Generalized complex geometry and topological sigma models

work in collaboration with

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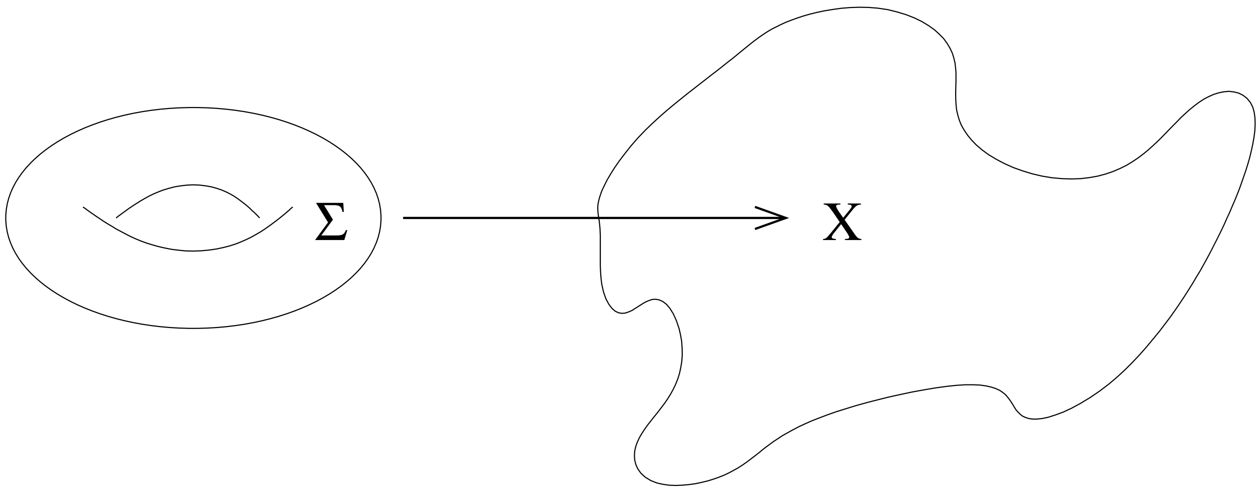

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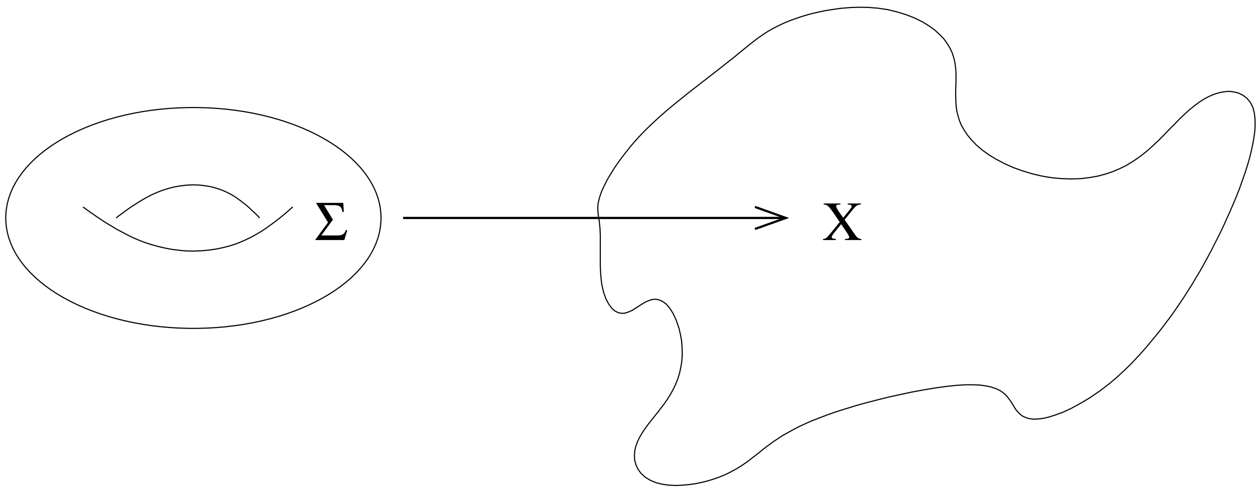

We take $X = \mathbb{R}^{1,3} \times M^6$.

