# D-branes and K-theory

or

what's a D-brane charge?

Matthias Traube[1]

[1]Max Planck Institut für Physik

Young scientists Workshop Ringberg

# An analogy: A particle

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- $A_\mu$ vector potential for background electric field
- $q$ electric charge of particle

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And what about non trivial geometry for $N_p$?

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- $N_p$ sufficiently complicated to suffice non trivial geoemtry

# Open Strings?

- Stack of $N$-parallel D-branes $\Rightarrow E \rightarrow N_p$ rank n complex vector bundle

$$\Rightarrow \text{Open strings} \Leftrightarrow E$$

# Action - the real deal

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$\Rightarrow \ ch(f_! E)$ is a charge current!

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This gives

$$S \sim N \int_{N_p} C_{p+1} \tag{5}$$

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There are three pieces:

1) $f_!$ pushes $E$ from $N_p$ to $S$
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So, $f_! E$ is an element in K-theory over $S \Rightarrow$ **D-brane charges are not just numbers but elements in K-theory over spacetime**

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**K-theory allows to subtract stuff**

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$$\Rightarrow \quad -[(a, b)] \equiv [(b, a)]$$

$$a \geq b \Rightarrow a = c + b \Rightarrow [(c + b, b)] = [(c, 0)] + [(b, b)] = [(c, 0)]$$

$$a \leq b \Rightarrow b = a + d \Rightarrow [(a, a + d)] = [(a, a)] + [(0, d)] = [(0, d)]$$

Hence the map $f : \mathbb{Z} \to K(\mathbb{N})$ is an isomorphism

$$f(n) = \begin{cases} [(n, 0)] & \text{if } n \geq 0 \\ [(0, n)] & \text{if } n < 0 \end{cases}$$

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$\Rightarrow K(\mathcal{V})$ has subtraction (in fact is a ring), i.e. $" - E"$ makes sense in $K(\mathcal{V})$.

For $K(\mathcal{V})$ charges it is possible that Dp-anti Dp-branes annihilate and leave lower dimensional branes.

# Take home message

1) Dp-branes are charged under all lower dimensional $C_q$.
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3) Open string effects and geometry of worldvolume lead naturally to K-theory as the right framework.

# Take home message

1) Dp-branes are charged under all lower dimensional $C_q$.
2) There is a mathematical framework describing the charges (K-theory).
3) Open string effects and geometry of worldvolume lead naturally to K-theory as the right framework.

The end (for real)

# For the D-brane aficionado

K-theory over spacetime is particularly nice in respect of the Sen conjecture: Every D-brane configuration arises from a D9-anti D9 brane condensate!

In $K(\mathcal{V})$ we have $[E] = [F] \Leftrightarrow \exists\, \underline{n}$ s.th. $E \oplus \underline{n} \cong F \oplus \underline{n}$, this is like saying things are equivalent iff we can add decoupled (trivial) theories s.th. they are equal (stably equivalent). This easily follows from thefact that $E \to M$ cplx. vector bundle over compact manfiold has a complementary cplx. vector bundle $\tilde{E} \to M$ s.th. $E \oplus \tilde{E} \cong \underline{m}$ for some $m \in \mathbb{N}$.

All of the above was for type IIB. For type IIA we get as charge space, roughly speaking, $K^1(S) = K(S^1 \wedge S)$, where $S^1 \wedge S$ is the smash product of spacetime with the unit circle.

# For the dedicated D-brane aficionado

There is an approach to K-theory using maps from the base space to a space of Fredholm operators in an infinite dimensional Hilbert space (c.f. Basic Bundle Theory and K-Cohomology Invariants, chapter 20, theorem 2.2). One can use this definition to define K-theory on non-compact spaces and furthermore incorporate non-trivial $H$-flux (twisted K-theory, Atiyah-Hirzebruch-Spectral-Sequence).