{u}{t} \\ 2\log\frac{u}{t} & \frac{C_d}{2}\log\frac{u}{t} & \frac{C_A}{2}\log\frac{tu}{s^2} \end{pmatrix}$$

Evolution of the soft functions

$$\Gamma^S_{IJ} = \left(\gamma_{\text{cusp}} c_S \log\left(\frac{\{Q\}}{\mu}, \{n_i\}\right) + \gamma_S\right)\delta_{IJ} + \gamma_{\text{cusp}} M_{IJ}(\{n_i\})$$

In general:

$$c_S \log\left(\frac{\{Q\}}{\mu}, \{n_i\}\right) = c_Q \log\frac{Q^2}{\mu^2} + c_T \log\frac{s}{-t}$$

where

$$c_Q = \frac{1}{2}\sum_{\text{initial}} C_R - \frac{1}{2}\sum_{\text{final}} C_R$$

$$c_T = \sum_{\text{initial}} C_R$$

$$\gamma_S = 0$$

$$M = \text{ same as the hard function}$$

Check on RG invariance

The factorization theorem in Laplace space implies

$$0 = \frac{d}{d\log\mu}\Big\{\mathbf{C}(\mu)\tilde{\mathbf{S}}\left(Q^2/\sqrt{s},\mu\right)\mathbf{C}^\dagger(\mu) \times \tilde{J}(Q,\mu)\,\tilde{J}(Q,\mu)\,\tilde{f}_1\left(\frac{Q^2}{-t},\mu\right)\tilde{f}_2\left(\frac{Q^2}{-t},\mu\right)\Big\} = 0$$

The expression proportional to γ_{cusp}, gives constraint:

$$\gamma_{\text{cusp}}\Big(2C_1\log\frac{Q^2}{-t} + 2C_2\log\frac{Q^2}{-t} - 2C_L\log\frac{Q^2}{\mu^2} - 2C_R\log\frac{Q^2}{\mu^2} + c_H\log\frac{s}{\mu^2} + c_Q\log\frac{Q^4}{s\mu^2} + 2c_T\log\frac{-t}{s}\Big)$$
$$+\gamma_{\text{cusp}}\Big(\lambda_K + \lambda_K' - \lambda_K - \lambda_K'\Big)$$
$$+2\gamma_{f_1} + 2\gamma_{f_2} - 2\gamma_{j_L} - 2\gamma_{j_R} + \gamma_H + \gamma_H^* - \gamma_S = 0$$

Check on RG invariance

The factorization theorem in Laplace space implies

$$0 = \frac{d}{d\log\mu}\Big\{\mathbf{C}(\mu)\tilde{\mathbf{S}}\left(Q^2/\sqrt{s},\mu\right)\mathbf{C}^\dagger(\mu)$$
$$\times\,\tilde{J}\left(Q,\mu\right)\tilde{J}\left(Q,\mu\right)\tilde{f}_1\left(\frac{Q^2}{-t},\mu\right)\tilde{f}_2\left(\frac{Q^2}{-t},\mu\right)\Big\} = 0$$

The expression proportional to γ_{cusp}, gives constraint:

$$c_H = \frac{1}{2}(C_1 + C_2 + C_L + C_R)$$
$$c_Q = \frac{1}{2}(-C_1 - C_2 + C_L + C_R))$$
$$c_T = (C_1 + C_2)$$

The rest gives constraint:

$$\gamma_S = 2\gamma_{f_1} + 2\gamma_{f_2} - 2\gamma_{j_L} - 2\gamma_{j_R} + \gamma_H + \gamma_H^*$$

final results

Threshold thrust can be related to p_T and y,

$$\tau = S - 2p_T\sqrt{S}\cosh y$$

Final resumed distribution,

$$\frac{d\sigma}{dp_T dy} = \frac{2}{p_T} \sum_{\substack{\text{channels} \\ 12\to LR}} \int_{\frac{P_T}{\sqrt{S}}e^{y}}^{1-\frac{P_T}{\sqrt{S}}e^{-y}} dv \int_{\frac{P_T}{\sqrt{S}}e^{y}}^{1-\frac{P_T}{\sqrt{S}}e^{-y}} dw \, [x_1 f_{1/N_1}(x_1,\mu)][x_2 f_{2/N_2}(x_2,\mu)]$$
$$\times \frac{1}{N_{\text{int}}}\frac{1}{\hat{\tau}} \sum_{K,K',\Gamma} d\sigma^{\Gamma} C_K^{\Gamma} C_{K'}^{\Gamma *}$$
$$\times \exp\left[4c_H S(\mu_h,\mu) - 2A_H(\mu_h,\mu) - \eta_H\left(\lambda_K + \lambda_K' - 2c_H \log\frac{s}{\mu_h^2}\right)\right]$$
$$\times \exp(4c_S S(\mu_s,\mu) - 2A_S(\mu_s,\mu))$$
$$\times \exp\left[-\eta_s\left(\lambda_K + \lambda_{K'} + c_T\frac{-t}{s}\right)\right] \widetilde{S}_{KK'}(\partial_{\eta_s},\mu_s)\left(\frac{\hat{\tau}}{\mu_s^2}\right)^{\frac{c_Q\eta_s}{2}} \frac{e^{\gamma_E\eta_s}}{\Gamma(\eta_s)}$$
$$\times \exp(-4c_L S(\mu_j,\mu) - 2A_{J_L}(\mu_j,\mu))\tilde{j}_L(\partial_{\eta_j},\mu_s)\left(\frac{\hat{\tau}}{\mu_j^2}\right)^{\eta_L}\frac{e^{\gamma_E\eta_L}}{\Gamma(\eta_L)}$$
$$\times \exp(-4c_R S(\mu_j,\mu) - 2A_{J_R}(\mu_j,\mu))\tilde{j}_R(\partial_{\eta_j},\mu_s)\left(\frac{\hat{\tau}}{\mu_j^2}\right)^{\eta_R}\frac{e^{\gamma_E\eta_R}}{\Gamma(\eta_R)}$$