Sigma models

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- integration only over ϕ gives the sigma model.
- supersymmetric models $\rightsquigarrow$ make ϕ a superfield containing worldsheet bosons and left- and right-moving fermions.

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- but computing physical observables can be reduced to geometrical questions.

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Before the twist: WS fermions $\psi_\pm$, which can be split into (anti-)holomorphic components w.r.t. $I_\pm$

$$\mathcal{P}_+\psi_+, \overline{\mathcal{P}}_+\psi_+, \mathcal{P}_-\psi_-, \overline{\mathcal{P}}_-\psi_-,$$

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where $\mathcal{P}_\pm = \frac{1}{2}(1 - iI_\pm)$, $\overline{\mathcal{P}}_\pm = \frac{1}{2}(1 + iI_\pm)$.

This corresponds to a splitting of the (complexified) tangent bundle of the target space

$$TM_{\mathbb{C}} = TM_+^{1,0} \oplus TM_+^{0,1} \oplus TM_-^{1,0} \oplus TM_-^{0,1}.$$

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$q_{V/A}$: vectorial/axial charge, J: spin before the twist, $J_{A/B}$: spin after the twist $= J + q_{V/A}/2$.

	q_V	q_A	J	J_A	J_B
$\mathcal{P}_+\psi_+$	-1	-1	$-\frac{1}{2}$	-1	-1
$\overline{\mathcal{P}}_+\psi_+$	$+1$	$+1$	$-\frac{1}{2}$	0	0
$\mathcal{P}_-\psi_-$	-1	$+1$	$+\frac{1}{2}$	0	$+1$
$\overline{\mathcal{P}}_-\psi_-$	$+1$	-1	$+\frac{1}{2}$	$+1$	0

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Note: A and B model are related by $I_- \leftrightarrow -I_-$.

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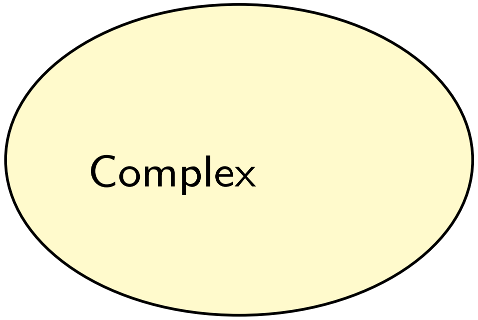

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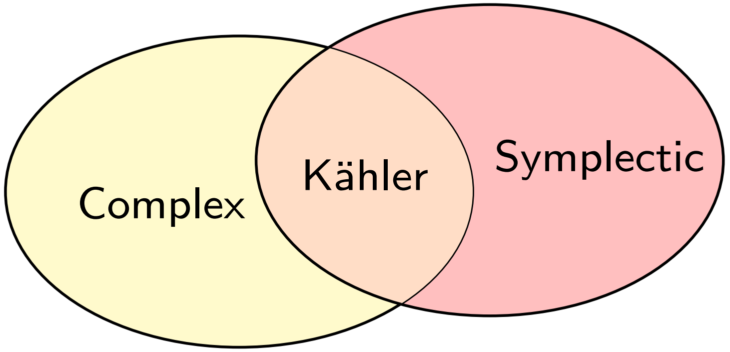