Conclusions

- We demonstrated the calculation of a simple (NNLL) dijet observable.

- Isolated which features are observable independent.

- The diagonalized basis is the best for color resummation.

- The final results for $\frac{d^2\sigma}{dp_T dy}$ take a relatively simple form.

Back Up

Hemisphere masses and Threshold Thrust

First Calculate the Hemisphere mass distribution

$$\frac{d^2\sigma}{dP_L^2 dP_R^2}$$

The thrust distribution (at threshold) is then

$$\frac{d\sigma}{d\tau} = \int dP_L^2 dP_R^2 \; \delta(\tau - P_L^2 - P_R^2) \frac{d^2\sigma}{dP_L^2 dP_R^2}$$

The thrust (at threshold) can be related to p_T and y.

$$\tau = S - 2p_T\sqrt{S}\cosh y$$

limited phenomenological interest, but demonstrates key ingredients for a realistic calculation.

- color mixing

- dependence on a simple cut

- a more exclusive jet introduces new scales (non-global logs, etc)

Matching

$q\bar{q} \to q'\bar{q}'$ (7 similar channels, 4 helicity structures)

$$\mathcal{O}_1^{\Gamma} = \left(\bar{\chi}_R Y_R^{\dagger} T^a Y_L P_{\pm} \chi_L\right)\left(\bar{\chi}_2 Y_2^{\dagger} T^a Y_1 P_{\pm} \chi_1\right)$$
$$\mathcal{O}_2^{\Gamma} = \left(\bar{\chi}_R Y_R^{\dagger} \mathbf{1} Y_L P_{\pm} \chi_L\right)\left(\bar{\chi}_2 Y_2^{\dagger} \mathbf{1} Y_1 P_{\pm} \chi_1\right)$$

where

$$\chi_i = W_{n_i}^{\dagger} \xi_{n_i,p_i}$$
$$Y_i = Y_{n_i} = P \exp\left(ig \int ds\ n \cdot A(s \cdot n)\right)$$
$$P_{\pm} = \gamma^{\mu} \frac{1 \pm \gamma_5}{2}, \qquad \Gamma = (+,+), (+,-), \cdots$$

Matching

$q\bar{q} \to q'\bar{q}'$ (7 similar channels, 4 helicity structures)

$$\mathcal{O}_1^{\Gamma} = \left(\bar{\chi}_R^{i_R}\Gamma\chi_L^{i_L}\right)\left(\bar{\chi}_2^{i_2}\Gamma'\chi_L^{i_1}\right)\left[\left(Y_R^{\dagger}T^aY_L\right)^{i_Ri_L}\left(Y_2^{\dagger}T^aY_1\right)^{i_2i_1}\right]$$

$$\mathcal{O}_2^{\Gamma} = \left(\bar{\chi}_R\Gamma\chi_L\right)\left(\bar{\chi}_2\Gamma'\chi_L\right)\left[\left(Y_R^{\dagger}\mathbf{1}Y_L\right)\left(Y_2^{\dagger}\mathbf{1}Y_1\right)\right]$$

where

$$\chi_i = W_{n_i}^{\dagger}\xi_{n_i,p_i}$$

$$Y_i = Y_{n_i} = P\exp\left(ig\int ds\ n\cdot A(s\cdot n)\right)$$

$$P_{\pm} = \gamma^{\mu}\frac{1\pm\gamma_5}{2}, \qquad \Gamma = (+,+),(+,-),\cdots$$

Diagonalization of the soft anomalous dimension

Γ^S is diagonalized by the same matrix as Γ^C.

$$\hat{\mathbf{S}}(Q,\mu) = \mathbf{w}\mathbf{S}(Q,\mu)\mathbf{w}^\dagger$$

For $q\bar{q} \to q'\bar{q}'$

$$\begin{pmatrix} \hat{S}_{++} & \hat{S}_{+-} \\ \hat{S}_{-+} & \hat{S}_{--} \end{pmatrix} = \begin{pmatrix} w_{1+} & w_{1-} \\ w_{2+} & w_{2-} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} w_{1+} & w_{1-} \\ w_{2+} & w_{2-} \end{pmatrix}^{-1}$$

The evolution equation simplifies,

$$\frac{d}{d\log\mu}\hat{S}_{+-}(Q,\mu) = -\left[\gamma_S + \gamma_{\text{cusp}}\left(\lambda_+ + \lambda_-^* - 4C_F \log\frac{-t}{s}\right)\right]\hat{S}_{+-}(Q,\mu)$$