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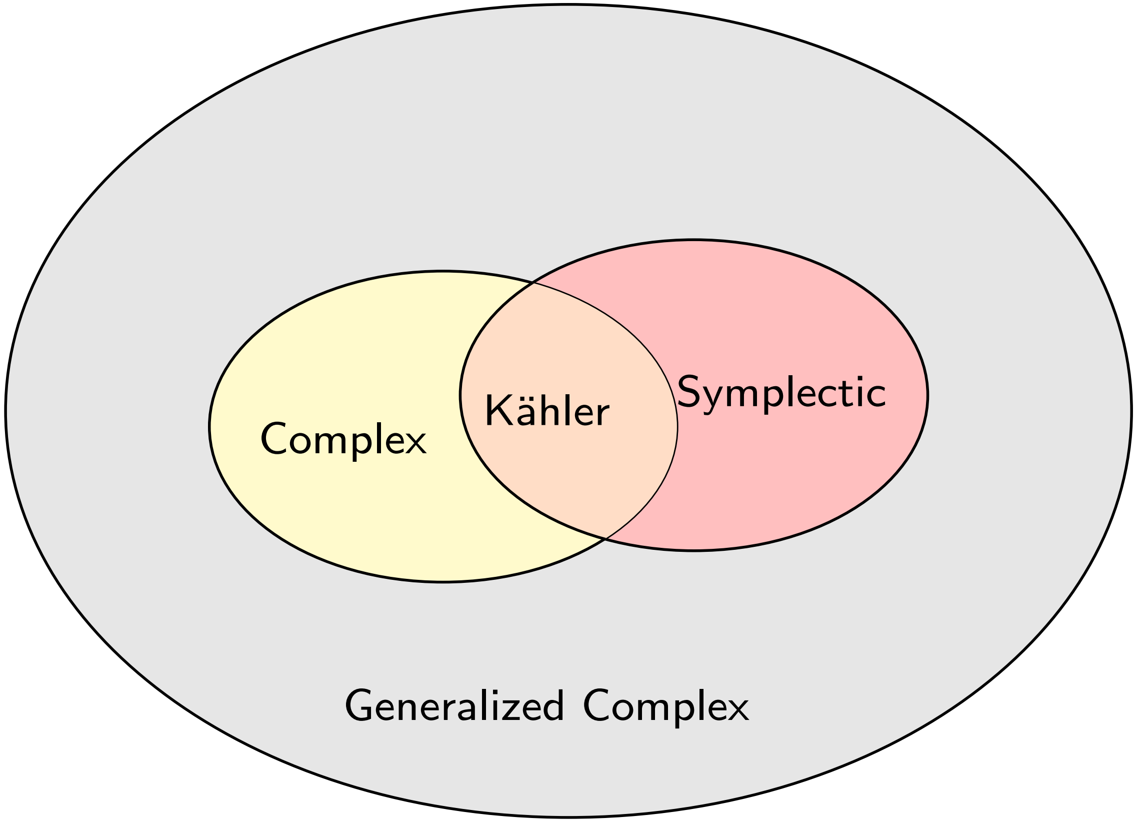

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	Standard	Generalized
	TM	$TM \oplus TM^*$
Elements	X	$X + \xi$
Structure group	$GL(2d)$	$O(2d, 2d)$
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$\exists$ a natural inner product on $T \oplus T^*$ given by

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\xi(X) + \eta(Y))$$

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- A Generalized complex structure (GCS) $\mathcal{J}$ can be defined as an endomorphism on $TM \oplus TM^*$, which satisfies

$$\mathcal{J}^2 = -1 \quad \text{and} \quad \mathcal{J}^* = -\mathcal{J}.$$

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Limiting cases reproducing standard complex and symplectic structures.

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From a standard Kähler structure (g, J, ω) we get $\mathcal{J}_J, \mathcal{J}_\omega$ and

$G = -\mathcal{J}_J \mathcal{J}_\omega$.

Generalized topological sigma models

Define a generalized Kähler structure by

$$\mathcal{J}_{1/2} = \frac{1}{2}\begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+^T \pm I_-^T) \end{pmatrix}.$$

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$\rightsquigarrow$ We can desribe the target space of the top. sigma models with a GKS.

Generalized B/A model:

$$\Psi_{1/2} := \frac{1}{2}(1 + i\mathcal{J}_{1/2})\begin{pmatrix} \psi_+ + \psi_- \\ g(\psi_+ - \psi_-) \end{pmatrix}.$$

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- Answer: From the generalized A or B model we get *both*. E.g. from the gen. B model we get the old B model by taking $I_+ = I_-$ and the old A model by $I_+ = -I_-$.

$$\mathcal{J}_1 = \frac{1}{2}\begin{pmatrix} I_+ + I_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(I_+^T + I_-^T) \end{pmatrix} \xrightarrow{B} \begin{pmatrix} I & 0 \\ 0 & -I^T \end{pmatrix}$$

$$\mathcal{J}_1 \xrightarrow{A} \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}$$

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- Kähler and complex structure moduli are exchanged.

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- What about the relation to "physical" string theory? Can we embed the setup to type IIA/B?
- What about lifts to M-theory (G2 manifolds)?
- Can we construct new types of topological branes using this framework?